Example
Let us consider the differential equation

\[ \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = (g + hx)e^{mx} \]

where the real numbers \( b, c, g, h \) and \( m \) are constants and \( m^2 + bm + c \neq 0 \). In this case we look for a “particular integral” of the form

\[ y_P = (u + vx)e^{mx}. \]

Differentiating using the Product Rule, we find that

\[ \frac{dy_P}{dx} = ve^{mx} + m(u + vx)e^{mx} = (v + mu + mvx)e^{mx} \]

and

\[ \frac{d^2y}{dx^2} = 2mve^{mx} + m^2(u + vx)e^{mx} = (2mv + m^2u + m^2vx)e^{mx} \]
and therefore
\[ \frac{d^2 y_P}{dx^2} + b \frac{dy_P}{dx} + cy_P \]
\[ = \left( 2mv + m^2 u + bv + (bm + c)u + (m^2 + bm + c)v \right) e^{mx}. \]

It follows that \( y_P \) solves the differential equation if and only if
\[ (2m + b)v + (m^2 + bm + c)u = g \]
and
\[ (m^2 + bm + c)v = h. \]

Solving the second of these equations for \( v \), we find that
\[ v = \frac{h}{m^2 + bm + c}. \]
Then solving the other equation for $u$, we find that

$$u = \frac{1}{m^2 + bm + c} (g - (2m + b)v)$$

$$= \frac{(m^2 + bm + c)g - (2m + b)h}{(m^2 + bm + c)^2}$$

Thus

$$y_P = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx}.$$
The general solution of the differential equation then takes the form

\[ y = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx} + y_C(x). \]

where the complementary function \( y_C \) satisfies the differential equation

\[ \frac{d^2y_C}{dx^2} + b \frac{dy_C}{dx} + cy_C = 0. \]
Example
Consider the differential equation

\[ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = (3 - 2x)e^{4x}. \]

This equation is of the form

\[ \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (g + hx)e^{mx} \]

with \( b = -2, \ c = 10, \ g = 3, \ h = -2 \) and \( m = 4 \). We have shown that equations of this form have a particular integral \( y_P \) that takes the form

\[ y_P = \frac{(m^2 + bm + c)(g + hx) - (2m + b)h}{(m^2 + bm + c)^2} e^{mx}. \]
Substituting the values of $b$, $c$, $g$, $h$ and $m$ into this equation, we find that

\[ m^2 + bm + c = 16 - 2 \times 4 + 10 = 18, \]
\[ (2m + b)h = (2 \times 4 - 2) \times (-2) = -12, \]

and therefore

\[ y_P = \frac{66 - 36x}{324} e^{4x} = \left( \frac{11}{54} - \frac{x}{9} \right) e^{4x}. \]
Now the auxiliary polynomial $z^2 - 2z + 10$ has roots $1 + \sqrt{-13}$ and $1 - \sqrt{-13}$. It follows that the complementary function $y_C$ for this differential equation takes the form

$$y_C(x) = e^x(A\cos 3x + B\sin 3x).$$

The general solution to the differential equation thus takes the form

$$y = \left(\frac{11}{54} - \frac{x}{9}\right)e^{4x} + e^x(A\cos 3x + B\sin 3x).$$
Example
Let us consider the differential equation

\[ \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = g \cos kx + h \sin kx \]

where the real numbers \(b, c, g, h\) and \(k\) are constants.

\[ \frac{d}{dx} (\cos kx) = -k \sin kx \quad \text{and} \quad \frac{d}{dx} (\sin kx) = k \cos kx. \]

We look for a particular integral \(y_P\) of the form

\[ y_P = u \cos kx + v \sin kx. \]
Differentiating, we find that

\[
\frac{dy_P}{dx} = kv \cos kx - ku \sin kx
\]

and

\[
\frac{d^2 y_P}{dx^2} = -k^2 u \cos kx - k^2 v \sin kx,
\]

and thus

\[
\frac{d^2 y_P}{dx^2} + b \frac{dy_P}{dx} + cy_P
= ((c - k^2)u + bkv) \cos kx + ((c - k^2)v - bku) \sin kx.
\]
Therefore $u$ and $v$ should be chosen to satisfy the equations

$$(c - k^2)u + bkv = g \quad \text{and} \quad (c - k^2)v - bku = h.$$  

It follows that

\begin{align*}
bkg + (c - k^2)h & = bk((c - k^2)u + bkv) + (c - k^2)((c - k^2)v - bku) \\
& = (b^2k^2 + (c - k^2)^2)v \\
(c - k^2)g - bkh & = (c - k^2)((c - k^2)u + bkv) - bk((c - k^2)v - bku) \\
& = (b^2k^2 + (c - k^2)^2)u.
\end{align*}
Thus

\[ u = \frac{(c - k^2)g - bkh}{b^2k^2 + (c - k^2)^2} \]

and

\[ v = \frac{bk g + (c - k^2)h}{b^2k^2 + (c - k^2)^2}, \]

and thus

\[ y_P = \frac{1}{b^2k^2 + (c - k^2)^2} \left( \left( (c - k^2)g - bkh \right) \cos kx 
\right.

\[ + \left( bk g + (c - k^2)h \right) \sin kx \right) \].
It follows that the general solution of the differential equation

$$\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = g \cos kx + h \sin kx$$

takes the form

$$y = \frac{1}{b^2k^2 + (c - k^2)^2} \left( ((c - k^2)g - bkh) \cos kx \\
+ (bkg + (c - k^2)h) \sin kx \right) + y_C,$$

where the complementary function $y_C$ satisfies the differential equation

$$\frac{d^2y_C}{dx^2} + b \frac{dy_C}{dx} + cy_C = 0.$$
Example
Consider the differential equation

\[ \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = 3 \cos 2x + 4 \sin 2x. \]

This equation is of the form

\[ \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = g \cos kx + h \sin kx \]

with \( b = -6, \ c = 9, \ k = 2, \ g = 3 \) and \( h = 4 \). We have shown that equations of this form have a particular integral \( y_P \) that takes the form

\[ y_P = \frac{1}{b^2k^2 + (c - k^2)^2} \left( ((c - k^2)g - bkh) \cos kx 
+ (bkg + (c - k^2)h) \sin kx \right). \]
Substituting the values of $b$, $c$, $k$, $g$ and $h$ into this equation, we find that

\begin{align*}
bk &= -12 \\
(c - k^2) &= 9 - 4 = 5 \\
b^2k^2 + (c - k^2)^2 &= 144 + 25 = 169, \\
(c - k^2)g - bkh &= 5 \times 3 - (-12) \times 4 = 15 + 48 = 63, \\
bkg + (c - k^2)h &= (-12) \times 3 + 5 \times 4 = -36 + 20 = -16.
\end{align*}

and therefore

\[ y_P = \frac{1}{169} (63 \cos 2x - 16 \sin 2x). \]
Now the auxiliary polynomial \( z^2 - 6z + 9 \) has a repeated root with value 3. It follows that the complementary function \( y_C \) for this differential equation takes the form

\[
y_C(x) = (A + Bx)e^{3x}.
\]

The general solution to the differential equation thus takes the form

\[
y = \frac{1}{169} (63 \cos 2x - 16 \sin 2x) + (A + Bx)e^{3x}.
\]