Additional Problem for MA2C03 concerning Vector Algebra

6. Let $A, B, C, D, E, F, G$ and $H$ be eight points in three-dimensional Euclidean space whose Cartesian coordinates are as follows:—

$$A = (1, 3, 2), \quad B = (3, 7, 3), \quad C = (4, 5, 1), \quad D = (6, 9, 2),$$
$$E = (1, 4, 7), \quad F = (3, 8, 8), \quad G = (4, 6, 6), \quad H = (6, 10, 7).$$

Note that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \mathbf{u},$$
$$\overrightarrow{AC} = \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{FH} = \mathbf{v},$$
$$\overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = \mathbf{w},$$

where

$$\mathbf{u} = (2, 4, 1), \quad \mathbf{v} = (3, 2, -1), \quad \mathbf{w} = (0, 1, 5).$$

It follows that $A, B, C, D, E, F, G$ and $H$ are the vertices of a parallelepiped in three-dimensional Euclidean space.

(a) Calculate the length of the line segments $BG BH$, and the cosine of the angle between these two line segments at the point $B$.

(6 points)

(b) Calculate the equation of the plane passing through the points $A, B$ and $F$, expressing the equation of the plane in the form $ax + by + cz = k$ for appropriate real constants $a, b, c$ and $k$.

(8 points)

(c) Find the volume of the parallelepiped with vertices at $A, B, C, D, E, F, G$ and $H$.

(6 points)