1. Use the Principle of Mathematical Induction to prove that
\[ \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \]
for all positive integers \( n \) and real numbers \( x \) satisfying \( x \neq 1 \).

When \( n = 1 \) the left hand side equals \( 1 + x \) and the right hand side equals \( \frac{1 - x^2}{1 - x} \). Now \( 1 - x^2 = (1 + x)(1 - x) \). Thus the right hand side equals \( 1 + x \), and thus equals the left hand side when \( n = 1 \). The result therefore holds when \( n = 1 \). (Alternatively one could not that the identity holds when \( n = 0 \), and use this as the basis for the induction proof.)

Suppose that
\[ \sum_{i=0}^{k} x^i = \frac{1 - x^{k+1}}{1 - x} \]
for some positive (or non-negative) integer \( k \). Then
\[ \sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1} = \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \]
\[ = \frac{1 - x^{k+1}}{1 - x} + \frac{x^{k+1}(1 - x)}{1 - x} \]
\[ = \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x} \]
\[ = \frac{1 - x^{k+2}}{1 - x}. \]

Thus if the required identity holds when \( n = k \) then it holds when \( n = k + 1 \). The identity therefore holds for all positive integers \( n \) by the Principle of Mathematical Induction.
2. Let $A$, $B$ and $C$ be sets. Prove that
\[
(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.
\]

We show that each element of $(A \setminus C) \cup (B \setminus C)$ is an element of $(A \cup B) \setminus C$, and that each element of $(A \cup B) \setminus C$ is an element of $(A \setminus C) \cup (B \setminus C)$.

Let $x \in (A \setminus C) \cup (B \setminus C)$. Then either $x \in A \setminus C$ or $x \in B \setminus C$. It follows that either $x \in A$ or $x \in B$, and therefore $x \in A \cup B$. But, in both cases just considered, $x \notin C$. Therefore $x \in (A \cup B) \setminus C$. This proves that
\[
(A \setminus C) \cup (B \setminus C) \subset (A \cup B) \setminus C.
\]

Now let $x \in (A \cup B) \setminus C$. Then $x \in A \cup B$ and $x \notin C$. But then either $x \in A$ and $x \in B$. If $x \in A$ then $x \in A \setminus C$, because $x \notin C$. Similarly, if $x \in B$ then $x \in B \setminus C$, because $x \notin C$. Thus $x \in (A \setminus C) \cup (B \setminus C)$. This proves that
\[
(A \setminus C) \cup (B \setminus C) \subset (A \cup B) \setminus C.
\]

Therefore
\[
(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.
\]

3. Let $Q$ be the relation on the set $\mathbb{R}^*$ of non-zero real numbers, where non-zero real numbers $x$ and $y$ satisfy $xQy$ if and only if $\frac{x^2}{y^2}$ is a rational number. Determine

(i) whether or not the relation $Q$ is reflexive,
(ii) whether or not the relation $Q$ is symmetric,
(iii) whether or not the relation $Q$ is anti-symmetric,
(iv) whether or not the relation $Q$ is transitive,
(v) whether or not the relation $Q$ is an equivalence relation,
(vi) whether or not the relation $Q$ is a partial order.

Note that a rational number is a number that can be expressed as a fraction $n/d$ whose numerator $n$ and denominator $d$ are integers. Not all real numbers are rational numbers: both $\sqrt{2}$ and $\pi$ are examples of real numbers that are not rational numbers.

[Justify your answers with short proofs and/or counterexamples.]
\[ \frac{x^2}{y^2} = 1 \] whenever \( x = y \), and the number 1 is a rational number. It follows that \( xQx \) for all real numbers \( x \). Thus the relation \( Q \) is reflexive.

If \( x, y \in \mathbb{R}^* \) and \( xQy \) then \( \frac{x^2}{y^2} = q \) for some non-zero rational number \( q \).

But then \( \frac{y^2}{x^2} = \frac{1}{q} \), and \( 1/q \) is also a rational number. It follows that \( yQx \). Thus the relation \( Q \) is symmetric.

Note that \( 1Q2 + 2Q1 \), but \( 1 \neq 2 \). This counter-example shows that the relation \( Q \) is not anti-symmetric.

If \( x, y, z \in \mathbb{R}^* \), \( xQy \) and \( yQz \) then \( \frac{x^2}{y^2} = q_1 \) and \( \frac{y^2}{z^2} = q_2 \) for some rational numbers \( q_1 \) and \( q_2 \). But then \( \frac{x^2}{z^2} = \frac{x^2}{y^2} \frac{y^2}{z^2} = q_1 q_2 \), and \( q_1 q_2 \) is a rational number. This shows that the relation \( Q \) is transitive.

The relation \( Q \) is an equivalence relation because it is reflexive, symmetric and transitive. Because this relation is not anti-symmetric, it is not a partial order.

4. Let \( f : [0, 4] \to [0, 10] \) be the function defined so that

\[
    f(x) = \begin{cases} 
    x^3 & \text{if } 0 \leq x \leq 2; \\
    x + 6 & \text{if } 2 < x \leq 4. 
    \end{cases}
\]

Determine whether or not this function is injective, and whether or not it is surjective, giving brief reasons for your answers.

Consider the behaviour of this function as \( x \) increases from 0 to 4. The value of the function increases continuously from 0 to 8 as \( x \) increases from 0 to 2. It then increases continuously from 8 to 10 as \( x \) increases from 2 to 4.

Let \( u, v \in [0, 10] \) satisfy \( u \leq v \) and \( f(u) = f(v) \). If \( f(u) \leq 8 \) then \( u, v \in [0, 2] \) and \( u = v = \sqrt[3]{f(u)} \). If \( f(u) > 8 \) then \( u, v \in (2, 4] \), and \( u = v = f(u) - 6 \). It follows that if \( u, v \in [0, 4] \) satisfy \( f(u) = f(v) \) then \( u = v \). Thus the function \( f \) is injective.

Let \( y \in [0, 10] \). If \( 0 \leq y \leq 8 \) then \( y = f(x) \), where \( x = \sqrt[3]{y} \). If \( 8 < y \leq 10 \) then \( y = f(x) \), where \( x = y - 6 \). Therefore each element \( y \) of \( [0, 10] \) is in the range of the function. Thus the function is surjective.