Course MA2C03: Michaelmas Term 2013.

Assignment I—Worked Solutions.

1. Use the Principle of Mathematical Induction to prove that

\[ \sum_{k=1}^{n} 7^k k = \frac{7}{36} \left( (6n - 1)7^n + 1 \right) \]

for all positive integers \( n \).

The required equality holds when \( n = 1 \), since both sides are then equal to 7. Suppose that the equality holds when \( n = m \) for some natural number \( m \), so that

\[ \sum_{k=1}^{m} 7^k k = \frac{7}{36} \left( (6m - 1)7^m + 1 \right) \]

Then

\[ \sum_{k=1}^{m+1} 7^k k = \sum_{k=1}^{m} 7^k k + 7^{m+1}(m+1) \]

\[ = \frac{7}{36} \left( (6m - 1)7^m + 1 \right) + 7^{m+1}(m+1) \]

\[ = \frac{7}{36} \left( (6m - 1)7^m + 1 + 36(m + 1)7^m \right) \]

\[ = \frac{7}{36} \left( 42m + 35 \right)7^m + 1 \]

\[ = \frac{7}{36} \left( 6m + 5 \right) \]

\[ = \frac{7}{36} \left( 6(m + 1) - 1 \right)5^{m+1} + 1 \]

and thus the equality holds when \( n = m + 1 \). It follows from the Principle of Mathematical Induction that the equality holds for all natural numbers \( n \).

2. Let \( A \) and \( B \) be sets. Prove that

\[ (A \cup B) \setminus (A \setminus B) = B. \]

We prove that every element of the set on the left hand side is an element of the set on the right hand side, and vice versa. Let \( x \in (A \cup B) \setminus (A \setminus B) \). Then \( x \in A \cup B \) and \( x \notin A \setminus B \). Now \( x \notin A \setminus B \)
implies that either \( x \notin A \) or else \( x \in A \cap B \). Thus \( x \in A \) and \( x \notin A \setminus B \) together imply that \( x \in A \cap B \) and thus \( x \in B \). Thus if \( x \in A \cup B \) and \( x \notin A \setminus B \) then \( x \in B \). We have thus shown that \( (A \cup B) \setminus (A \setminus B) \) is a subset of \( B \).

Now let \( x \in B \). Then \( x \in A \cup B \) and \( x \notin A \setminus B \), and thus \( x \in (A \cup B) \setminus (A \setminus B) \). We have thus shown that \( B \) is a subset of \( (A \cup B) \setminus (A \setminus B) \). Therefore \( (A \cup B) \setminus (A \setminus B) = B \), as required.

3. Let \( S \) be the relation on the set \( \mathbb{Z} \) of integers, where integers \( x \) and \( y \) satisfy \( xSy \) if and only if \( x^3 - x \geq y^3 - y \). Determine

   (i) whether or not the relation \( S \) is reflexive,
   (ii) whether or not the relation \( S \) is symmetric,
   (iii) whether or not the relation \( S \) is anti-symmetric,
   (iv) whether or not the relation \( S \) is transitive,
   (v) whether or not the relation \( S \) is an equivalence relation,
   (vi) whether or not the relation \( S \) is a partial order.

[Justify your answers with short proofs and/or counterexamples.]

If integers \( x \) and \( y \) satisfy \( x = y \) then \( x^3 - x = y^3 - y \), and therefore \( xSy \). Thus \( xSx \) for all integers \( x \). We conclude that the relation \( S \) on the set \( \mathbb{Z} \) of integers is reflexive.

The relation \( S \) on \( \mathbb{Z} \) is not symmetric. Indeed if \( x = 3 \) and \( y = 2 \) then \( x^3 - x = 24 \) and \( y^3 - y = 6 \). Thus \( xSy \), but \( y \not\in Sx \).

The relation \( S \) is not anti-symmetric. Note that \( x^3 - x = 0 \) when \( x = 0 \) and \( x = 1 \) (and also when \( x = -1 \)). It follows that 0S1 and 1S0 but 0 \( \neq 1 \).

The relation \( S \) is transitive. Indeed let \( x, y \) and \( z \) be integers satisfying \( xSy \) and \( ySz \). Then \( x^3 - x \geq y^3 - y \geq z^3 - z \), and therefore \( x^3 - x \geq z^3 - z \), and thus \( xSz \).

The relation \( S \) on \( \mathbb{Z} \) is not an equivalence relation because it is not symmetric.

The relation \( S \) on \( \mathbb{Z} \) is not a partial order relation because it is not anti-symmetric.
4. Let \( f : [1, 4] \rightarrow [0, 6] \) be the function defined so that \( f(x) = x^2 - 4x + 4 \) for all \( x \in [1, 4] \). Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers. (Note that \([1, 4]\) denotes the set of all real numbers between 1 and 4 inclusive, and therefore includes fractions such as \( \frac{3}{2} \) and irrational numbers like \( \sqrt{2} \) and \( \pi \).)

We consider the behaviour of the function \( f \) on the interval \([1, 4]\). Now \( f'(x) = 2x - 4 \) (where \( f'(x) \) denotes the derivative of the function \( f \) at \( x \) for all \( x \in [1, 4] \)). It follows that \( f'(x) < 0 \) when \( 1 \leq x < 2 \) and \( f'(x) > 0 \) when \( 2 < x \leq 4 \). Thus the function \( f \) is strictly decreasing on the interval \([1, 2]\) and is strictly increasing on the function \([2, 4]\). Also \( f(1) = 1 \), \( f(2) = 0 \) and \( f(4) = 4 \). The function \( f \) therefore will not be injective, and indeed \( f(1) = f(3) = 1 \).

Now the range of the function is the interval \([0, 4]\), since \( f \) maps the interval \([1, 2]\) onto the whole of the interval \([0, 1]\), and maps the interval \([2, 4]\) onto the whole of the interval \([0, 4]\). Therefore there does not exist any \( x \in [1, 4] \) satisfying \( f(x) = 5 \), though 5 is an element of the codomain \([0, 6]\) of the function. Therefore the function \( f : [1, 4] \rightarrow [0, 6] \) is not surjective.