Course MA2C01: Michaelmas Term 2011.

Assignment II.

To be handed in by Wednesday 1st February, 2012.
Please include both name and student number on any work handed in.

1. (a) Let $A$ be the set $\mathbb{C} \times \mathbb{C}$ consisting of all ordered pairs $(z, w)$, where $z$ and $w$ are complex numbers. Let $\times$ denote the binary operation on $A$ defined by $(z, w) \times (u, v) = (zu - wv, zv + wu)$ for all complex numbers $z$, $w$, $u$ and $v$. Prove that $(A, \times)$ is a monoid. What is its identity element? Prove that an element $(z, w)$ of $A$ is invertible if and only if $z^2 + w^2 \neq 0$.

(b) Let $(\mathbb{C}, \times)$ be the monoid consisting of the set of complex numbers with the usual operation of multiplication, and let $f: \mathbb{C} \rightarrow A$ be the function from $\mathbb{C}$ to $A$ which sends the complex number $x + iy$ to the ordered pair $(x, y)$ for all real numbers $x$ and $y$. Is the function $f$ a homomorphism from $(\mathbb{C}, \times)$ to $(A, \times)$? Is this function an isomorphism?

2. (a) Describe the formal language over the alphabet $\{0, 1\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$ and whose productions are the following:

\[
\langle S \rangle \rightarrow 0 \\
\langle S \rangle \rightarrow 00\langle S \rangle \\
\langle S \rangle \rightarrow \langle S \rangle 11
\]

Is this context-free grammar a regular grammar?

(b) Give the specification of a finite state acceptor for the language over the alphabet $\{a, b, c\}$ consisting of all finite strings, such as $aabbc$, $aabbbbc$ and $aaabbbbc$, that consist of two or more occurrences of the character $a$, followed by two or more occurrences of the character $b$, followed by a single occurrence of the character $c$. You should in particular specify the starting state, the finishing state or states, and the transition table for this finite state acceptor.

(c) Give the specification of a regular grammar to generate the language over the alphabet $\{a, b, c\}$ that was defined in (b).
3. Consider a graph $G$ with vertices $a, b, c, d$ and $e$, and edges $ab, ad, bc, bd, cd, ce$ and $de$. Let $V$ denote the set of vertices of this graph, so that $V = \{a, b, c, d, e\}$.

(a) Draw a diagram showing the vertices and edges of this graph.

(b) Determine the degrees of each of the vertices of the graph.

(c) Is this graph regular? [If not, briefly explain why not.]

(d) Is this graph complete? [If not, briefly explain why not.]

(e) Is this graph connected? [If not, briefly explain why not.]

(f) Does this graph have an Eulerian circuit? [If so, give an example. If not, briefly explain why not.]

(g) Does this graph have an Eulerian trail that starts at some vertex of the graph, and ends at some other vertex? [If so, give an example. If not, briefly explain why not.]

(h) Does this graph have a Hamiltonian circuit? [If so, give an example.]

(i) Is this graph a tree? [If not, briefly explain why not.]

(j) Does this graph have a spanning tree? [If so, give an example. If not, briefly explain why not.]

(k) Does there exist a function $\theta: V \to V$ from the set $V$ of vertices of the graph to itself that is an isomorphism from the graph $G$ to itself and that satisfies $\theta(a) = e$. [If so, give an example. If not, briefly explain why not.]

(l) Does there exist a function $\varphi: V \to V$ from the set $V$ of vertices of the graph to itself that is an isomorphism from the graph $G$ to itself and that satisfies $\varphi(c) = d$. [If so, give an example. If not, briefly explain why not.]