Course MA2C01: Michaelmas Term 2009.
Worked Solutions for Assignment II.

1. Let $c$ be a fixed positive integer, and let $\otimes$ denote the binary operation on the set $\mathbb{Z}$ of integers defined by the formula

$$x \otimes y = xy + c(x + y) + c^2 - c$$

for all integers $x$, $y$ and $z$.

(a) Is $(\mathbb{Z}, \otimes)$ a semigroup? [Justify your answer.]

**Solution.** $(\mathbb{Z}, \otimes)$ is a semigroup if and only if the binary operation $\otimes$ is associative. Now

$$((x \otimes y) \otimes z) = (x \otimes y) \otimes (z \otimes y) = (x \otimes y) \otimes (yz + c(y + z) + c^2 - c)$$

Thus the operation $\otimes$ on $\mathbb{Z}$ is associative, and therefore $(\mathbb{Z}, \otimes)$ is a semigroup.

(b) Is $(\mathbb{Z}, \otimes)$ a monoid? If so, what is its identity element?

**Solution.** The semigroup $(\mathbb{Z}, \otimes)$ is a monoid if and only if it has an identity element $e$. If so, this identity element must satisfy

$$xe + cx + ce + c^2 - c = x.$$

for all $x \in \mathbb{Z}$. But then

$$(e + c - 1)(x + c) = 0$$

for all $x \in \mathbb{Z}$. Examination of this formula shows that there is an identity element $e$, and moreover $e = 1 - c$. Thus $(\mathbb{Z}, \otimes)$ is a monoid with identity element $1 - c$. 
(c) Which of the elements of \( \mathbb{Z} \) are invertible? Is \((\mathbb{Z}, \otimes)\) a group?

An integer \( x \) is invertible in this monoid if and only if there exists some integer \( y \) such that \( x \otimes y = 1 - c \). Now

\[
x \otimes y = 1 - c
\]

\[
\iff xy + c(x + y) + c^2 - c = 1 - c
\]

\[
\iff (x + c)(y + c) = 1
\]

Thus \( x \) is invertible if and only if \( c \neq -c \) and \( 1/(x + c) \) is an integer. It follows that the invertible elements of the monoid are \( 1 - c \) and \(-1 - c \). The monoid \((\mathbb{Z}, \otimes)\) is not a group since it has elements that are not invertible.

2. Construct a regular grammar that generates the language \( L \) over the alphabet \( \{0, 1\} \), where

\[
L = \{1, 1000, 1000000, 1000000000, \ldots\},
\]

so that a string of binary digits belongs to \( L \) if and only if it consists of the digit 1 followed by a string of \( 3n \) zeroes, for some non-negative integer \( n \). You should specify your formal grammar in Backus-Naur form.

**Solution.** Non terminals: \( \langle S \rangle, \langle A \rangle, \langle B \rangle, \langle C \rangle \).

Start symbol: \( \langle S \rangle \).

Productions:

\[
\begin{align*}
\langle S \rangle & \rightarrow 1 \langle A \rangle \\
\langle A \rangle & \rightarrow 0 \langle B \rangle | \epsilon \\
\langle B \rangle & \rightarrow 0 \langle C \rangle \\
\langle C \rangle & \rightarrow 0 \langle A \rangle
\end{align*}
\]
3. Answer the following questions concerning the graph with vertices a, b, c, d, e and f pictured above. [Justify all your answers.]

(a) Is the graph complete?

Solution. Not complete. There is no edge from a to f.

(b) Is the graph regular?

Solution. Regular. All vertices are of degree 3.

(c) Is the graph connected?

Solution. Connected. All vertices may be joined to a by a path of length at most 2.

(d) Does the graph have an Eulerian circuit?

Solution. No. Were a Eulerian circuit to exist, the degrees of all vertices would need to be even. This is not the case.

(e) Does the graph have a Hamiltonian circuit?

Solution. Yes. a b e d f c a is one such circuit.

(f) Give an example of a spanning tree for the graph, specifying the vertices and edges of the spanning tree.

Solution. One such spanning tree has vertices a, b, c, d, e and f and edges a b, b c, b e, e d, e f. (There are many others. Note that any spanning tree is connected, has all six vertices, and has five edges.)
(g) Given an example of an isomorphism between the graph pictured above and that pictured below. (You should specify the isomorphism as a function between the sets \{a, b, c, d, e, f\} and \{u, v, w, x, y, z\} of vertices of the two graphs.)

**Solution.**

One such isomorphism

\[
\varphi: \{a, b, c, d, e, f\} \rightarrow \{u, v, w, x, y, z\}
\]

is defined so that

\[
\varphi(a) = x, \quad \varphi(b) = y, \quad \varphi(c) = w, \quad \varphi(d) = u, \quad \varphi(e) = z, \quad \varphi(f) = v.
\]

(Any such isomorphism must send edges to edges and thus must send triangles to triangles. Thus the vertices of the triangle \(a b c\) must either be mapped to the vertices of the triangle \(x y w\), in some order which in this case is arbitrary, or else to the vertices of the triangle \(u z w\). Moreover \(d\) is adjacent to \(a\), and therefore \(\varphi(d)\) must be adjacent to \(\varphi(a)\), and similarly for \(e\) and \(f\).)