40. Introduction to Harmonic Analysis

40.1. Basic Trigonometrical Identities and Integrals

The following trigonometric identities satisfied by the sine and cosine functions are basic and well-known:

\[
\begin{align*}
\cos^2 A + \sin^2 A &= 1, \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B, \\
\cos 2A &= \cos^2 A - \sin^2 A, \\
\sin(A + B) &= \sin A \cos B + \cos A \sin B, \\
\sin 2A &= 2 \sin A \cos A, \\
\cos^2 A &= \frac{1}{2} (1 + \cos 2A), \\
\sin^2 A &= \frac{1}{2} (1 - \cos 2A), \\
2 \cos A \cos B &= \cos(A + B) + \cos(A - B), \\
2 \sin A \cos B &= \sin(A + B) + \sin(A - B), \\
2 \sin A \sin B &= \cos(A - B) - \cos(A + B),
\end{align*}
\]
On differentiating the sine and cosine function, we find that

\[
\frac{d}{dx} \sin qx = q \cos qx \\
\frac{d}{dx} \cos qx = -q \sin qx.
\]

for all real numbers \(q\).

It follows that

\[
\int \sin qx = -\frac{1}{q} \cos qx + C \\
\int \cos qx = \frac{1}{q} \sin qx + C,
\]

for all non-zero real numbers \(q\), where \(C\) is a constant of integration.
Proposition 40.1

Let $j$ and $k$ be positive integers. Then

\[
\begin{align*}
\int_0^{2\pi} \cos jx \, dx &= 0, \\
\int_0^{2\pi} \sin jx \, dx &= 0, \\
\int_0^{2\pi} \cos jx \cos kx \, dx &= \begin{cases} 
\pi & \text{if } j = k, \\
0 & \text{if } j \neq k,
\end{cases} \\
\int_0^{2\pi} \sin jx \sin kx \, dx &= \begin{cases} 
\pi & \text{if } j = k, \\
0 & \text{if } j \neq k,
\end{cases} \\
\int_0^{2\pi} \sin jx \cos kx \, dx &= 0.
\end{align*}
\]
**Proof**

First we note that

\[
\int_0^{2\pi} \cos jx \, dx = \left[ \frac{1}{j} \sin jx \right]_0^{2\pi} = \frac{1}{j} (\sin 2j\pi - 0) = 0
\]

and

\[
\int_0^{2\pi} \sin jx \, dx = \left[ -\frac{1}{j} \cos jx \right]_0^{2\pi} = -\frac{1}{j} (\cos 2j\pi - 1) = 0
\]

for all non-zero integers \( j \), since \( \cos 2j\pi = 1 \) and \( \sin 2j\pi = 0 \) for all integers \( j \).
Let $j$ and $k$ be positive integers. It follows from basic trigonometrical identities that

$$\int_0^{2\pi} \cos jx \cos kx \, dx = \frac{1}{2} \int_0^{2\pi} (\cos((j - k)x) + \cos((j + k)x)) \, dx.$$ 

and

$$\int_0^{2\pi} \sin jx \sin kx \, dx = \frac{1}{2} \int_0^{2\pi} (\cos((j - k)x) - \cos((j + k)x)) \, dx$$

But

$$\int_0^{2\pi} \cos((j + k)x) \, dx = 0$$

(since $j + k$ is a positive integer, and is thus non-zero).
Also
\[ \int_{0}^{2\pi} \cos((j - k)x) \, dx = 0 \text{ if } j \neq k, \]
and
\[ \int_{0}^{2\pi} \cos((j - k)x) \, dx = 2\pi \text{ if } j = k \]
(since \( \cos((j - k)x) = 1 \) when \( j = k \)). It follows that
\[ \int_{0}^{2\pi} \cos jx \cos kx \, dx = \int_{0}^{2\pi} \sin jx \sin kx \, dx = \frac{1}{2} \int_{0}^{2\pi} \cos((j - k)x) \, dx \]
\[ = \begin{cases} \pi & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \]
Also
\[ \int_{0}^{2\pi} \sin jx \cos kx \, dx = \frac{1}{2} \int_{0}^{2\pi} (\sin((j + k)x) + \sin((j - k)x)) \, dx = 0 \]
for all positive integers \( m \) and \( n \). (Note that \( \sin((j - k)x) = 0 \) in the case when \( j = k \)).
40.2. Fourier Coefficients

We consider the theory of harmonic analysis, in which functions are approximated by sums of trigonometric functions. Let $p$ and $q$ be real numbers satisfying $p < q$. Let us denote by $I(p, q)$ the set whose elements are those real-valued functions on the interval

$$\{x \in \mathbb{R} : p \leq x \leq q\}$$

that are integrable and that have finitely many points of discontinuity in the interval.
We restrict attention to the case where $p = 0$ and $q = 2\pi$. Given $f, g \in \mathcal{I}(0, 2\pi)$, we define

$$(f, g) = \frac{1}{\pi} \int_{0}^{2\pi} f(x)g(x) \, dx$$

Note that

$$(f + h, g) = (f, g) + (h, g) \quad \text{and} \quad (f, g + h) = (f, g) + (f, h)$$

for all $f, g, h \in \mathcal{I}(0, 2\pi)$. Moreover $(f, g) = (g, f)$, and

$$(cf, g) = (f, cg) = c(f, g)$$

for all $f, g \in \mathcal{I}(0, 2\pi)$ and for all real numbers $c$. Also let

$$\|f\| = \sqrt{(f, f)} = \left(\frac{1}{\pi} \int_{0}^{2\pi} f(x)^2 \, dx\right)^{\frac{1}{2}}.$$
If \( f \in \mathcal{I}(0, 2\pi) \), and if \( \|f\| = 0 \) then either \( f(x) = 0 \) for all real numbers \( x \) satisfying \( 0 \leq x \leq l \) or else the set of values of \( x \) for which \( f(x) \neq 0 \) is a finite set whose elements are points of discontinuity of the function \( f \). It follows that if \( f, g \in \mathcal{I}(0, 2\pi) \) and if \( \|f - g\| = 0 \) then either \( f(x) = g(x) \) for all real numbers \( x \) satisfying \( 0 \leq x \leq l \) or else the set of values of \( x \) for which \( f(x) \neq g(x) \) is a finite set whose elements are points of discontinuity either of the function \( f \) or else of the function \( g \).

In general \( \|f - g\| \) can be regarded as a measure of the “closeness” of the functions \( f \) and \( g \). It is but one of many such measures of closeness in widespread use by mathematicians.
let $c_j(x) = \cos jx$ for all non-negative integers $j$, and let $s_j(x) = \sin jx$ for all positive integers $j$. Then $c_0(x) = 1$ for all $x$, and therefore

\[
(c_0, c_0) = \frac{1}{\pi} \int_0^{2\pi} (c_0(x))^2 \, dx = 2.
\]

Also if $j$ is a positive integer then

\[
(c_0, c_j) = (c_j, c_0) = \frac{1}{\pi} \int_0^{2\pi} \cos jx \, dx = 0,
\]

\[
(c_0, s_j) = (s_j, c_0) = \frac{1}{\pi} \int_0^{2\pi} \sin jx \, dx = 0.
\]
Next let $j$ and $k$ be positive integers. It follows from Proposition 40.1 that

$$(c_j, c_k) = \frac{1}{\pi} \int_0^{2\pi} \cos jx \cos kx \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

$$(s_j, s_k) = \frac{1}{\pi} \int_0^{2\pi} \sin jx \sin kx \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

$$(s_j, c_k) = (c_j, s_k) = 0$$
Proposition 40.2

Let $f(x)$ be a real-valued function of the real variable $x$ defined for $0 \leq x \leq 2\pi$. Suppose that there exist constants $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ such that

$$f(x) = \frac{1}{2} a_0 + \sum_{j=1}^{N} a_j \cos jx + \sum_{j=1}^{N} b_j \sin jx$$

for all $x$. Then

$$a_j = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos jx \, dx$$

for $j = 0, 1, \ldots, N$ and

$$b_j = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin jx \, dx$$

for $j = 1, 2, \ldots, N$. 
Proof
The function $f$ satisfies

$$f(x) = \frac{1}{2} a_0 c_0 + \sum_{k=1}^{N} a_k c_k(x) + \sum_{k=1}^{N} b_k s_k(x),$$

where the functions $c_0, c_1, \ldots, c_N$ and $s_1, s_2, \ldots, s_N$ are defined as described above. It follows that

$$(f(x), c_0) = \frac{1}{2} a_0 (c_0, c_0) + \sum_{k=1}^{N} a_j (c_k, c_0) + \sum_{k=1}^{N} b_k (s_k, c_0).$$

But $(c_k, c_0) = 0$ and $(s_k, c_0) = 0$ for all positive integers $k$. It follows that

$$(f(x), c_0) = \frac{1}{2} a_0 (c_0, c_0) = a_0.$$
Next let $j$ be a positive integer. Then

$$(f(x), c_j) = \frac{1}{2} a_0(c_0, c_j) + \sum_{k=1}^{N} a_k(c_k, c_j) + \sum_{k=1}^{N} b_k(s_k, c_j).$$

But $(c_0, c_j) = 0$, $(s_k, c_j) = 0$ for all integers $k$, and $(c_k, c_j) = 0$ unless $j = k$. It follows that

$$(f(x), c_j) = a_j.$$

Similarly

$$(f(x), s_j) = \frac{1}{2} a_0(c_0, s_j) + \sum_{k=1}^{N} a_k(c_k, s_j) + \sum_{k=1}^{N} b_k(s_k, s_j) = b_j.$$

The result follows.
Now let $f(x)$ be an integrable function, defined for values of the real variable $x$ satisfying $0 \leq x \leq 2\pi$, that is either continuous throughout its domain or else has at most finitely many points of discontinuity there. Let

$$p(x) = \frac{1}{2} a_0 + \sum_{k=1}^{N} a_k c_k(x) + \sum_{k=1}^{N} b_k s_k(x),$$

where $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are the Fourier coefficients of $f$, determined so that $a_0 = \langle f, c_0 \rangle$, $a_k = \langle f, c_k \rangle$ and $b_k = \langle f, s_k \rangle$ for $k = 1, 2, \ldots, N$. Then

$$\langle f - p, c_0 \rangle = \langle f, c_0 \rangle - \frac{1}{2} a_0 (c_0, c_0) = \langle f, c_0 \rangle - a_0 = 0,$$
$$\langle f - p, c_j \rangle = \langle f, c_j \rangle - \langle p, c_j \rangle = \langle f, c_j \rangle - a_j = 0,$$
$$\langle f - p, s_j \rangle = \langle f, s_j \rangle - \langle p, s_j \rangle = \langle f, s_j \rangle - b_j = 0.$$
Let $u_0, u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ be arbitrary real numbers, and let

$$q(x) = \frac{1}{2}u_0 + \sum_{k=1}^{N} u_k c_k(x) + \sum_{k=1}^{N} v_k s_k(x).$$

Then

$$(f - p, q) = \frac{1}{2}u_0(f - p, c_0) + \sum_{k=1}^{N} u_k (f - p, c_k) + \sum_{k=1}^{N} v_k (f - p, s_k) = 0,$$

and $(q, f - p) = (f - p, q) = 0$. It follows that

$$(f - p - q, f - p - q)$$

$$= (f - p, f - p) - (f - p, q) - (q, f - p) + (q, q)$$

$$= (f - p, f - p) + (q, q).$$
Thus

$$\|f - p - q\|^2 = \|f - p\|^2 + \|q\|^2.$$ 

Now, taking $\|f - p - q\|$ as a measure of the closeness of the function $p + q$ to the function $f$, we see that the function $p + q$ is closest to $f$ with respect to this measure when $q = 0$. 

Thus if we seek to approximate \( f \) by a function of the form

\[
p(x) = \frac{1}{2} a_0 + \sum_{j=1}^{N} a_j \cos jx + \sum_{j=1}^{N} b_j \sin jx,
\]

where coefficients \( a_0, a_1, \ldots, a_N \) and \( b_1, b_2, \ldots, b_N \) are to be determined to as to achieve a good fit, we see that the values of these coefficients that result in an approximating function that is closest to the function \( f \), where distance from \( f \) is measured by the quantity \( \| f - p \| \), precisely when the coefficients \( a_0, a_1, \ldots, a_N \) and \( b_1, b_2, \ldots, b_N \) are the Fourier coefficients of \( f \), defined such that

\[
a_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx \, dx
\]

for \( j = 0, 1, \ldots, N \) and

\[
b_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin jx \, dx
\]

for \( j = 1, 2, \ldots, N \).