38.2. The Scalar Product

Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples \((u_1, u_2, u_3)\) and \((v_1, v_2, v_3)\) respectively. The scalar product of the vectors \( \mathbf{u} \) and \( \mathbf{v} \) is defined to be the real number \( \mathbf{u} \cdot \mathbf{v} \) defined by the formula

\[
\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.
\]

In particular,

\[
\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2,
\]

for any vector \( \mathbf{u} \), where \(|\mathbf{u}|\) denotes the length of the vector \( \mathbf{u} \).
Note that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Also

$$(\mathbf{s}\mathbf{u} + \mathbf{t}\mathbf{v}) \cdot \mathbf{w} = \mathbf{s}\mathbf{u} \cdot \mathbf{w} + \mathbf{t}\mathbf{v} \cdot \mathbf{w},$$

$\mathbf{u} \cdot (\mathbf{s}\mathbf{v} + \mathbf{t}\mathbf{w}) = \mathbf{s}\mathbf{u} \cdot \mathbf{v} + \mathbf{t}\mathbf{u} \cdot \mathbf{w}$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and real numbers $s$ and $t$. 
Proposition 38.1

Let \( \mathbf{u} \) and \( \mathbf{v} \) be non-zero vectors in three-dimensional space. Then their scalar product \( \mathbf{u} \cdot \mathbf{v} \) is given by the formula

\[
\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta,
\]

where \( \theta \) denotes the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \).

Proof

Suppose first that the angle \( \theta \) between the vectors \( \mathbf{u} \) and \( \mathbf{v} \) is an acute angle, so that \( 0 < \theta < \frac{1}{2} \pi \). Let us consider a triangle \( ABC \), where \( \overrightarrow{AB} = \mathbf{u} \) and \( \overrightarrow{BC} = \mathbf{v} \), and thus \( \overrightarrow{AC} = \mathbf{u} + \mathbf{v} \). Let \( ADC \) be the right-angled triangle constructed as depicted in the figure below, so that the line \( AD \) extends \( AB \) and the angle at \( D \) is a right angle.
Note:

\[ AD = |u| + |v| \cos \theta, \]
\[ CD = |v| \sin \theta, \]
\[ |u + v|^2 = AC^2 = AD^2 + CD^2 \quad \text{(Pythagoras).} \]
Then the lengths of the line segments $AB$, $BC$, $AC$, $BD$ and $CD$ may be expressed in terms of the lengths $|u|$, $|v|$ and $|u + v|$ of the displacement vectors $u$, $v$ and $u + v$ and the angle $\theta$ between the vectors $u$ and $v$ by means of the following equations:

$$AB = |u|, \quad BC = |v|, \quad AC = |u + v|,$$

$$BD = |v| \cos \theta \quad \text{and} \quad DC = |v| \sin \theta.$$ 

Then

$$AD = AB + BD = |u| + |v| \cos \theta.$$ 

The triangle $ADC$ is a right-angled triangle with hypotenuse $AC$. It follows from Pythagoras’ Theorem that

$$|u + v|^2 = AC^2 = AD^2 + DC^2 = (|u| + |v| \cos \theta)^2 + |v| \sin^2 \theta$$

$$= |u|^2 + 2|u||v| \cos \theta + |v|^2 \cos^2 \theta + |v|^2 \sin^2 \theta$$

$$= |u|^2 + |v|^2 + 2|u||v| \cos \theta,$$

because $\cos^2 \theta + \sin^2 \theta = 1$. 

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

and therefore

$$|\mathbf{u} + \mathbf{v}|^2 = (u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2$$
$$= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2 + u_3^2 + 2u_3v_3 + v_3^2$$
$$= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1v_1 + u_2v_2 + u_3v_3)$$
$$= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}.$$ 

On comparing the expressions for $|\mathbf{u} + \mathbf{v}|^2$ given by the above equations, we see that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$ when $0 < \theta < \frac{1}{2} \pi$. 
The identity \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta \) clearly holds when \( \theta = 0 \) and \( \theta = \pi \). Pythagoras’ Theorem ensures that it also holds when the angle \( \theta \) is a right angle (so that \( \theta = \frac{1}{2} \pi \)). Suppose that \( \frac{1}{2} \pi < \theta < \pi \), so that the angle \( \theta \) is obtuse. Then the angle between the vectors \( \mathbf{u} \) and \( -\mathbf{v} \) is acute, and is equal to \( \pi - \theta \). Moreover \( \cos(\pi - \theta) = -\cos \theta \) for all angles \( \theta \). It follows that

\[
\mathbf{u} \cdot \mathbf{v} = -\mathbf{u} \cdot (-\mathbf{v}) = -|\mathbf{u}| |\mathbf{v}| \cos(\pi - \theta) = |\mathbf{u}| |\mathbf{v}| \cos \theta
\]

when \( \frac{1}{2} \pi < \theta < \pi \). We have therefore verified that the identity \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta \) holds for all non-zero vectors \( \mathbf{u} \) and \( \mathbf{v} \), as required. \( \blacksquare \)
Corollary 38.1

Two non-zero vectors \( \mathbf{u} \) and \( \mathbf{v} \) in three-dimensional space are perpendicular if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Proof
It follows directly from Proposition 38.1 that \( \mathbf{u} \cdot \mathbf{v} = 0 \) if and only if \( \cos \theta = 0 \), where \( \theta \) denotes the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \). This is the case if and only if the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular.
Example
We can use the scalar product to calculate the angle $\theta$ between the vectors $(2, 2, 0)$ and $(0, 3, 3)$ in three-dimensional space. Let $\mathbf{u} = (2, 2, 0)$ and $\mathbf{v} = (0, 3, 3)$. Then $|\mathbf{u}|^2 = 2^2 + 2^2 = 8$ and $|\mathbf{v}|^2 = 3^2 + 3^2 = 18$. It follows that $(|\mathbf{u}| |\mathbf{v}|)^2 = 8 \times 18 = 144$, and thus $|\mathbf{u}| |\mathbf{v}| = 12$. Now $\mathbf{u} \cdot \mathbf{v} = 6$. It follows that

$$6 = |\mathbf{u}| |\mathbf{v}| \cos \theta = 12 \cos \theta.$$ 

Therefore $\cos \theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3} \pi$. 

Example
We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point $P$ on the unit sphere about the origin $O$ in three-dimensional space may be expressed in terms of angles $\theta$ and $\varphi$ as follows:

$$P = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

The angle $\theta$ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0, 0, 1)$. Thus the angle $\frac{1}{2} \pi - \theta$ represents the ‘latitude’ of the point $P$, when we regard the point $(0, 0, 1)$ as the ‘north pole’ of the sphere. The angle $\varphi$ measures the ‘longitude’ of the point $P$. 
Now let $P_1$ and $P_2$ be points on the unit sphere, where

$$P_1 = (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1),$$

$$P_2 = (\sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2, \cos \theta_2).$$

We wish to find the angle $\psi$ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points $P_1$ and $P_2$ from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 38.1, we see that

$$\cos \psi = \overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$$

$$= \sin \theta_1 \sin \theta_2 \cos \varphi_1 \cos \varphi_2 + \sin \theta_1 \sin \theta_2 \sin \varphi_1 \sin \varphi_2$$

$$+ \cos \theta_1 \cos \theta_2$$

$$= \sin \theta_1 \sin \theta_2 \left(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2\right) + \cos \theta_1 \cos \theta_2$$

$$= \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) + \cos \theta_1 \cos \theta_2.$$
Example
Let $X$ be a plane in three-dimensional space, and let $\mathbf{p}$ be a vector that is perpendicular to the plane $X$. Let $O$ be the origin of a Cartesian coordinate system in three-dimensional space, and let $\mathbf{v}$ and $\mathbf{w}$ be the position vectors $\overrightarrow{OV}$ and $\overrightarrow{OW}$ of points $V$ and $W$ respectively lying in the plane $X$. Then the vector $\mathbf{p}$ is perpendicular to the displacement vector $\overrightarrow{VW}$. Now $\overrightarrow{VW} = \mathbf{w} - \mathbf{v}$. It follows that

$$(\mathbf{w} - \mathbf{v}) \cdot \mathbf{p} = 0$$

(see Corollary 38.1), and therefore $\mathbf{v} \cdot \mathbf{p} = \mathbf{w} \cdot \mathbf{p}$. Identifying the points of the plane $X$ with their position vectors $\mathbf{r}$ with respect to the origin $O$ of the Cartesian coordinate system, we find that it follows from this that there exists a real number $k$ such that

$$X = \{ \mathbf{r} \in \mathbb{R}^3 : \mathbf{r} \cdot \mathbf{p} = k \}.$$
Let \( \mathbf{r} = (x, y, z) \) and \( \mathbf{p} = (a, b, c) \). The point \( \mathbf{r} \) belongs to the plane \( X \) if and only if \( \mathbf{r} \cdot \mathbf{p} = k \). It follows that

\[
X = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = k\}.
\]

Suppose that the vector \( \mathbf{r} \) is the position vector of an arbitrary point \( R \) of three-dimensional space. We wish to determine the distance from this point to the plane \( X \). Now the line through the point \( \mathbf{r} \) parallel to the vector \( \mathbf{p} \) cuts the plane \( X \) in a single point. Therefore there exists a unique real number \( t \) for which \( \mathbf{r} + tp \in X \). For this value of \( t \) the equation

\[(\mathbf{r} + tp) \cdot \mathbf{p} = k\]

is satisfied. Then

\[
\mathbf{r} \cdot \mathbf{p} = t|\mathbf{p}|^2 = k,
\]

and therefore

\[
t = \frac{1}{|\mathbf{p}|^2}(k - \mathbf{r} \cdot \mathbf{p}).
\]
Let \( \mathbf{w} = \mathbf{r} + t\mathbf{p} \), where \( t \) has the value determined above that ensures that \( \mathbf{w} \in X \). Let \( \mathbf{v} \) be an arbitrary point that lies on the plane \( X \). Then the displacement vector \( \mathbf{v} - \mathbf{w} \) from \( W \) to \( V \) is perpendicular to the vector \( \mathbf{p} \). Now

\[
\mathbf{v} - \mathbf{r} = t\mathbf{p} + (\mathbf{v} - \mathbf{w}).
\]

It follows, either directly from Pythagoras’ Theorem, or else from an equivalent calculation using scalar products (using the result of Corollary 38.1) that

\[
|\mathbf{v} - \mathbf{r}|^2 = t^2|\mathbf{p}|^2 + |\mathbf{v} - \mathbf{w}|^2.
\]

It follows that

\[
|\mathbf{v} - \mathbf{r}| \geq t|\mathbf{p}|
\]

and that

\[
|\mathbf{v} - \mathbf{r}| = t|\mathbf{p}| \iff \mathbf{v} = \mathbf{w}.
\]
Thus the point \( w \) is the closest point of the plane \( X \) to the point \( R \) with position vector \( r \). It follows that the distance \( d(r, X) \) from the point \( R \) to the plane \( X \) is the length \( |w - r| \) of the vector \( w - r \). Thus

\[
d(r, X) = t|p| = \frac{1}{|p|} |k - r \cdot p|.
\]

Let \( r = (x, y, z) \) and \( p = (a, b, c) \). Then

\[
d(r, X) = \frac{|k - ax - by - cz|}{\sqrt{a^2 + b^2 + c^2}}.
\]
Example

Suppose that we wish to determine the equation of a cone in three-dimensional space. Let $O$ be the origin of a Cartesian coordinate system, let $V$ be the apex of the cone, let $v$ be the position vector of $V$, so that $v = \overrightarrow{OV}$, and let $b$ be a vector pointed into the axis of the cone. Let $\theta$ be a fixed angle between zero and a right angle. The cone consists of those points $R$ for which the displacement vector $\overrightarrow{VR}$ makes an angle $\theta$ with the vector $b$. It follows from Proposition 38.1 that $r$ is the position vector of a point lying on the cone if and only if

$$(r - v) \cdot b = |r - v| |b| \cos \theta.$$
Squaring both sides of this identity, we find that

\[ ((r - v) \cdot b)^2 = \|r - v\|^2 \|b\|^2 \cos^2 \theta. \]

Let

\[ r = (x, y, z), \quad v = (v_x, v_y, v_z) \quad \text{and} \quad b = (b_x, b_y, b_z). \]

Then the equation of the cone becomes

\[
((x - v_x)b_x + (y - v_y)b_y + (z - v_z)b_z)^2 = C \left( (x - v_x)^2 + (y - v_y)^2 + (z - v_z)^2 \right),
\]

where \( C = \|b\|^2 \cos^2 \theta \). Note that this constant \( C \) must satisfy the inequalities \( 0 \leq C < \|b\|^2 \).