Selected Propositions from Euclid's *Elements of Geometry*
Books II, III and IV (T.L. Heath's Edition)

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SELECTED PROPOSITIONS FROM EUCLID'S *ELEMENTS*, BOOK II

Definitions

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

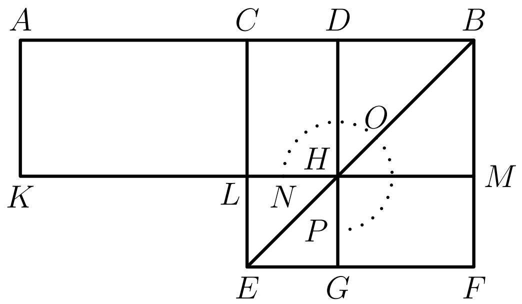
2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

Proposition 5

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

For let the square $CEFB$ be described on CB [I. 46], and let BE be joined; through D let DG be drawn parallel to either CE or BF, through H again let KM be drawn parallel to either AB or EF, and again through A let AK be drawn parallel to either CL or BM [I. 31].

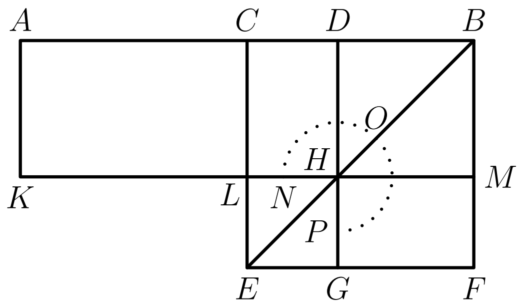


Then, since the complement CH is equal to the complement HF [I. 43], Let DM be added to each; therefore the whole CM is equal to the whole DF. But CM is equal to AL, since AC is also equal to CB [I. 36]; therefore AL is also equal to DF. Let CH be added to each; therefore the whole AH is equal to the gnomon NOP. But AH is the rectangle AD, DB, for DH is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on CD, be added to each; therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD. But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB; therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc.

Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK II, PROPOSITION 5



In this proposition $ABMK$ is a rectangle divided into smaller rectangles by line segments $[CL]$ and $[DH]$ parallel to the sides $[AK]$ and $[BM]$ of the containing rectangle. The conditions of the proposition require that the line segment $[AB]$ be bisected at the point C, so that $|AC| = |CB|$, and that D be a point of the line segment $[AB]$ that does not bisect the segment. They also require that $|BD| = |BM|$. Thus the rectangles $ACLK$ and $CBML$ are equal to one another (*Elements*, I, 36), and $DBMH$ is a square.

The proposition in effect claims that the the sum of (the areas of) the rectangle $ADHK$ and a square constructed on the line segment $[CD]$ is equal to (the area of) a square constructed on the line segment $[CB]$.

In modern algebraic notation let $x = |BC|$ and $y = |CD|$. Then $|AD| = x + y$ and $|DB| = x - y$. The proposition therefore corresponds to the algebraic identity

$$(x + y)(x - y) + y^2 = x^2.$$

Euclid proceeds by completing a square $CBFE$ on the line segment $[BC]$ (*Elements*, I, 46). The diagonal $[BE]$ of that square is also drawn. and a line segment $[DG]$ is drawn parallel to $[CE]$ (*Elements*, I, 31), joining the top and bottom sides of the square $CDFE$ and passing through the point where the diagonal $[BE]$ intersects $[KM]$.

Euclid shows that the rectangle $ADHK$ is equal in area to the *gnomon* NOP that is formed from the union of the rectangle $CDHL$, the rectangle $HMFG$ and the square $DBMH$. He deduces from this that the sum of (the areas of) the rectangle $ADHK$ and the square $LHGE$ is equal to (the area of) the square $CBEF$, which yields the required result.

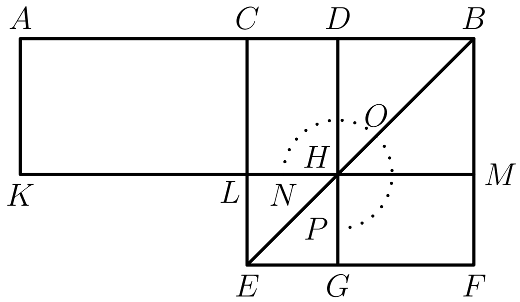
The proposition asserts that

> "the rectangle contained by AD, DB together with the square on CD is equal to the square on CB."

This result may be presented in symbolic notation as follows:

$$|AD| \times |DB| + |CD|^2 = |CB|^2,$$

where $|AD| \times |DB|$ denotes the area of a rectangle with sides equal (in length) to the line segments $[AD]$ and $[DB]$, and where $|CD|^2$ and $|CB|^2$ denote the areas of squares whose sides are equal to the line segments $[CD]$ and $[CB]$ respectively.



The proof may be summarized symbolically as follows.

$$\text{area}(HMFG) = \text{area}(CDHL) \quad (\textit{Elements}, \text{I}, 43).$$

Also

$$\text{area}(ACLK) = \text{area}(CBML) = \text{area}(CDHL) + \text{area}(DBMH).$$

Combining these results, we find that

$$\text{area}(ACLK) = \text{area}(HMFG) + \text{area}(DBMH) = \text{area}(DBFG).$$

Now

$$\text{area}(ADHK) = \text{area}(ACLK) + \text{area}(CDHL).$$

Therefore

$$\text{area}(ADHK) = \text{area}(CDHL) + \text{area}(DBFG) = \text{area}(\text{gnomon } NOP).$$

Now $\text{area}(ADHK) = |AD| \times |DB|$ and $\text{area}(LHGE) = |CD|^2$. Therefore

$$\begin{aligned} |AD| \times |DB| + |CD|^2 &= \text{area}(\text{gnomon } NOP) + \text{area}(LHGE) \\ &= \text{area}(CBEF) = |CB|^2, \end{aligned}$$

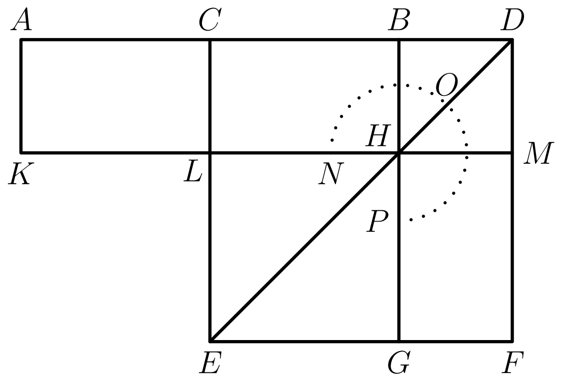
as required.

Proposition 6

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line; I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

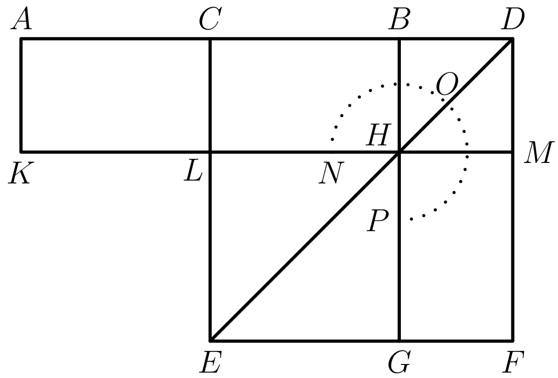
For let the square $CEFD$ be described on CD [i. 46], and let DE be joined; through the point B let BG be drawn parallel to either EC or DF, through the point H let KM be drawn parallel to either AB or EF, and further through A let AK be drawn parallel to either CL or DM [i. 31].



Then, since AC is equal to CB, AL is also equal to CH [i. 36]. But CH is equal to HF [i. 43]. Therefore AL is also equal to HF. Let CM be added to each; therefore the whole AM is equal to the gnomon NOP. But AM is the rectangle AD, DB, for DM is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on BC, be added to each; therefore the rectangle contained by AD, DB together with the square on CB is equal to the gnomon NOP and LG. But the gnomon NOP and LG are the whole square $CEFD$, which is described on CD; therefore the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

Therefore etc.

Q.E.D.



In this proposition $ADMK$ is a rectangle divided into smaller rectangles by line segments $[CL]$ and $[DM]$ parallel to the sides $[AK]$ and $[DM]$ of the containing rectangle. The conditions of the proposition require that the line segment $[AB]$ be bisected at the point C, so that $|AC| = |CB|$, and also require that $|BD| = |DM|$. Thus the rectangles $ACKL$ and $CBHL$ are equal to one another (*Elements*, I, 36), and $BDMH$ is a square.

The proposition in effect claims that the the sum of (the areas of) the rectangle $ADMK$ and a square constructed on the line segment $[BC]$ is equal to (the area of) a square constructed on the line segment $[CD]$.

In modern algebraic notation let $x = |BC|$ and $y = |BD|$. Then $|CD| = x + y$. Also $|AC| = x$, because $[AB]$ is bisected at the point C, and therefore

$$|AD| = 2x + y.$$

The proposition therefore corresponds to the algebraic identity

$$(2x + y)y + x^2 = (x + y)^2.$$

Euclid proceeds by completing a square $CDFE$ on the line segment $[CD]$ (*Elements*, I, 46). The diagonal $[DE]$ of that square is also drawn. and a line segment $[BG]$ is drawn parallel to $[CE]$ (*Elements*, I, 31), joining the top and bottom sides of the square $CDFE$ and passing through the point where the diagonal $[DE]$ intersects $[KM]$.

Euclid shows that the rectangle $ADMK$ is equal in area to the gnomon NOP that is formed from the union of the rectangle $CBHL$, the rectangle $HMFG$ and the square $BDHM$. He deduces from that that the sum of (the areas of) the rectangle $ADMK$ and the square $LHGE$ is equal to (the area of) the square $CBEF$, which yields the required result.

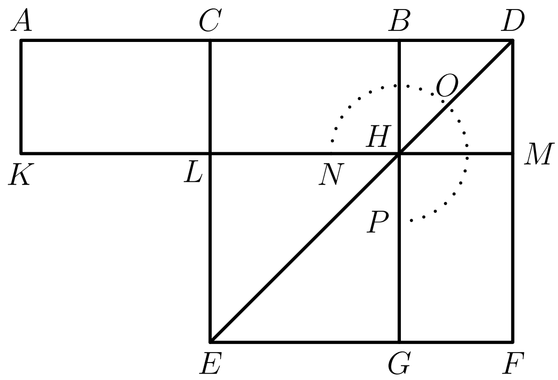
The proposition asserts that

> "the rectangle contained by AD, DB together with the square on CB is equal to the square on CD."

This result may be presented in symbolic notation as follows:

$$|AD| \times |BD| + |BC|^2 = |CD|^2,$$

where $|AD|\times|BD|$ denotes the area of a rectangle with sides equal (in length) to the line segments $[AD]$ and $[BD]$, and where $|BC|^2$ and $|CD|^2$ denote the areas of squares whose sides are equal to the line segments $[BC]$ and $[CD]$ respectively.



The proof may be summarized symbolically as follows.

$$\text{area}(HMFG) = \text{area}(CBHL) \quad (\textit{Elements}, \text{I}, 43).$$

Also

$$\text{area}(ACLK) = \text{area}(CBHL),$$

because the point C bisects $[AB]$. Therefore

$$\text{area}(ACLK) = \text{area}(HMFG),$$

and thus

$$\begin{aligned} \text{area}(ADMK) &= \text{area}(ACLK) + \text{area}(CBHL) + \text{area}(BDMH) \\ &= \text{area}(HMFG) + \text{area}(CBHL) + \text{area}(BDMH) \\ &= \text{area}(\text{gnomon } NOP) \end{aligned}$$

Moreover $\text{area}(ADMK) = |AD|\times|BD|$, $\text{area}(LHGE) = |CB|^2$ and $\text{area}(CDFE) = |CD|^2$. It follows that

$$\begin{aligned} |AD| \times |BD| + |CB|^2 &= \text{area}(\text{gnomon } NOP) + \text{area}(LHGE) \\ &= \text{area}(CDFE) = |CD|^2, \end{aligned}$$

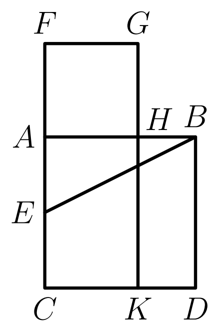
as required.

Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let AB be the given straight line; thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square $ABDC$ be described on AB; let AC be bisected at the point E, and let BE be joined; let CA be drawn through to F, and let EF be made equal to BE; let the square FH be described on AF, and let GH be drawn through to K. I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH.



For, since the straight line AC has been bisected at E, and FA added to it, the rectangle contained by CF, FA together with the square on AE is equal to the square on EF [II. 6]. But EF is equal to EB; therefore the rectangle CF, FA together with the square on AE is equal to the square on EB. But the squares on BA, AE are equal to the square on EB, for the angle at A is right [I. 47]: therefore the rectangle CF, FA together with the square on AE is equal to the squares on BA, AE. Let the square on AE be subtracted from each; therefore the rectangle CF, FA which remains is equal to the square on AB.

Now the rectangle CF, FA is FK, for AF is equal to FG; and the square on AB is AD; therefore FK is equal to AD. Let AK be subtracted from each; therefore FH which remains is equal to HD. And HD is the rectangle AB, BH, for AB is equal to BD; and FH is the square on AH; therefore the rectangle contained by AB, BH is equal to the square on HA. Therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA.

Q.E.F.

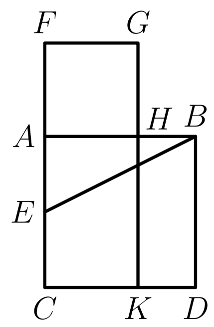
Note (DRW) on Euclid's *Elements*, Book II, Proposition 11

It is required to construct a point H in the line segment for which

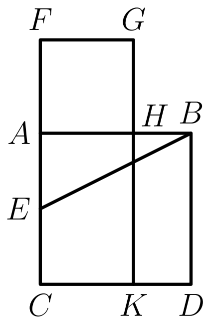
$$|AB| \times |HB| = |AH|^2.$$

To achieve this, the square $ABDC$ is constructed on the line segment $[AB]$ (*Elements*, I, 46), the side $[AC]$ of that square is bisected at $[E]$, and the side $[CA]$ is produced beyond A to a point F located so that $|EF| = |EB|$. (Note that the point F lies on a circle with centre E passing through the point B.) The square $FGHA$ is then constructed on $[AF]$. This square has a corner at the point H on the line segment $[AB]$ for which $|AH| = |AF|$. Proposition 6 of Book II and Proposition 47 of Book I (Pythagoras's Theorem) of Euclid's *Elements* can then by applied to prove that

$$|AB| \times |HB| = |AH|^2.$$



The proof that the square $FGHA$ is equal in area to the rectangle $HBDK$ may be summarized symbolically as follows.



$$|AE| = |EC| = \frac{1}{2}|AB|.$$

and $|AH| = |AF|$. Also $|EF| = |EB|$. Therefore

$$\begin{aligned} \text{area}(FGKC) + |AE|^2 &= |CF| \times |AF| + |AE|^2 \\ &= |EF|^2 \quad (\textit{Elements}, \text{II}, 6) \\ &= |EB|^2 \\ &= |AB|^2 + |AE|^2 \quad (\textit{Elements}, \text{I}, 47). \end{aligned}$$

Subtracting $|AE|^2$ from both sides, we find that

$$\text{area}(FGKC) = \text{area}(ABDC).$$

If we then subtract (the area of) the rectangle $AHKC$ from both sides, we find that

$$\text{area}(FGHA) = \text{area}(HBKD),$$

and thus

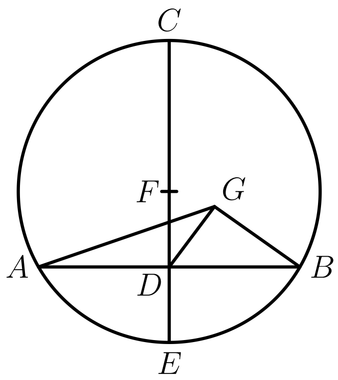
$$|AH|^2 = |AB| \times |HB|,$$

as required.

Proposition 1

To find the centre of a given circle.

Let ABC be the given circle; thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D; from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F; I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre, and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common, the two sides AD, DG are equal to the two sides BD, DG respectively; and the base GA is equal to the base GB, for they are radii; therefore the angle ADG is equal to the angle DGB [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the the equal angles is right [I Def. 10]; therefore the angle GDB is right.

But the angle FDB is also right; Therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

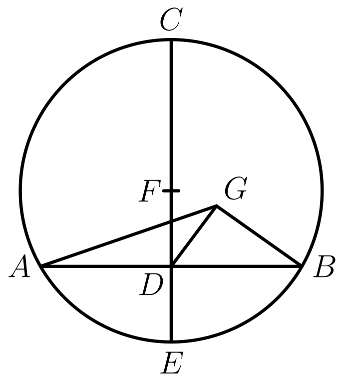
Therefore the point F is the centre of the circle ABC.

Porism. From this, it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

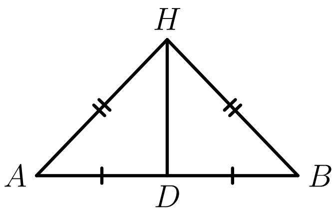
Q.E.F.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 1

T.L. Heath, in the commentary included with his translation of Euclid's *Elements of Geometry*, credits Todhunter with the observation that Euclid's construction tacitly assumes that the point D that bisects the line segment $[AB]$ joining two distinct points A and B on the circumference of the given circle lies within the circle. Heath notes that, even if Euclid's text were interpreted in a way consistent with allowing the point D to fall outside the circle, nevertheless it would need to be established that the perpendicular bisector of the line segment $[AB]$ does in fact intersect the circle in two points. As it happens, the result that the point D bisecting $[AB]$ lies within the circle is in fact and immediate consequence of the following proposition (*Elements*, III, 2). Thus ideally Euclid should have placed Proposition 2 of Book III before Proposition 1.



Let A and B be two distinct points in the plane, and let H be a point in the plane that is distinct from the midpoint D of the line segment $[AB]$ with endpoints A and B but is equidistant from the points A and B. Then the sides $[HA]$ and $[AD]$ of the triangle $\triangle HAD$ are equal in length to the sides Then the sides $[HB]$ and $[BD]$ of the triangle $\triangle HBD$, and the line segment $[HD]$ is a common side of both triangles.



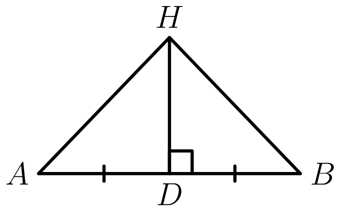
It follows from the SSS Congruence Rule (*Elements*, I, 8) that the triangles $\triangle HDA$ and $\triangle HDB$ are congruent to one another, and therefore the angles $\angle HDA$ and $\angle HDB$ are equal to one another. It then follows from the definition of *right angles* that $\angle HDA$ is a right angle, and thus the line DH is the perpendicular bisector of the line segment $[AB]$.

It follows directly from the result just obtained is that, given any two distinct points on the circumference of a circle, the centre of a circle must lie on the perpendicular bisector of the line segment or *chord* joining those two points. This result is separately stated by Euclid as a *porism* (or immediate corollary) following the proof of the main proposition.

Euclid's argument to the effect that the centre of the circle must lie on the perpendicular bisector of the chord AB joining two distinct points A and B employs the proof-technique of *reductio ad absurdum* ("proof by contradiction"), but may be paraphrased as follows.

The SSS Congruence Rule ensures that, given any point equidistant from A and B, the line joining that point to the midpoint D of the chord $[AB]$ must meet the chord at right angles, for the reasons set out immediately above. Thus if G is a point that does not lie on the perpendicular bisector of the chord $[AB]$ then the line passing through the points G and D does not intersect the chord $[AB]$ at right angles. and therefore the point G cannot be equidistant from the points A and B. In particular, such a point G cannot be the centre of the circle. It follows that the centre of the circle must lie on the perpendicular bisector of the chord $[AB]$.

Now let H be a point on the perpendicular bisector of the line segment $[AB]$. It then follows from the SAS congruence rule (*Elements*, I, 4) that the triangles $\triangle HDA$ and $\triangle HDB$ are congruent to one another, and therefore $|HA| = |HB|$.

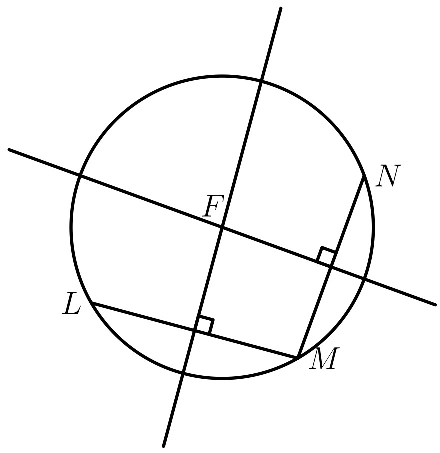


Thus points on the perpendicular bisector of $[AB]$ are equidistant from the points A and B.

We conclude therefore that a point H of the plane is equidistant from the points A and B of that plane if and only if it lies on the perpendicular bisector of the the line segment $[AB]$. In particular, given two points A and B on the circumference of a circle, the centre of the circle must lie on the perpendicular bisector of the chord $[AB]$ joining those two points.

The arguments presented above establish the following result: *given two distinct points A and B, and given a third point H, the point H is equidistant from the points A and B if and only if it lies on the perpendicular bisector of the line segment $[AB]$ joining the points A and B.*

A well-known alternative construction for finding the centre of a circle begins by taking three distinct points on the circumference of the circle. The centre of the circle must lie on the perpendicular bisector of the line segment joining any two distinct points on the circumference of the circle. Therefore the perpendicular bisector of the line segment joining the first and second of the three points taken on the circumference intersects the perpendicular bisector joining the second and third of these points at the centre of the circle.

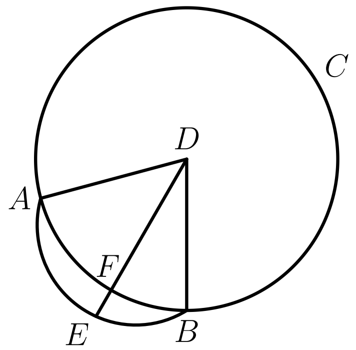


Proposition 2

If on the circumference of a given circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A and B be taken at random on its circumference; I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.



Then since DA is equal to DB, the angle DAE is also equal to the angle DBE [I. 5]. And, since one side AEB of the triangle DAE is produced, the angle DEB is greater than the angle DAE [I. 16]. But the angle DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE. And the greater angle is subtended by the greater side [I. 19]; therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

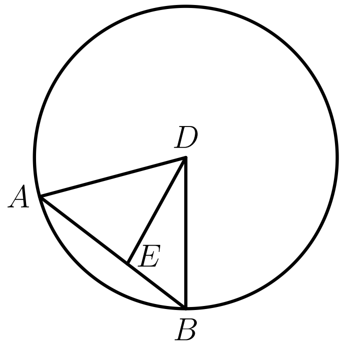
Similarly we can prove that neither will it fall on the circumference itself; therefore it will fall within. Therefore etc.

Q.E.D.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 2

Euclid proves a signficant case of this proposition, showing the the line segment joining two distinct points A and B on a given circle cannot pass outside the circle, using the proof technique of *reductio and absurdum* ("proof by contradiction"). A more direct proof strategy can be adopted.

Let A and B be two distinct points on a circle with centre D and let E be a point on the line segment $[AB]$ that lies between A and B.



The triangle $\angle DAB$ is an isosceles triangle, because $[DA]$ and $[DB]$ are radii of a circle and are thus equal in length. Therefore $|\angle DAB| = |\angle DBA|$ (*Elements*, I, 5). However the exterior angle $\angle DEB$ of the triangle $\triangle DAE$ at E is greater than the opposite (or remote) interior angle $\angle DAE$ of that triangle at A (*Elements*, I, 16). Now $\angle DAE$ and $\angle DBE$ denote the same angles as $\angle DAB$ and $\angle DBA$ respectively, because the point E lies between A and B on the line segment joining these points. Also the angles $\angle DAB$ and $\angle DBA$ are equal, as we noted above. Therefore $|\angle DAE| = |\angle DBE|$, and thus the angle $\angle DEB$ of the triangle $\triangle DEB$ at the vertex E is greater than the angle $\angle DBE$ of this triangle at the vertex B. It follows that the side $[DB]$ of this triangle opposite the vertex E is longer than the side $[DE]$ of the triangle opposite the vertex B (*Elements*, I, 19). Thus $|DE|$ is less than the radius of the circle, and therefore the point E lies inside the circle.

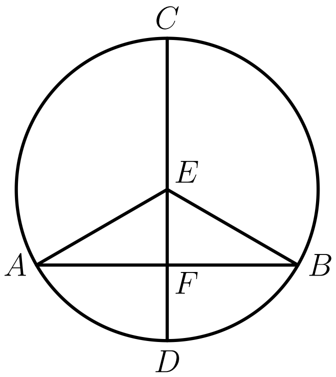
This proves that the straight line segment joining two distinct points A and B on the circumference of a circle falls within the circle.

Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F; I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common, two sides are equal to two sides; and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10] therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles; I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction, since EA is equal to EB, the angle EAF is also equal to the angle EBF [I. 5].

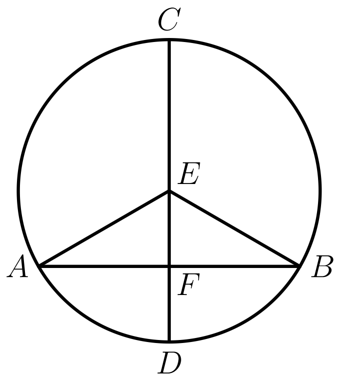
But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles; therefore they will also have the remaining sides equal to the remaining sides [I. 26]; therefore AF is equal to FB.

Therefore etc.

Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK III, PROPOSITION 3

The first part of the proof of Proposition 3 of Book III in essence repeats the argument, already present in the proof of Proposition 1 of Book III, that applies the SSS Congruence Rule (*Elements*, III, 3) in order to show that the line joining the centre of the circle to the midpoint of the chord $[AB]$ joining two distinct points A and B on the circumference of the circle must intersect the chord at right angles. Indeed the Porism that follows the proof of Proposition 1 may be viewed as a statement to the effect that the centre of the circle must lie on the perpendicular bisector of the chord. It follows that, in the configuration depicted below, the line passing through the both the centre E of the circle and the midpoint F of the chord $[AB]$ must coincide with the perpendicular bisector of the chord $[AB]$, and therefore must bisect that chord at right angles.

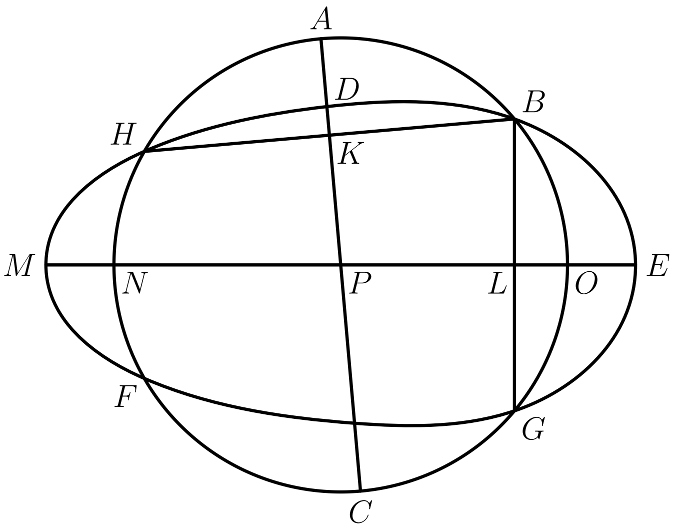


The conclusion of the proof of Proposition 3 however presents a new argument not already covered in its essentials in the proof of Proposition 1. Let a line passing through the centre E of the circle intersect the chord $[AB]$ at right angles at some point F of the chord. The triangle EAB is an isosceles triangle, because $[EA]$ and $[EB]$ are radii of the circle, and therefore $|\angle EAB| = |\angle EBA|$. It follows that the angles of the triangle $\triangle EAF$ at A and F are equal to the angles of the triangle $\triangle ABF$ at B and F respectively. Moreover the side $[EA]$ of the first triangle is equal to the corresponding side $[EB]$ of the second triangle. Applying the AAS Congruence Rule (*Elements*, I, 26), we deduce that the triangles $\triangle EAF$ and $\triangle EBF$ are congruent, and therefore $|AF| = |BF|$. Thus the line EF bisects the chord $[AB]$ at the point F.

Proposition 10

A circle does not cut a circle at more points than two.

For, if possible, let the circle ABC cut the circle DEF at more points than two, namely B, G, F, H; let BH, BG be joined and bisected at the points K, L, and from K, L let KC, LM be drawn at right angles to BH, BG and carried through to the points A, E.



Then, since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles, the centre of the circle ABC is on AC [III. 1, Por.].

Again, since in the smae circle ABC a straight line NO cuts a straight line BG into two equal parts and at right angles, the centre of the circle ABC is on NO.

But it was also proved to be on AC, and the straight lines AC, NO meet at no point except at P; therefore the point P is the centre of the circle ABC.

Similarly we can prove that P is also the centre of the circle DEF; therefore the two circles ABC, DEF which cut one another have the same centre P: which is impossible. [III. 5].

Therefore etc.

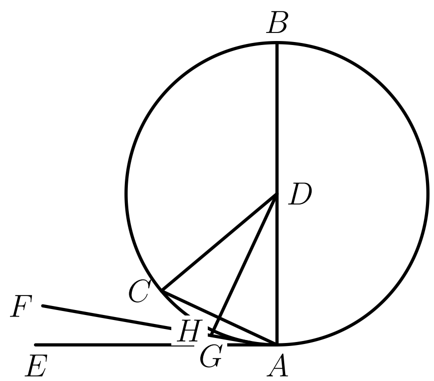
Q.E.D.

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.



Since DA is equal to DC, the angle DAC is also equal to the angle ACD [I. 5].

But the angle DAC is right; therefore the angle ACD is also right: thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible [I. 17].

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference; therefore it will fall outside.

Let it fall as AE; I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right, and the angle DAG is less than a right angle, AD is greater than DG [I. 19].

But DA is equal to DH; therefore DH is greater than DG, the less than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle, and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilinear angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contined by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed; therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.—

PORISM. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Q.E.D.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 16

Before examining Euclid's proof of Proposition 16 in Book III of the *Elements of Geometry*, we discuss the intersection of lines and circles.

A line cannot intersect a circle in more than two distinct points. Indeed suppose that a given line were to intersect a circle in three distinct points P, Q and R. Then the centre of the circle would lie on the perpendicular bisectors of each of the line segments $[PQ]$, $[PR]$ and $[QR]$ (*Elements*, III, 1). These perpendicular bisectors would be distinct from one another. However these perpendicular bisectors would all be perpendicular to the given line and would therefore be parallel to one another (*Elements*, I, 28), and therefore the centre of the circle could not lie on more than one of these perpendicular bisectors. Thus the assumption that the given line intersects the circle in three or more points would lead to a contradiction, and therefore a line cannot intersect a circle in more than two points.

This result can also be seen as follows. Suppose that a given line were to intersect a circle in three distinct points P, Q and R, with the point Q lying between P and R. The line segment $[PR]$ would then fall within the circle (*Elements*, III, 2), and therefore the point Q would lie within the circle, contradicting the assumption that it is a point at which the given line meets the circle.

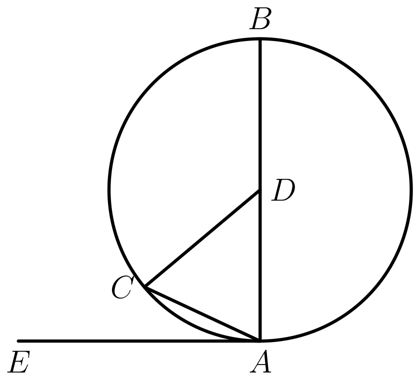
A line that passes through points lying within a given circle must meet that circle in exactly two points. Euclid implicitly relies on this assumption in presenting and justifying the geometric construction for dropping a perpendicular from a given point to a given line (*Elements*, I, 12). A line meeting a circle at two distinct points is said to *cut* the circle at those points of intersection. A line cuts a given circle at a given point if, in the neighbourhood of that point, it passes through points that lie within the circle. A line segment or ray is said to cut a given circle at a given point if it forms part of a line cutting the circle at that point.

A line that meets a given circle in a single point cannot therefore pass through any points that lie within the circle. Such a line is said to *touch* the circle at the point where it meets the circle, and moreover such a line is said to be a *tangent line* to the circle at the point where it cuts the circle.

In summary, if a given line meets a given circle, then either the line meets the circle at a single point, in which case it touches the circle at that point and does not pass through any point lying within the circle, or else the ilne meets the circle at two distinct points, in which case it cuts the circle at each of those points. Moreover, in the case where the line meets the circle in exactly two points, all points of the line that lie between the two points of intersection lie within the circle.

We now turn our attention to the specifics of Euclid's proof of Proposition 16. The following argument is essentially a reformulation of Euclid's argument.

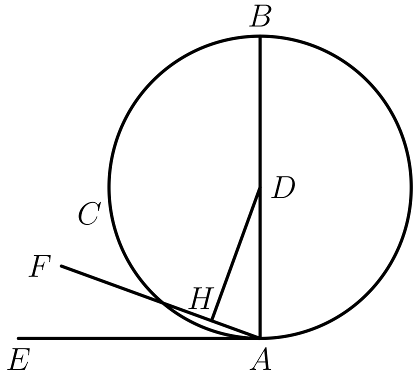
Let a circle be drawn with centre D passing through the point A. Suppose that a line AC cuts that circle at the point A and also at another point C, as depicted in the figure that accompanies the proof. (We do not at this stage make any further assumption regarding the angle between this line and the diameter AB.)



The triangle $\triangle DAC$ is an isosceles triangle, with equal sides $[DA]$ and $[DC]$. The angles $\angle DAC$ and $\angle DCA$ of this triangle at A and C must therefore be equal (*Elements*, I, 5), and the sum of these two angles must be less than two right angles (*Elements*, I, 17). It follows that the angles $\angle DAC$ and $\angle DCA$ must each be less than a right angle. We conclude from this that no line cutting the circle at A and some other point can be perpendicular to the diameter $[AB]$. Moreover any line through A that does not cut the circle at A must touch the circle at A. It follows therefore that any line, such as AE, that is perpendicular to the diameter $[AB]$ at A must touch the circle at A.

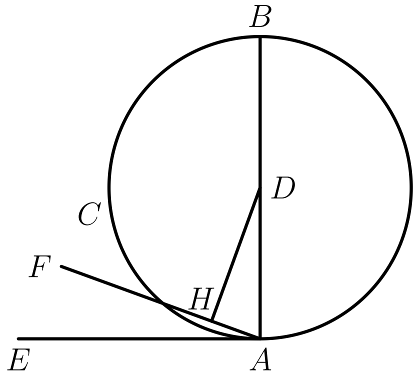
We have seen that a line that cuts the circle at A must make an angle with $[AD]$ less than a right angle on the side of $[AD]$ on which it passes within the circle. Euclid's argument that a line cannot be interposed between the tangent line AE and the circle amounts to a converse of this result.

Indeed let $[AF]$ be a line segment that makes an acute angle at A with the radius $[AD]$. (We make no further assumptions about this line segment.) A perpendicular can be dropped from the centre D to the line $[AF]$, intersecting the line $[AF]$ at some point H (*Elements*, I, 12).



Then $\triangle DAH$ is a triangle for which the angle at A is acute and the angle at H is right. It follows from this that the side $[DA]$ of this triangle opposite the greater angle $\angle DHA$ is greater than the side $[DH]$ opposite the smaller angle $\angle DAH$ (*Elements*, I, 19). Therefore the point H lies closer to the centre D of the circle than the point A, and therefore lies within the circle. Thus the line segment $[AF]$ cuts the circle at A. It follows directly that no line can be interposed between the tangent line $[AE]$ and the circumference of the circle, as Euclid claims.

In the conclusion of the statement of this proposition, Euclid asserts that "the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle". Now the general definition of *angle* given in the definitions commencing Book I of Euclid's *Elements of Geometry* applies to an angle formed by a straight lines and a circular arc at a point at which they meet one another. The "angle of the semicircle" at A is the angle formed by the diameter $[AB]$ and the circular arc BAC. In a small neighbourhood of the point A, the interior of the angle "of the semicircle" BAC consists of all points of that neighbourhood that lie within the semicircle.



Let $\angle BAF$ be an acute "rectilinear angle" at the point A, on the same side of the diameter $\angle BA$ as the semicircle BAC. All points within a sufficiently small neighbourhood of the point A that lie within the rectilinear angle $\angle BAF$ also lie within the semicircle BAC, and therefore lie within the "angle of the semicircle" at the point A. Thus, in the language of Euclid, the "angle of the semicircle" BAC at A is greater than the rectilinear angle $\angle BAF$.

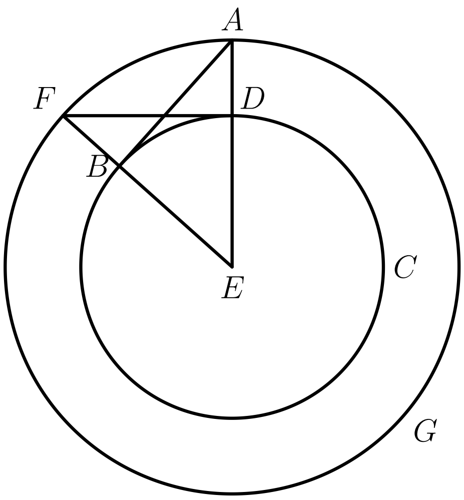
The "remaining angle" referred to in the statement of the proposition is the "horn angle" formed by the tangent line AE and the circular arc ACB. All points within this horn angle in a sufficiently small neighbourhood of the point A lie within the the rectilinear angle $\angle EAF$, no matter how small this rectilinear angle. Therefore the the "horn angle" at A formed by the tangent line AE and the circular arc ACB is less than any rectilinear angle.

Proposition 17

From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; thus it is required to draw from the point A a straight line touching the circle BCD.

For let the centre E of the circle be taken [iii. 1]. let AE be joined, and with centre E and distance EA let the circle AFG be described; from D let DF be drawn at right angles to EA, and let AF, AB be joined; I say that AB has been drawn from the point A touching the circle BCD.



For, since E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two sides FE, ED: and they contain a common angle, the angle at E; therefore the base DF is equal to the base AB, and the triangle DEF is equal to the triangle BEA, and the remaining angles to the remaining angles [1. 4]; therefore the angle EDF is equal to the angle EBA.

But the angle EDF is right; therefore the angle EBA is also right.

Now EB is a radius; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [iii. 16, Por.] therefore AB touches the circle BCD.

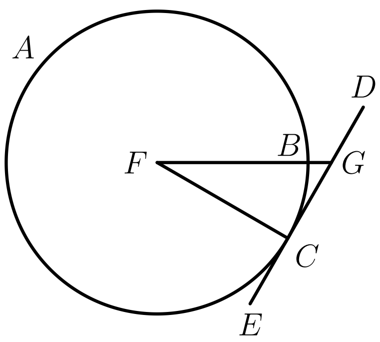
Therefore from the given point A the straight line AB has been drawn touching the circle BCD.

Proposition 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C; I say that FC is perpendicular to DE.

For, if not, let FG be drawn from F perpendicular to DE.



Then, since the angle FGC is right, the angle FCG is acute [I. 17]; and the greater angle is subtended by the greater side; therefore FC is greater than FG.

But FC is equal to FB; therefore FB is also greater than FG, the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC; therefore FC is perpendicular to DE. Therefore, etc.

Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK III, PROPOSITION 3

It would be possible to deduce Proposition 18 fairly directly from Proposition 16: if the line meets the circle at a point C on the circumference of the circle, but not at right angles to the radius joining the centre F of the circle to the point C, then it must make an acute angle with the radius $[FC]$ on one or other side of the radius $[FC]$, and the line would therefore pass within the circle, because the "angle of the semicircle is greater [...] than any acute rectilinear angle". It follows that if a line touches the circle, not passing within the circle, then it must meet the circle at right angles to the radius at the point of intersection.

Nevertheless the proof of Proposition 18 presented by Euclid may be regarded as more straightforward than the statement and proof of Proposition 16.

We now review some of the principal results concerning tangent lines included in Propositions 16 and 18 of Book III of Euclid's *Elements of Geometry.*

A foundation for these results is provided by the following two well-known results that are firmly established in Book I of the *Elements of Geometry*:

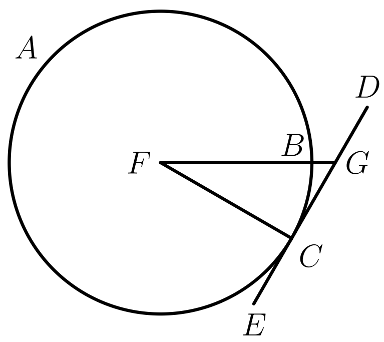
(i) given a line, and given a point not on the line, a perpendicular can be dropped from the given point to some point on the given line so that the given line and the perpendicular dropped onto it meet at right angles at their point of intersection;

(ii) in a right-angled triangle, the hypotenuse is longer than either of the two other sides.

The geometric construction for dropping a perpendicular onto a given line is presented in Proposition 12 of Book I of Euclid's *Elements of Geometry.* The result concerning the hypotenuse of a right-angled triangle may be justified in at least two ways. On the one hand it clearly follows as an immediate consequence of Pythagoras's Theorem (*Elements*, I, 47). Alternatively one can justify this by noting that any angle of right-angled triangle other than the right angle itself must be an acute angle, because the sum of any two angles of a triangle taken together must be less than two right angles (*Elements*, I, 17). But, in a triangle, the greater angle is subtended by the greater side (*Elements*, I, 19). The hypotenuse is subtended by the right angle in a right-angled triangle. It is therefore longer than either of the two other sides, because those other sides are subtended by acute angles that are less than a right-angle.

We now apply these results in order to show that a line meeting a circle at some point of the circumference touches the circle at that point if and only if it is perpendicular to the radius joining the the centre of the circle to that point on the circumference.

Let DE be a given line, and let F be a given point that does not lie on the line DE. A perpendicular can be dropped from the given point F to the given line DE, meeting that line at some point C on the line (see (i) above). If G is some point on the line DE that is distinct from the point C then $\triangle FGC$ is a right-angled triangle with its right angle at the vertex C. The hypotenuse $[FG]$ of this triangle is then longer than the perpendicular $[FC]$ dropped from F to the line DE. It follows that the point C is the closest point in the line DE to the point F. Moreover every other point on the line DE is further away from the point F than the point C.



It follows from the observations just made that a circle centred on the point F and passing through the point C will not intersect the line DE at any other point of that line. No point of the line DE lies within the circle. Therefore the line DE will touch at the point C the circle centred on the point F and passing through C.

Thus if a circle is given, with centre F, and if a point C is taken on its circumference, then the line passing through the point C at right angles to the radius $[FC]$ will touch the circle at C. This is the result stated in the Porism to Proposition 16 of Book III of Euclid's *Elements of Geometry.*

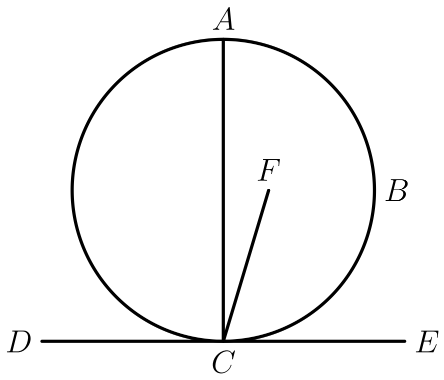
Conversely if a circle centred on a point F touches a line DE at a point C on the circumference of that circle, then that line DE must be perpendicular to the radius $[FC]$ at the point C. For if the line DE were not perpendicular to the radius FC then the perpendicular dropped from the point F to the line DE would meet the line DE at some point G distinct from the point C, and that point G would be closer to the centre of the circle than the point C, and thus the line would not touch the circle at C, contrary to hypothesis. This is the result stated in Proposition 18 of Book III of Euclid's *Elements of Geometry.*

Proposition 19

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line DE touch the circle ABC at the point C, and from C let CA be drawn at right angles to DE; I say that the centre of the circle is on AC.

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.



Since a straight line D touches the circle ABC, and FC has been joined from the point of contact, FC is perpendicular to DE [iii. 18]; therefore the angle FCE is right.

But the angle ACE is also right; therefore the angle ACE is equal to the angle ACE, the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except a point on AC. Therefore, etc.

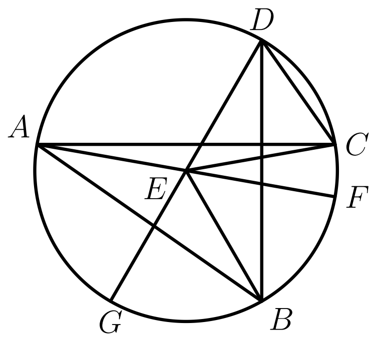
Q.E.D.

Proposition 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle at its centre, and the angle BAC an angle at the circumference, and let them have the same circumference BC as base; I say that the angle BEC is double of the angle BAC.

For let AE be joined and drawn through to F.



Then, since EA is equal to EB, the angle EAB is also equal to the angle EBA [I. 5]; therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA [I. 32]; therefore the angle BEF is also double of the angle EAB.

For the same reason the angle FEC is also double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.

Again let another straight line be inflected, and let there be another angle BDC; let DE be joined and produced to G.

Similarly then we can prove that the angle GEC is double of the angle EDC, of which the angle GEB is double of the angle EDB; therefore the angle BEC which remains is double of the angle BDC. Therefore, etc.

Q.E.D.

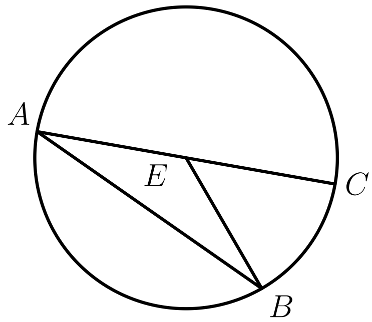
Note (DRW) on Euclid's *Elements*, Book III, Proposition 20

Let a circle be given, together with two distinct points B and C on its circumference that are not the endpoints of a diameter of the circle. These points B and C are the endpoints of a short arc subtending an angle $\angle BEC$ at the centre of the circle. Euclid claims that the angle $\angle BEC$ subtended by this arc BC at the centre is double the angle subtended by this arc at a point A on the circumference of the circle. Now "rectilineal angles" in Euclid's *Elements of Geometry* are "ordinary" angles less than two right angles, and indeed the concept of "reflex angle" does not appear in Euclid's *Elements of Geometry.* Accordingly the result in the form stated by Euclid is only valid in cases where the centre E and the point A on the circumference of the circle both lie on the same side of the line BC passing through the points B and C.

The full proof of the result in this configuration where the points A and E lie on the same side of the line BC falls naturally into three cases, depending on the location of the centre of the circle with respect to the triangle $\triangle ABC$: the centre E of the circle might lie on one or other of the sides $[AB]$ and $[AC]$ of the triangle $\triangle ABC$; the centre E might lie inside the triangle $\triangle ABC$; the centre E might lie outside the triangle $\triangle ABC$.

Euclid considers the second and third of these cases, and the geometrical figure accompanying Euclid's proof is applicable to both these cases. The discussion below uses simpler figures, based on Euclid's figure, that are appropriate to the separate discussion of the three cases.

Consider the first case in which the centre E of the circle lies on one or other of the sides $[AB]$ and $[AC]$ of the triangle $\triangle ABC$. This case is not explicitly considered by Euclid. We may suppose, without loss of generality, that E lies on the side $[AC]$, as depicted in the following figure.



In this case the triangle $\triangle EAB$ is an isosceles triangle with equal sides $[EA]$ and $[EB]$, and therefore

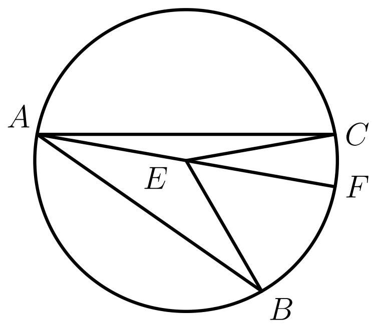
$$|\angle EAB| = |\angle EBA|$$

are equal (*Elements*, I, 5). The external angle $\angle BEC$ of the triangle $\triangle EAB$ at E is equal to the sum of the two opposite (or remote) angles $\angle EAB$ and $\angle EBA$ of this triangle at A and B (*Elements*, I, 32). Each of these two opposite angles is equal to $\angle BAC$. Therefore

$$|\angle BEC| = |\angle EAB| + |\angle EBA| = 2 \times |\angle BAC|.$$

The required result has thus been established in the case where the centre E of the circle lies on a side of the triangle $\triangle ABC$.

The next case to consider is that in which the centre E of the circle lies in the interior of the triangle $\triangle ABC$. Let the line segment AE be produced beyond E to a point F on the circumference of the circle. This is the first of the cases that Euclid explicitly considers.



Applying the result established in the previous case to the short arcs joining B to F and F to C, we see that

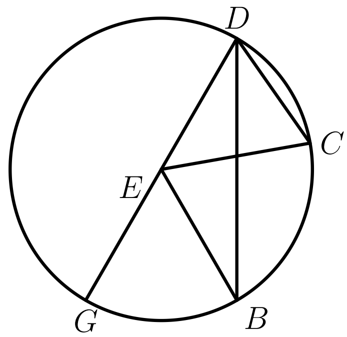
$$|\angle BEF| = 2 \times |\angle BAF| \quad \text{and} \quad |\angle CEF| = 2 \times |\angle CAF|.$$

It follows that

$$\begin{aligned} |\angle BEC| &= |\angle BEF| + |\angle CEF| = 2 \times |\angle BAF| + 2 \times |\angle CAF| \\ &= 2 \times \angle BAC. \end{aligned}$$

The result is thus now established in the case where the centre E of the circle lies inside the triangle $\triangle ABC$.

It remains to prove the result in the second of the two cases explicitly considered by Euclid. In this case the angle $\angle BEC$ subtended at the centre of the circle by the short arc from B to C is compared to the angle $\angle BDC$ subtended by that arc at a point D on the circumference of the circle that lies on the same side of the line BC as the centre of the circle but is situated so that the centre E of the circle lies outside the triangle $\triangle DBC$. In this configuration, let the line segment $[DE]$ be produced beyond E to a point G lying on the circumference of the circle, as depicted in the figure below. We must show that $|\angle BEC| = 2 \times |\angle BDC|$.



Now
$$|\angle GDC| = |\angle GDB| + |\angle BDC|$$
and
$$|\angle GEC| = |\angle GEB| + |\angle BEC|.$$
Moreover, applying the result obtained in the first case considered, we find that
$$|\angle GEB| = 2 \times |\angle GDB| \quad \text{and} \quad |\angle GEC| = 2 \times |\angle GDC|.$$
Therefore
$$\begin{aligned} |\angle GEB| + |\angle BEC| &= |\angle GEC| = 2 \times |\angle GDC| \\ &= 2 \times |\angle GDB| + 2 \times |\angle BDC| \\ &= |\angle GEB| + 2 \times |\angle BDC|, \end{aligned}$$
and therefore
$$|\angle BEC| = 2 \times |\angle BDC|.$$

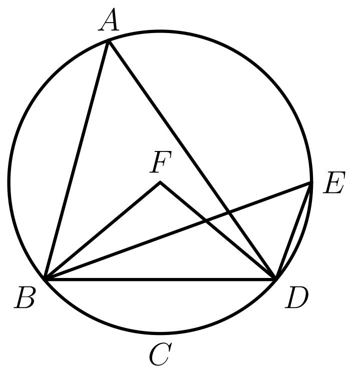
The result stated by Euclid has therefore been verified in all relevant cases, subject to the implicit requirement that the circular arc in question lies on the opposite side of the line joining its endpoints to both the centre of the circle and the point on the circumference at which the angle subtended by the arc is to be considered.

Proposition 21

In a circle the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD, BED be angles in the same segment $BAED$; I say that the angles BAD, BED are equal to one another.

For let the centre of circle $ABCD$ be taken, and let it be F; let BF, FD be joined.



Now, since the angle BFD is at the centre, and the angle BAD at the circumference, and they have the same circumference BCD as base, therefore the angle BFD is double of the angle BAD [III. 20]

For the same reason the angle BFD is also double of the angle BED; therefore the angle BAD is equal to the angle BED.

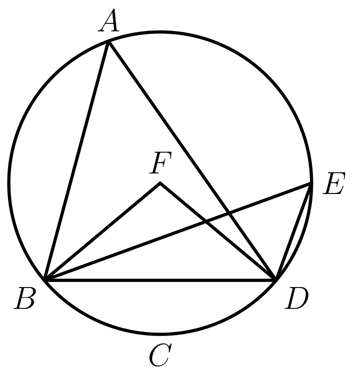
Therefore, etc.

Q.E.D.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 21

Let a circle be given, together with two distinct points B and D on its circumference, and let A and E be points on the circumference of the circle that both lie on the same side of the line BD. The points B, D and A then determine a segment of the circle bounded by the circular arc $BAED$ and the straight line segment $[BD]$. The points B, D and E determine the same segment of the circle. Both angles $\angle BAD$ and $\angle BED$ are angles *in* the segment in question, according to the terminology adopted by Euclid and set out in the *Definitions* for Book III of the *Elements of Geometry*.

The proposition states that if the segment determined by the points B, D and A (as described above) coincides with the segment determined by the points B, D and E (so that A and E are points of the circumference of the circle that lie on the same side of the line BD), then the angles $\angle BAD$ and $\angle BED$ are equal. However Euclid only considers explicitly the case in which the centre F of the circle lies on the same side of the line BD as the points A and E and therefore lies in the interior of the segment.



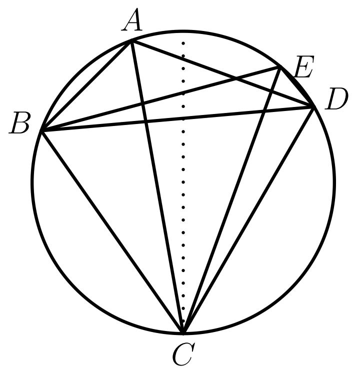
In this case considered explicitly by Euclid, the segment $BAED$ is said to be *greater than a semicircle*, and the centre of the circle lies in its interior. In this configuration the previous proposition (*Elements*, III, 20) ensures that

$$2 \times |\angle BAD| = |\angle BFD| = 2 \times |\angle BED|,$$

from which it follows that the angles $\angle BAD$ and $\angle BED$ are equal in the case where A, E and F all lie on the same side of the line BD.

The result in the general case can be deduced from that in this special case. The argument below is adapted from that presented by Robert Simson in his edition of Euclid's *Elements of Geometry*, published in 1756 (see Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, Volume 2, page 50).

Let a point C be taken on the circle that is the endpoint of a diameter of the circle whose other endpoint lies on the arc $BAED$.



The centre of the circle then lies on the same side of the line BC as the points A and E. Also the centre of the circle lies on the same side of the line DC as the points A and E. It therefore follows from the special case of the proposition explicitly considered by Euclid that

$$|\angle BAC| = |\angle BEC| \quad \text{and} \quad |\angle DAC| = |\angle DEC|.$$

Therefore

$$|\angle BAD| = |\angle BAC| + |\angle DAC| = |\angle BEC| + |\angle DEC| = |\angle BED|.$$

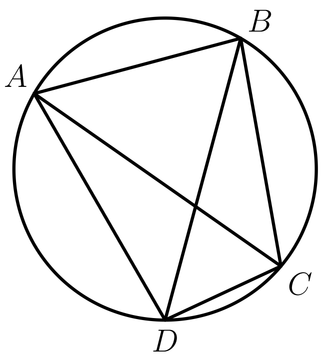
This establishes the result in all cases.

Proposition 22

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral in it; I say that the opposite angles are equal to two right angles.

Let AC, BD be joined.



Then, since in any triangle the three angles are equal to two right angles [I. 32], the three angles CAB, ABC, BCA of the triangle ABC are equal to two right angles.

But the angle CAB is equal to the angle BDC, for they are in the same segment $BADC$ [III. 21]; and the angle ACB is equal to the angle ADB, for they are in the same segment $ADCB$; therefore the whole angle ADC is equal to the angles BAC, ACB.

Let the angle ABC be added to each; therefore the angles ABC, BAC, ACB are equal to the angles ABC, ADC.

But the angles ABC, BAC, ACB are equal to two right angles; therefore the angles ABC, ADC are also equal to two right angles.

Similarly we can prove that the angles BAD, DCB are also equal to two right angles.

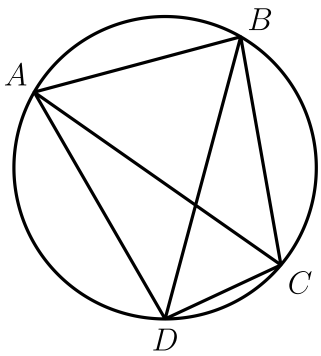
Therefore, etc.

Q.E.D.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 22

We present the proof using more symbolic notation. Let A, B, C and D be distinct points lying on the circumference of a circle that are vertices of a quadrilateral $ABCD$, as depicted in the figure.

(Such a quadrilateral with vertices on the circumference of a circle is said to be a *cyclic quadrilateral*.)



We must show that

$$|\angle ABC| + |\angle ADC| = |\angle BAD| + |\angle BCD| = \text{two right angles.}$$

Now

$$|\angle CAB| + |\angle ABC| + |\angle BCA| = \text{two right angles}$$

(*Elements*, I, 32). But

$$|\angle ADC| = |\angle CDB| + |\angle BDA|,$$

and moreover

$$|\angle CDB| = |\angle CAB| \quad \text{and} \quad |\angle BDA| = |\angle BCA|$$

(*Elements*, III, 21). It follows that

$$\begin{aligned} |\angle ABC| + |\angle ADC| &= |\angle ABC| + |\angle CDB| + |\angle BDA| \\ &= |\angle ABC| + |\angle CAB| + |\angle BCA| \\ &= \text{two right angles.} \end{aligned}$$

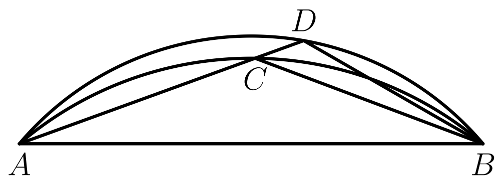
Similarly

$$|\angle BAD| + |\angle BCD| = \text{two right angles.}$$

Proposition 23

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line AB let two similar and unequal segments of circles ACB, ADB be constructed on the same side; let ACD be drawn through, and let CB, DB be joined.



Then, since the segment ACB is similar to the segment ADB, and similar segments of circles are those which admit equal angles [III. Def. 11], the angle ACB is equal to the angle ADB, the exterior to the interior: which is impossible [I. 16]. Therefore, etc.

Q.E.D.

Proposition 24

Similar segments of circles on equal straight lines are equal to one another.

For let AEB, CFD be similar segments of circles on equal straight lines AB, CD; I say that the segment AEB is equal to the segment CFD.

For, if the segment AEB be applied to CFD, and if the point A be placed on C and the straight line AB on CD, the point B will also coincide with the point D, because AB is equal to CD; and, AB coinciding with CD, the segment AEB will also coincide with CFD.

For, if the straight line AB coincide with CD but the segment AEB do not coincide with CFD, it will either fall within it, or outside it; or it will fall awry, as CGD, and a circle cuts a circle at more points than two: which is impossible [III. 10].

Therefore, if the straight line AB be applied to CD, the segment AEB will not fail to coincide with CFF also; therefore it will coincide with it and will be equal to it.

Therefore, etc.

Q.E.D.

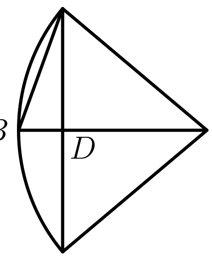
Proposition 25

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let ABC be the given segment of a circle; thus it is required to describe the complete circle belonging to the segment ABC, that is, of which it is a segment.

For let AC be bisected at D, let DB be drawn from the point D at right angles to AC, and let AB be joined; the angle ABD is then greater than, equal to, or less than the angle BAD.

First let it be greater; and on the straight line BA, and at the point A on it, let the angle BAE be constructed equal to the angle ABD; let DB be drawn through to E, and let EC be joined.



Then, since the angle ABE is equal to the angle BAE, the straight line EB is also equal to EA [I. 6].

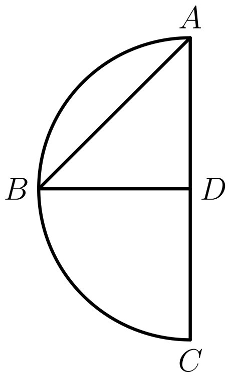
And, since AD is equal to DC, and DE is common, the two sides AD, DE are equal to the two sides CD, DE respectively; and the angle ADE is equal to the angle CD, for each is right; therefore the base AE is equal to the base CE; therefore the three straight lines AE, EB, EC are equal to one another.

Therefore the circle drawn with centre E and distance one of the straight line AE, EB, EC will also pass through the remaining points and will have been completed. [III. 9]

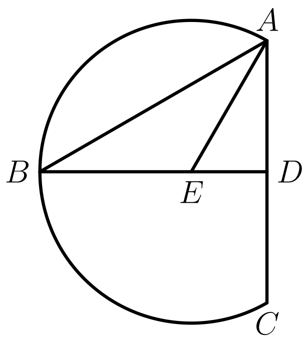
Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment ABC is less than a semicircle, because the centre E happens to be outside it.

Similarly, even if the angle ABD be equal to the angle BAD, AD being equal to each of the two BD, DC, the three straight lines DA, DB, C will be equal to one another, D will be the centre of the completed circle, and ABC will clearly be a semicircle.



But, if the angle ABD be less than the angle BAD, and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD, the centre will fall on DB within the segment ABC, and the segment ABC will clearly be greater than a semicircle.



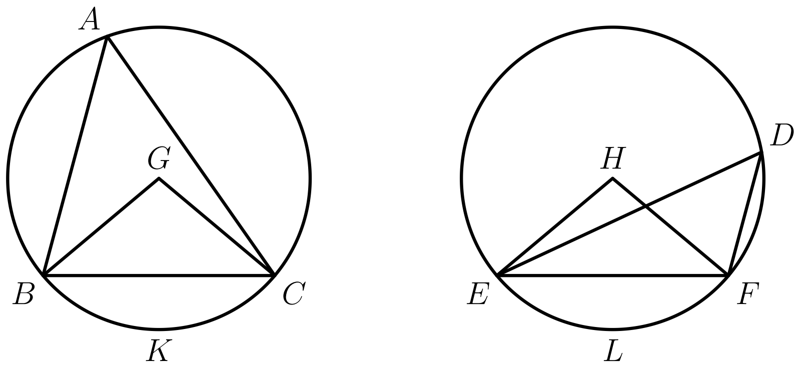
Therefore, given a segment of a circle, the complete circle has been described.

Q.E.F.

Proposition 26

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let ABC, DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC, EHF, and at the circumferences the angles BAC, EDF; I say that the circumference BKC is equal to the circumference ELF.



For let BC, EF be joined.

Now, since the circles ABC, DEF are equal, the radii are equal.

Thus the two straight lines BG, GC are equal to the two straight lines EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal to the base EF [I. 4]. And, since the angle at A is equal to the angle at D, the segment BAC is similar to the segment EDF [III. Def. 11]; and there are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another [III. 24]; therefore the segment BAC is equal to EDF.

But the whole circle ABC is also equal to the whole circle DEF; therefore the circumference BKC which remains is equal to the circumference ELF.

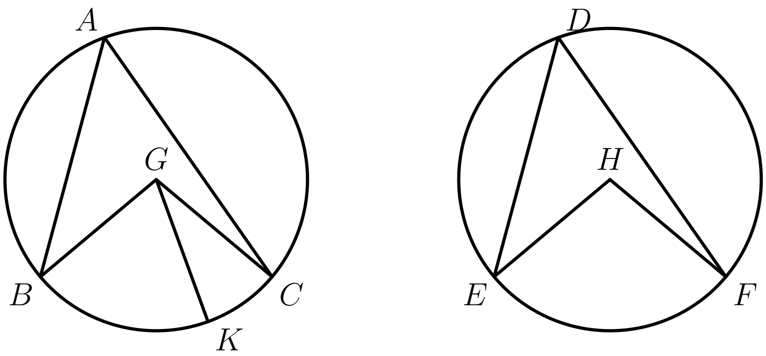
Therefore etc.

Q.E.D.

Proposition 27

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles ABC, DEF, on equal circumferences BC, EF, let the angles BGC, EHF stand at the centres G, H, and the angles BAC, EDF at the circumferences; I say that the angle BGC is equal to the angle EHF, and the angle BAC is equal to the angle EDF.



For, if the angle BGC is unequal to the angle EHF, one of them is greater.

Let the angle BGC be greater: and on the straight line BG, and at the point G on it, let the angle BGK be constructed equal to the angle EHF.

Now equal angles stand on equal circumferences, when they are at the centres [III. 26]; therefore the circumference BK is equal to the circumference EF.

But EF is equal to BC; Therefore BK is also equal to BC, the less to the greater: which is impossible.

Therefore the angle BGC is not unequal to the angle EHF; therefore it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF [III. 20]; therefore the angle at A is also equal to the angle at D.

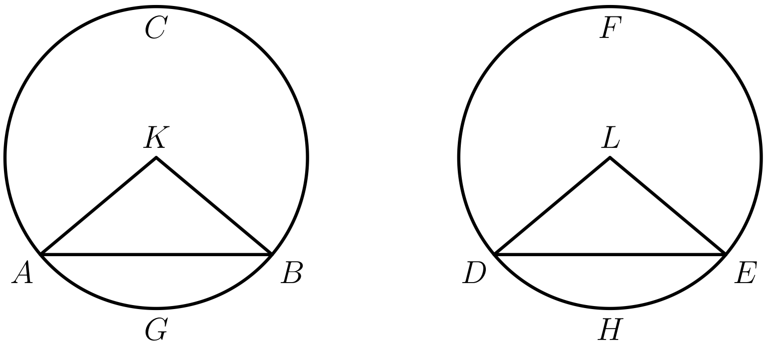
Therefore etc.

Q.E.D.

PROPOSITION 28

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC, DEF be equal circles, and in the circles let AB, DE be equal straight lines cutting off ACB, DFE as greater circumferences and AGB, DHE as lesser; I say that the greater circumference ACB is equal to the greater circumference DFE, and the less circumference AGB to DHE.



For let the centres K, L of the circles be taken, and let AK, KE, DL, LE be joined.

Now, since the circles are equal, the radii are also equal; therefore the two sides AK, KB are equal to the two sides DL, LE; and the base AB is equal to the base DE; therefore the angle AKB is equal to the angle DLE [I. 8].

But equal angles stand on equal circumferences, when they are at the centres [III. 26]; therefore the circumference AGB is equal to DHE.

And the whole circle ABC is also equal to the whole circle DEF; therefore the circumference ACB which remains is also equal to the circumference DFE which remains.

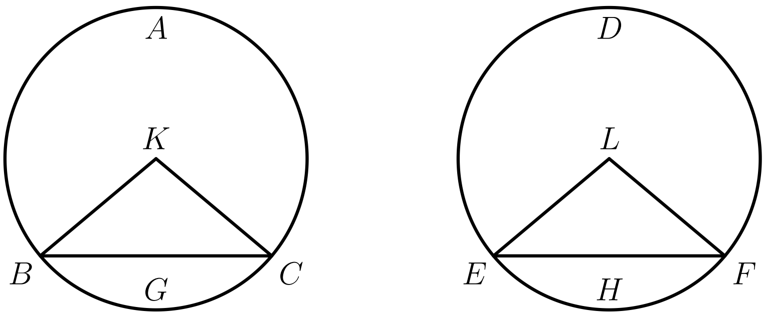
Therefore etc.

Q.E.D.

PROPOSITION 29

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and in them let equal circumferences BGC, EHF be cut off; and let the straight lines BC, EF be joined; I say BC is equal to EF.



For let the centres of the circles be taken, and let them be K, L; let BK, KC, EL, LF be joined.

Now, since the circumference BGC is equal to the circumference EHF, the angle BKC is also equal to the angle ELF. III. 27

And, since the circles ABC, DEF are equal, the radii are also equal; therefore the two sides BK, KC are equal to the two sides EL, LF; and they contain equal angles; therefore the base BC is equal to the base EF [I. 4].

Therefore etc.

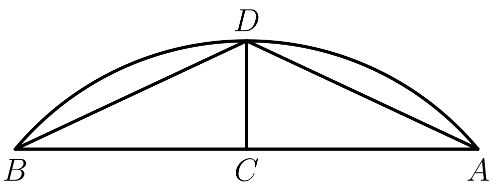
Q.E.D.

Proposition 30

To bisect a given circumference.

Let ADB be a given circumference; thus it is required to bisect the circumference ADB.

Let AB be joined and bisected at C; from the point C let CD be drawn at right angles to the straight line AB, and let AD, DB be joined.



Then, since AC is equal to CB, and CD is common, the two sides AC, CD are equal to the two sides BC, CD; and the angle ACD is equal to the angle BCD, for each is right; therefore the base AD is equal to the base DB [I. 4].

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less [III. 28]; and each of the circumferences AD, DB is less than a semicircle; therefore the circumference AD is equal to the circumference DB.

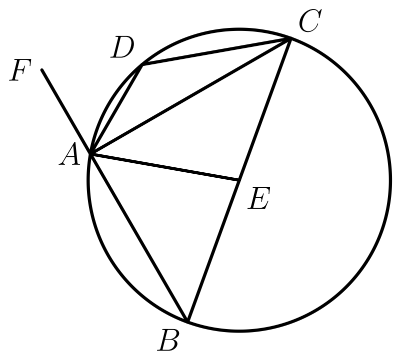
Therefore the given circumference has been bisected at the point D.

Q.E.F.

Proposition 31

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Let $ABCD$ be a circle, let BC be its diameter, and E its centre, and let BA, AC, AD, DC be joined; I say that the angle BAC in the semicircle is right, the angle in the segment ABC greater than the semicircle is less than a right angle, and the angle ADC in the segment ADC less than the semicircle is greater than a right angle.



Let AE be joined, and let BA be carried through to F.

Then, since BE is equal to EA, the angle ABE is aso equal to the angle BAE [I. 5]. Again, since CE is equal to EA, the angle ACE is also equal to the angle CAE [I. 5]. Therefore the whole angle BAC is equal to the two angles ABC, ACB. But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC, ACB [I. 32]; therefore the angle BAC is also equal to the angle FAC; therefore each is right; therefore the angle BAC inthe semicircle BAC is right.

Next, since in the triangle ABC the two angles ABC, BAC are less than two right angles, and the angle BAC is a right angle, the angle ABC is less than a right angle; and its is the angle in the segment ABC greater than the semicircle.

Next, sicne $ABCD$ is a quadrilateral in a circle, and the opposite angles of quadrilaterals in circles are equal to two right angles [III, 22], while the angle ABC is less than a right angle, therefore the angle ADC which remains is greater than a right angle; and it is the angle in the segment ADC less than the semicircle.

I say further than the angle of the greater segment, namely that contained by the circumference ABC and the straight line AC, is greater than a right angle; and the angle of the less segment, namely that contained by the circumference ADC and the straight line AC, is less than a right angle.

This is at once manifest.

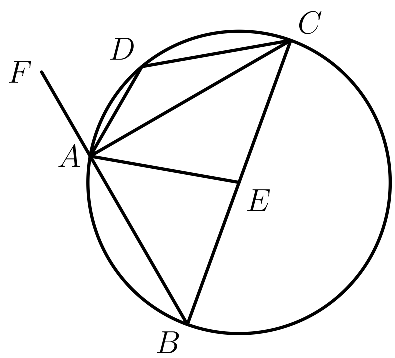
For, since the angle contained by the straight lines BA, AC is right, the angle contained by the circumference ABC and the straight line AC is greater than a right angle. Again, sicne the angle contained by the straight lines AC, AF is right, the angle contained by the straight line CA and the circumference ADC is less than a right angle.

Therefore etc.

Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK III, PROPOSITION 31

Euclid first proves that an angle in a semicircle is a right angle. In the figure accompanying the proof, the circular arc BAC and the diameter $[BC]$ together bound a semicircle, and, in accordance with the definitions introducing Book III, the angle $\angle BAC$ represents the angle in that semicircle. Euclid proves that this angle $\angle BAC$ is a right angle.



The proof may be presented more symbolically as follows. Let E be the centre of the circle. The triangles $\triangle EAB$ and $\triangle EAC$ are isosceles triangles, because $|EA| = |EB| = |EC|$. It follows that $|\angle EAB| = |\angle EBA|$ and $|\angle EAC| = |\angle ECA|$ (*Elements*, I, 5). Therefore

$$\begin{aligned} |\angle BAC| &= |\angle EAB| + |\angle EAC| = |\angle EBA| + |\angle ECA| \\ &= |\angle CBA| + |\angle BCA|. \end{aligned}$$

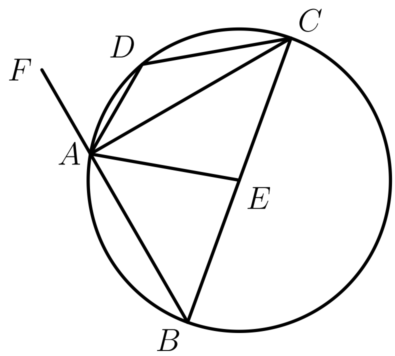
Also

$$|\angle FAC| = |\angle CBA| + |\angle BCA|,$$

because $\angle FAC$ is an exterior angle of the triangle $\triangle ABC$ at A and is therefore equal to the sum of the interior and opposite (or remote) angles of this triangle that are located at the vertices B and C (*Elements*, I, 32). It follows that $|\angle BAC| = |\angle FAC|$. Thus the two supplementary angles $\angle BAC$ and $|\angle FAC|$ are equal, and therefore (by definition of right angles), each of them is equal to a right angle.

Euclid has now proved that "the angle in a semicircle is right". Using the same diagram, one can also see that "in a greater segment" than a semicircle is less than a right angle and an angle "in a less segment" than a semicircle is greater than a right angle.

Indeed the line segment $[AC]$ together with the circular arc CBA together constitute the boundary of a segment. This segment is "greater than a semicircle" because a semicircle can be fitted within it, and the figure is applicable to any configuration involving a segment greater than a semicircle. The rectilinear angle $\angle ABC$ is, by definition, the angle in this segment. Now $\angle BAC$ has been shown to be a right angle and the sum of the angles $\angle BAC$ and $\angle ABB$ must be less than two right angles (*Elements*, I, 17). It follows that $\angle ABC$ must be less than a right angle. Thus the angle in a segment greater than a semicircle has been shown to be less than a right angle.



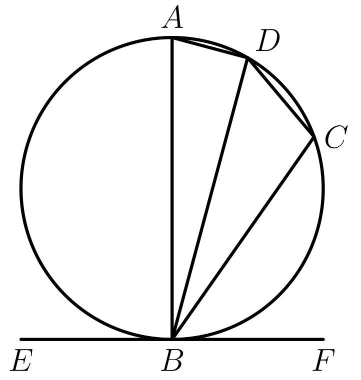
Next we consider the angle $\angle ADC$ located at a point D that lies on the other side of the line AC from the point B. The line segment $[AC]$ together with the circular CDA together constitute the boundary of a segment. This segment is less than a semicircle, because it fits within the semicircle $BCDA$. Moreover the figure is applicable to any configuration involving a segment less than a semicircle. Now the sum of the angles $\angle ADC$ and $\angle ABC$ is equal to two right angles, because $ADCB$ is a cyclic quadrilateral (*Elements*, III, 22). But $\angle ABC$ is less than a right angle. It follows that the angle $\angle ADC$ in the segment ACD must be more than a right angle. Thus the angle in a segment less than a semicircle has been shown to be greater than a right angle.

In this proposition Euclid also considers the "angle of the greater segment". This is the angle formed by the line segment $[AC]$ and the circular arc ABC at A. It is not a rectilinear angle. From the figure we see that it is greater than the right angle $\angle BAC$. Euclid considers also the "angle of the less segment". This is the angle formed by the line segment $[AC]$ and the circular arc ADC at A. From the figure we see that it is less than the right angle $\angle FAC$.

Proposition 32

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

For let a straight line EF touch the circle $ABCD$ at the point B, and from the point B let there be drawn across, in the circle $ABCD$, a straight line BD cutting it; I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle, that is, that the angle FBD is equal to the angle constructed in the segment BAD, and the angle EBD is equal to the angle constructed in the segment DCB.



For let BA be drawn from B at right angles to EF, let a point C be taken at random on the circumference BD, and let AD, DC, CB be joined.

Then, since a straight line EF touches the circle $ABCD$ at B, and BA has been drawn from the point of contact at right angles to the tangent, the centre of the circle $ABCD$ is on BA [III. 19]. Therefore BA is a diameter of the circle $ABCD$; therefore the angle ADB, being an angle in a semicircle, is right. [III. 31]. Therefore the remaining angles BAD, ABD, are equal to one right angle. [I. 32]. But the angle ABF is also right; therefore the angle ABF is equal to the angles BAD, ABD. Let the angle ABD be subtracted from each; therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle.

Next, since $ABCD$ is a quadrilateral in a circle, its opposite angles are equal to two right angles [III. 22]. But the angles DBF, DBE are also equal to two right angles; therefore the angles DBF, DBE are equal to the angles BAD, BCD, of which the angle BAD was proved equal to the angle DBF; therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle.

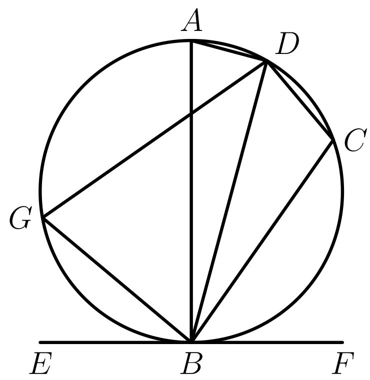
Therefore etc.

Q.E.D.

Note (DRW) on Euclid's *Elements*, Book III, Proposition 32

The line EF in the figure is tangent to the circle, touching the circle at the point B. A line segment $[BA]$ is taken perpendicular to the line EF at the endpoint B that meets the circle again at the other endpoint A. This line segment $[AB]$ then passes through the centre of the circle (*Elements*, III, 17), and is thus a diameter of the circle.

Let the point B of contact be the endpoint of a chord $[BD]$. It is required to show that the angle $\angle DBF$ is equal to the angle in the semicircle BAD bounded by the arc BAD and the chord $[BD]$. Now the angle in this semicircle is by definition the angle $\angle BGD$ taken at any point G of the arc BAD distinct from the endpoints B and D.



Now all angles $\angle BGD$ taken at points G of the arc BAD are equal to one another (*Elements*, III, 21). It follows from this that $|\angle BGD| = |\angle BAD|$ for all points G distinct from B and D that lie on the arc BAD. Thus, in order to show that the angle $\angle DBF$ is equal to the angle in "the alternate segment" BAD, it suffices to show that the angles $\angle BAD$ and $\angle DBF$ are equal to one another.

Now the angle in a semicircle is a right angle (*Elements*, III, 31). Therefore $\angle ADB$ is a right angle, and therefore the sum of the angles $\angle BAD$ and $\angle ABD$ is also a right angle (*Elements*, III, 32). The angle $\angle ABF$ is also a right angle, by construction. Therefore

$$|\angle BAD| + |\angle ABD| = \text{one right angle} = |\angle DBF| + |\angle ABD|.$$

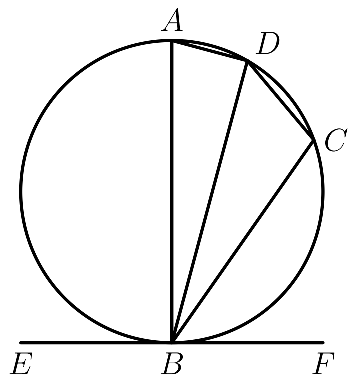
Subtracting the angle $\angle ABD$, we conclude that $|\angle BAD| = |\angle DBF|$. Thus the angle $\angle DBF$ made by the chord $[BD]$ and the tangent $[BF]$ is equal to the angle in the alternate segment BAD.

Now the sum of any two opposite angles of the cyclic quadrilateral $BADC$ is equal to two right angles, and so

$$|\angle BAD| + |\angle DCB| = \text{two right angles } ((\textit{Elements}, \text{III}, 22)).$$

Also $\angle DBF$ and $\angle DBE$ are supplementary angles, and therefore

$$|\angle DBF| + |\angle DBE| = \text{two right angles } ((\textit{Elements}, \text{I}, 13)).$$



Thus

$$|\angle DBF| + |\angle DBE| = |\angle BAD| + |\angle DCB|.$$

We have already shown that

$$|\angle DBF| = |\angle BAD|.$$

It follows that

$$|\angle DBE| = |\angle DCB|.$$

The proposition follows.

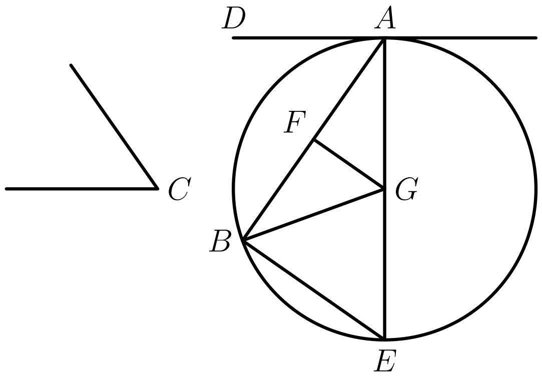
Proposition 33

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilinear angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle; thus it is required to describe on the given straight line AB a segment of a circle admitting an angle equal to the angle at C.

The angle at C is then acute, or right, or obtuse.

First let it be acute, and, as in the first figure, on the straight line AB, and at the point A, let the angle BAD be constructed equal to the angle at C; therefore the angle BAD is also acute. Let AE be drawn at right angles to DA, let AB be bisected at F, let FG be drawn from the point F at right angles to AB, and let GB be joined.

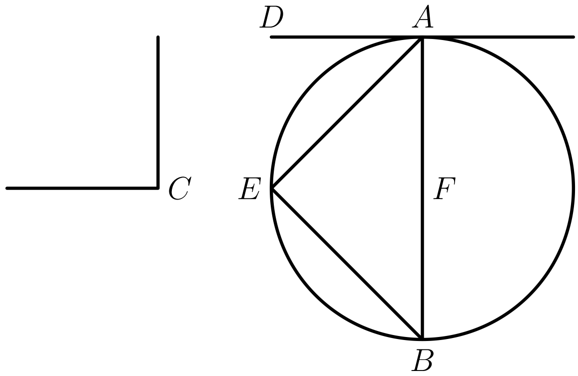


Then, since AF is equal to FB, and FG is common, the two sides AF, FG are equal to the two sides BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base BG [i. 4]. Therefore the circle described with centre G and distance GA will pass through B also. Let it be drawn, and let it be ABE; let EB be joined.

Now, since AD is drawn from A, the extremity of the diameter AE, at right angles to AE [iii. 16, Por.]. Since then a straight line AD touches the circle ABE, and from the point of contact at A a straight line AB is drawn across in the circle ABE, the angle DAB is equal to the angle AEB in the alternate segment of the circle [iii. 32]. But the angle DAB is equal to the angle at C; therefore the angle at C is also equal to the angle AEB.

Therefore on the given straight line AB the segment AEB of a circle has been described admitting the angle AEB equal to the given angle, the angle at C.

Next let the angle at C be right; and let it be again be required to describe on AB a segment of a circle admitting an angle equal to the right angle at C. Let the angle BAD be constructed equal to the right angle at C, as is the case in the second figure; Let AB be bisected at F, and with centre F and distance either FA or FB let the circle AEB be described.

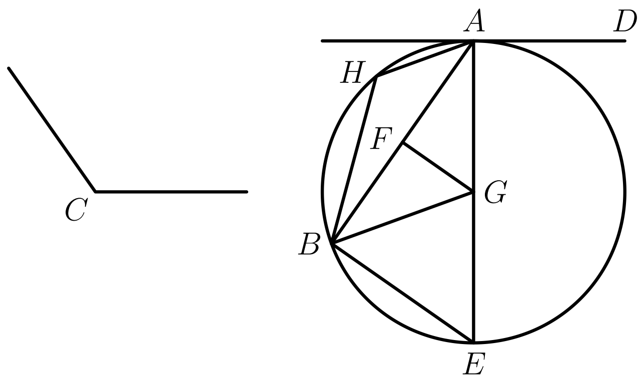


Therefore the straight line AD touches the circle ABE, because the angle at A is right [III. 16, Por]. And the angle BAD is equal to the angle in the segment AEB, for the latter too is itself a right angle, being an angle in a semicircle [III. 31]. But the angle BAD is also equal to the angle at C. Therefore the angle AEB is also equal to the angle at C.

Therefore again the segment AEB of a circle has been described on AB admitting an angle equal to the angle at C.

Next, let the angle at C be obtuse; and on the straight line AB, and at the point A, let the angle BAD be constructed equal to it, as in the case in the third figure; let AE be drawn at right angles to AD, let AB be again bisected at F, let FG be drawn at right angles to AB, and let GB be joined.

Then, since AF is again equal to FB; and FG is common, the two sides AF, FG are equal to the two sides BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base BG [I. 4]. Therefore the circle described with centre G and distance GA will pass through B also; let it so pass, as in AEB.



Now, since AD is drawn at right angles to the diameter AE from its extremity, AD touches the circle AEB [III. 16, Por.]. And AB has been drawn across from the point of contact at A; therefore the angle BAD is equal to the angle constructed in the alternate segment AHB of the circle [III. 32]. But the angle BAD is equal to the angle at C. Therefore the angle in the segment AHB is also equal to the angle at C.

Therefore on the given straight line AB, the segment AHB of a circle has been described admitting an angle equal to the angle at C.

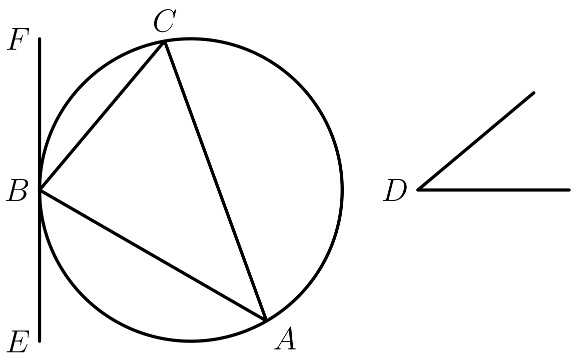
Q.E.F.

Proposition 34

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

Let ABC be the given circle, and the angle at D the given rectilineal angle; thus it is required to cut off from the circle ABC a segment admitting an angle equal to the given rectilineal angle, the angle at D.

Let EF be drawn touching ABC at the point B, and on the straight line FB, and at the point B on it, let the angle FBC be constructed equal to the angle at D [I. 23].



Then, since a straight line EF touches the circle ABC, and BC has been drawn across from the point of contact at B, the angle FBC is equal to the angle constructed in the alternate segment BAC [III. 32].

But the angle FBC is equal to the angle at D; therefore the angle in the segment BAC is equal to the angle at D.

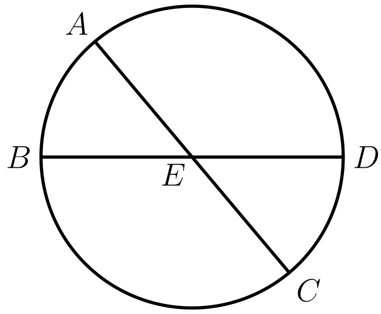
Therefore from the given circle ABC the segment ABC has been cut off admitting an angle equal to the given rectlineal angle, the angle at D.

Q.E.F.

PROPOSITION 35

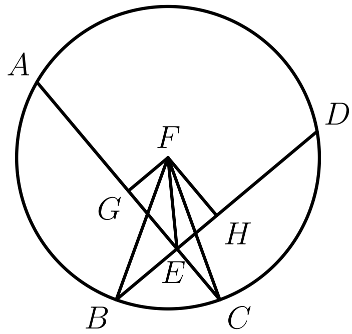
If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle $ABCD$ let the two straight lines AC, BD cut one another at the point E; I say that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.



If now AC, BD are through the centre, so that E is the centre of the circle $ABCD$, it is manifest that, AE, EC, DE, EB being equal, the rectangle contained by AE, EC is also equal to the rectangle contained by DE, EB.

Next let AC, DB not be through the centre; let the centre of $ABCD$ be taken, and let it be F; from F let FG, FH be drawn perpendicular to the straight lines AC, DB, and let FB, FC, FE be joined.



Then, since a straight line GF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AG is equal to GC. Since, then, the straight line AC has been cut into equal parts at G and into unequal parts at E, the rectangle contained by AE, EC together with the square on EG is equal to the square on GC [II. 5]. Let the square on GF be added; therefore the rectangle AE, EC together with the squares on GE, GF is equal to the squares on CG, GF.

But the square on FE is equal to the squares on EG, GF, and the square on FC is equal to the squares on CG, GF [I. 47]; therefore the rectangle AE, EC together with the square on FE is equal to the square on FC. And FC is equal to FB; therefore the rectangle AE, EC together with the square on EF is equal to the square on FB.

For the same reason, also, the rectangle DE, EB together with the square on FE is equal to the square on FB. But the rectangle AE, EC together with the square on FE was also proved equal to the square on FB; therefore the rectangle AE, EC together with the square on FE is equal to the rectangle DE, EB together with the square on FE. Let the square on FE be subtracted from each; therefore the rectangle contained by AE, EC which remains is equal to the rectangle contained by DE, EB.

Therefore etc.

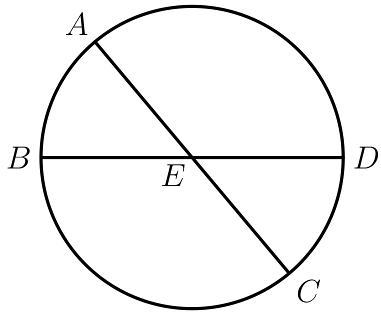
Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK III, PROPOSITION 35

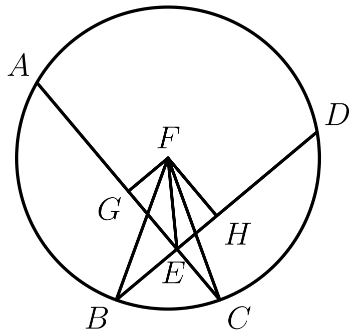
Let A, B, C and D be points that lie on a given circle, and suppose that the line segments $[AC]$ and $[BD]$ both intersect at a point E. The proposition asserts that

$$|AE| \times |EC| = |DE| \times |EB|.$$

The result is clear in the case when the intersection point E is located at the centre of the circle, because in that case $[AE]$, $[BE]$, $[CE]$ and $[DE]$ are all radii of the circle.



It remains to consider the case then the point E where $[AC]$ and $[BD]$ intersect is not located at the centre of the circle. In this case let perpendiculars $[FG]$ and $[FH]$ be drawn from the centre F of the given circle to the line segments $[AC]$ and $[BD]$, meeting those line segments at the points G and H respectively (*Elements*, I, 12). Then the point G bisects the line line segment $[AC]$, and the point H bisects the line segment $[BD]$ (*Elements*, III, 3). Let the points B, C and E all be joined to the centre F of the circle, as shown in the figure below.

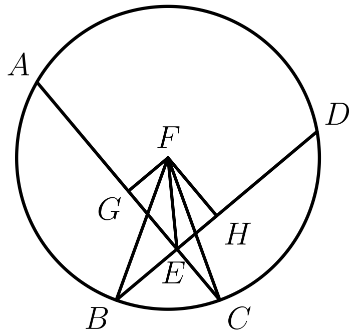


Now the line segment $[AC]$ is cut into equal parts at the point G and into unequal parts at the point E. It follows that

$$|AE| \times |EC| + |EG|^2 = |GC|^2 \quad (\textit{Elements}, \text{II}, 5).$$

This can be verified algebraically. Indeed if $|GC| = u$ and $|EG| = v$ then $|AE| = u + v$ and $|EC| = u - v$, and therefore

$$|AE| \times |EC| + |EG|^2 = (u + v)(u - v) + v^2 = u^2 = |GC|^2.$$



Now Pythagoras's Theorem (*Elements*, I, 47) ensures that

$$|EG|^2 + |GF|^2 = |FE|^2 \quad \text{and} \quad |CG|^2 + |GF|^2 = |FC|^2.$$

It follows that

$$\begin{aligned} |AE| \times |EC| + |FE|^2 &= |AE| \times |EC| + |EG|^2 + |GF|^2 \\ &= |GC|^2 + |GF|^2 = |FC|^2. \end{aligned}$$

Similarly

$$|DE| \times |EB| + |FE|^2 = |FB|^2.$$

But $|FB| = |FC|$, because $[FB]$ and $[FC]$ are both radii of the given circle. Therefore

$$|AE| \times |EC| + |FE|^2 = |DE| \times |EB| + |FE|^2.$$

Subtracting $|FE|^2$ from both sides, we find that

$$|AE| \times |EC| = |DE| \times |EB|,$$

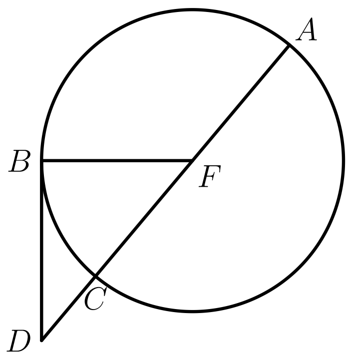
as required.

Proposition 36

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

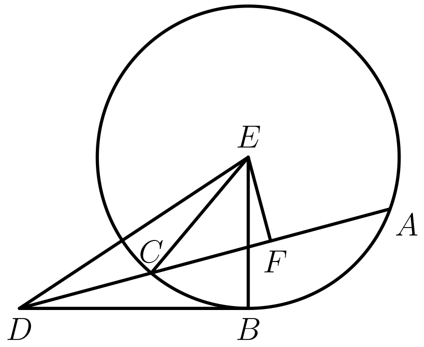
For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let BD touch it; I say that the rectangle contained by AD, DC is equal to the square on DB.

Then DCA is either through the centre or not through the centre.



First let it be through the centre, and let F be the centre of the circle ABC; let FB be joined; therefore the angle FBD is right [III. 18]. And, since AC has been bisected at F, and CD is added to it, the rectangle AD, DC together with the square on FC is equal to the square on FD [II. 6]. But FC is equal to FB; therefore the rectangle AD, DC together with the square on FB is equal to the square on FD. And the squares on FB, BD are equal to the square on FD [I. 47]; therefore the rectangle AC, DC together with the square on FB is equal to the squares on FB, BD. Let the square FB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on the tangent DB.

Again, let DCA not be through the centre of the circle ABC; let the centre E be taken, and from E let EF be drawn perpendicular to AC; let EB, EC, ED be joined.



Then the angle EBD is right [III. 18]. And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AF is equal to FC.

Now, since the straight line AC has been bisected at the point F, and CD is added to it, the rectangle contained by AD, DC together with the square on FC is equal to the square on FD [II. 6]. Let the square on FE be added to each; therefore the rectangle AD, DC together with the squares on CF, FE is equal to the squares on FD, FE. But the square on EC is equal to the squares on CF, FE, for the angle EFC is right [I. 47]; and the square on ED is equal to the squares on DF, FE; therefore the rectangle AD, DC together with the square on EC is equal to the square on ED. And EC is equal to EB; therefore the rectangle AD, DC together with the square on EB is equal to the square on ED. But the squares on EB, BD are equal to the square on ED, for the angle EBD is right [I. 47]; therefore the rectangle AD, DC together with the square on EB is equal to the squares on EB, BD. Let the square on EB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on DB.

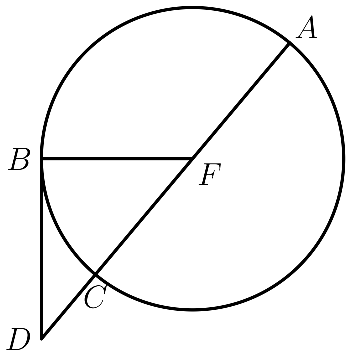
Therefore etc.

Q.E.D.

The conditions of the proposition specify that a point D is taken outside a given circle, and that a line segment $[DA]$ is drawn from D to a point A of the circle, cutting the circle in two points A and C, where C lies between D and A. Another line segment $[DB]$ is drawn from D to a point B of the circle, touching the circle at a point B. (The line DB is then a tangent line to the circle at B.) The proposition asserts that, in this situation,

$$|AD| \times |DC| = |DB|^2.$$

Euclid first considers the case in which the line segment $[DA]$ passes through the centre of the circle.



Let F denote the point at the centre of the given circle. Then F bisects the line segment $[AC]$. Proposition 6 of Book II of Euclid's *Elements* ensures that, in symbolical notation,

$$|AD| \times |DC| + |FC|^2 = |FD|^2.$$

To verify this algebraically, let $|FC| = u$ and $|DC| = v$. Then $|AD| = 2u+v$ and $|FD| = u + v$, and therefore

$$|AD| \times |DC| + |FC|^2 = (2u + v)v + u^2 = (u + v)^2 = |FD|^2.$$

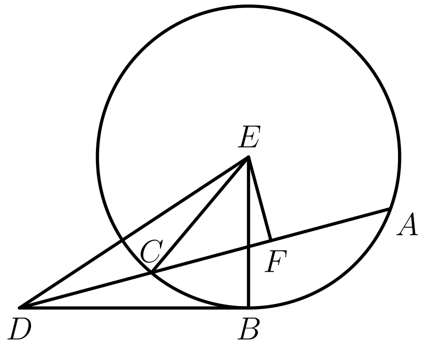
Also $\angle DBF$ is a right angle (*Elements*, III, 18). Pythagoras's Theorem (*Elements*, I, 47) therefore ensures that

$$|FD|^2 = |DB|^2 + |FB|^2.$$

Now $|FC| = |FB|$, because $[FC]$ and $[FB]$ are both radii of the given circle. It follows that, in this case

$$|AD| \times |DC| = |DB|^2,$$

Having completed discussion of the case where the line from D passes through the centre of the circle, Euclid considers the case of a line segment from the point D, cutting the circle at the points A and C, which does not pass through the centre of the circle.



The line segment $[AC]$ is bisected at the point F, and the points F and C are joined to the centre E of the circle. Because $|FC| = |FA|$, Proposition 6 of Book II of Euclid's *Elements* ensures that, in symbolical notation,

$$|AD| \times |DC| + |FC|^2 = |FD|^2.$$

The angle $\angle DFE$ is a right angle (*Elements*, III, 3). Therefore, making several applications of Pythagoras's Theorem (*Elements*, I, 47), we find that

$$\begin{aligned} |DE|^2 &= |FD|^2 + |FE|^2 \quad (\textit{Elements}, \text{I}, 47) \\ &= |AD| \times |DC| + |FC|^2 + |FE|^2 \quad (\textit{Elements}, \text{II}, 6) \\ &= |AD| \times |DC| + |CE|^2 \quad (\textit{Elements}, \text{I}, 47). \end{aligned}$$

Moreover the line DB is tangent to the circle, and therefore $\angle DBE$ is a right angle (*Elements*, III, 18). Therefore

$$\begin{aligned} |DE|^2 &= |DB|^2 + |BE|^2 \quad (\textit{Elements}, \text{I}, 47) \\ &= |DB|^2 + |CE|^2 \quad (\text{because } |BE| = |CE|) \end{aligned}$$

Thus

$$|AD| \times |DC| + |CE|^2 = |DE|^2 = |DB|^2 + |CE|^2,$$

Subtracting $|CE|^2$ from both sides, it follows that

$$|AD| \times |DC| = |DB|^2,$$

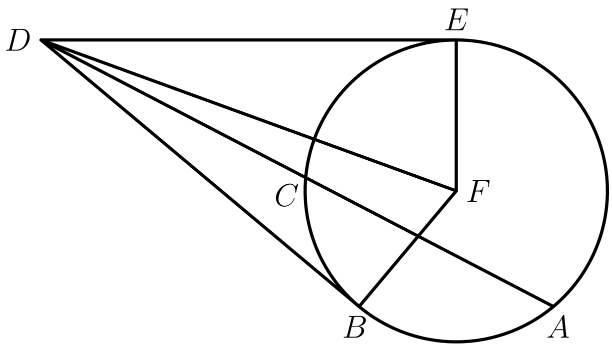
as required.

Proposition 37

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which fall on it will touch the circle.

For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let DB fall on it; and let the rectangle AD, DC be equal to the square on DB.

I say that DB touches the circle ABC.



For let DE be drawn touching ABC; let the centre of the circle ABC be taken, and let it be F; let FE, FB, FD be joined. Thus the angle FED is right [III. 18]. Now, since DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE [III. 36] But the rectangle AD, DC was also equal to the square on DB; therefore the square on DE is equal to the square on DB; therefore DE is equal to DB. And FE is equal to FB; therefore the two sides DE, EF are equal to the two sides DB, BF; and FD is the common base of the triangles; therefore the angle DEF is equal to the angle DBF [I. 8]. But the angle DEF is right; therefore the angle DBF is also right. And FB produced is a diameter; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle [III. 16, Por]; therefore DB touches the circle.

Similarly this can be proved to be the case even if the centre be on AC.

Therefore etc.

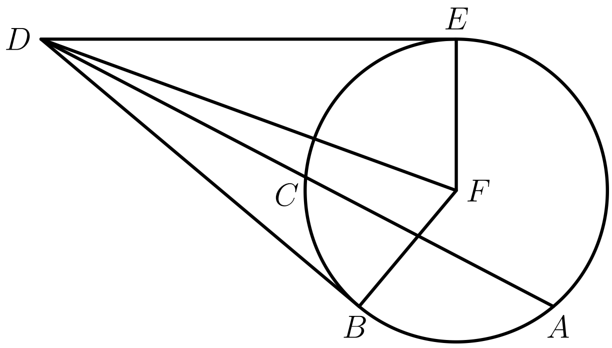
Q.E.D.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK III, PROPOSITION 37

The conditions of the proposition specify that a point D is taken outside a given circle, and that a line segment $[DA]$ is drawn from D to a point A of the circle, cutting the circle in two points A and C, where C lies between D and A. Another line segment $[DB]$ is drawn from D to a point B of the circle. The proposition asserts that if

$$|AD| \times |DC| = |DB|^2$$

then the line segment $[DB]$ touches the given circle at the point B (and is thus tangent to the circle at B).



Now a point E can be found on the given circle such that the line segment $[DE]$ touches the circle at E (*Elements*, III, 17). The points B, D and E are then joined to the centre F of the circle. Because $[DE]$ touches the circle at E the angle $\angle DEF$ is a right angle (*Elements*, III, 18). It follows that

$$|AD| \times |DC| = |DE|^2 \quad (\textit{Elements}, \text{III}, 36).$$

But $|AD| \times |DC| = |DB|^2$. Therefore $|DE| = |DB|$.

Now $|FE| = |FB|$, because the points B and E both lie on the given circle centred on the point F. Therefore the three sides of the triangle $\triangle DFB$ are respectively equal to the corresponding sides of the triangle $\triangle DFE$. The SSS Congruence Rule (*Elements*, I, 8) therefore guarantees that $\angle DBF = \angle DEF$. But $\angle DEF$ is a right angle, therefore $\angle DBF$ is also a right angle. Therefore the line segment $[DB]$ touches the given circle at the point B (*Elements*, III, 16), as required.

Definitions

1. A rectilineal figure is said to be **inscribed in a rectilineal figure** when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2. Similarly a figure is said to be **circumscribed about a figure** when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3. A rectilineal figure is said to be **inscribed in a circle** when each angle of the inscribed figure lies on the circumference of the circle. A rectilineal figure is said to be **circumscribed about a circle**, when each side of the circumscribed figure touches the circumference of the circle. Similarly a circle is said to be **inscribed in a figure** when the circumference of the circle touches each side of the figure in which it is circumscribed. A circle is said to be **circumscribed abut a figure** when the circumference of the circle passes through each angle of the figure about which it is circumscribed. A straight line is said to be **fitted into a circle** when its extremities are on the circumference of the circle.

Proposition 1

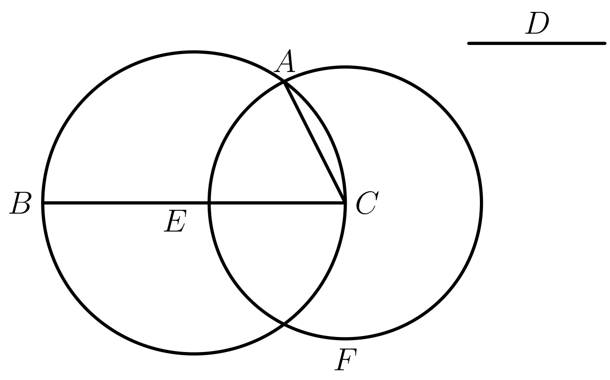
Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given line not greater than the diameter of the circle; thus it is required to fit into the circle ABC a straight line equal to the straight line D.

Let a diameter BC of the circle be drawn.

Then, if BC is equal to D, that which was enjoined will have been done; for BC has been fitted into the circle ABC equal to the straight line D.

But, if BC is greater than D, let CE be made equal to D, and with centre C and distance CE let the circle EF be described; let CA be joined.



Then, since the point C is the centre of the circle EAF, CA is equal to CE. But CE is equal to D; therefore D is also equal to CA.

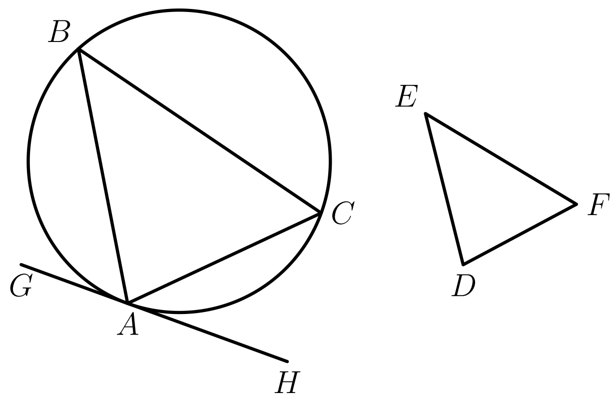
Therefore into the given circle ABC there has been fitted CA equal to the given straight line D.

Proposition 2

In a given circle to inscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to inscribe in the circle ABC a triangle equiangular with the triangle DEF.

Let GH be drawn touching the circle ABC at A [III. 16, Por.]; on the straight line AH, and at the point A on it, let the angle HAC be constructed equal to the angle DEF, and on the straight line AG, and at the point A on it, let the angle GAB be constructed equal to the angle DFE [I. 23]; let BC be joined.



Then, since a straight line AH touches the circle ABC, and from the point of contact at A the straight line AC is drawn across in the circle, therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle [III. 32]. But the angle HAC is equal to the angle DEF; therefore the angle ABC is also equal to the angle DEF. For the same reason the angle ACB is also equal to the angle DFE; therefore the remaining angle BAC is also equal to the remaining angle EDF.

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle.

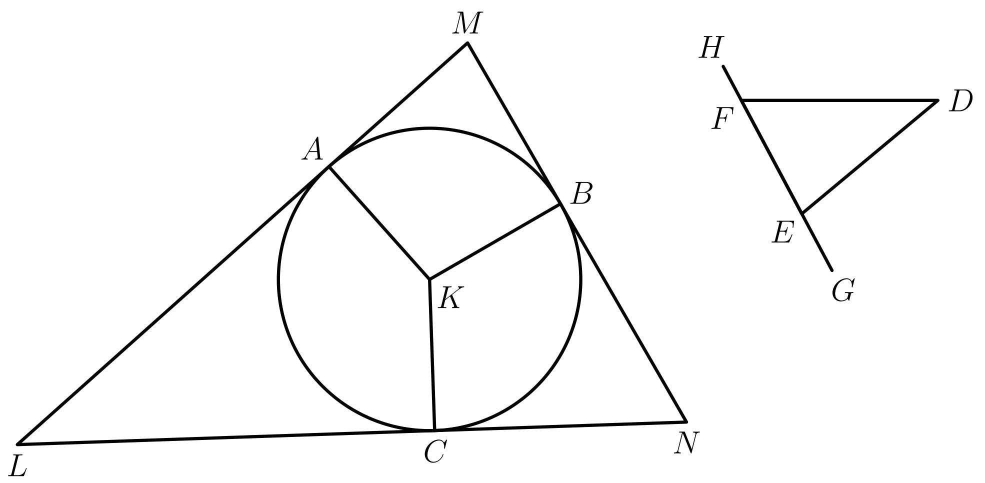
Q.E.F.

Proposition 3

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to circumscribe about the circle ABC a triangle equiangular with the triangle DEF.

Let EF be produced in both directions to the points G, H, let the centre K of the circle ABC be taken [iii. 1], and let the straight line KB be drawn across at random; on the straight line KB, and at the point K on it, let the angle BKA be constructed equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C let LAM, MBN, NCL be drawn touching the circle ABC iii. 16, Por..



Now, since LM, MN, NL touch the circle ABC at the points A, B, C, and KA, KB, KC have been joined from the centre K to the points A, B, C, therefore the angles at the points A, B, C are right [iii. 18]. And, since the four angles of the quadrilateral $AMBK$ are equal to four right angles, inasmuch as $AMBK$ is in fact divisible into two triangles, and the angles KAM, KBM are right; therefore the remaining angles AKB, AMB are equal to two right angles. But the angles DEG, DEF are also equal to two right angles. [i. 13]; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which the angle AKB is equal to the angle DEG; therefore the angle AMB which remains is equal to the angle DEF which remains.

Similarly it can be proved that the angle LNB is also equal to the angle DFE; therefore the remaining angle MLN is equal to the angle EDF.

Therefore the triangle LMN is equiangular with the triangle DEF; and it has been circumscribed about the circle ABC. Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

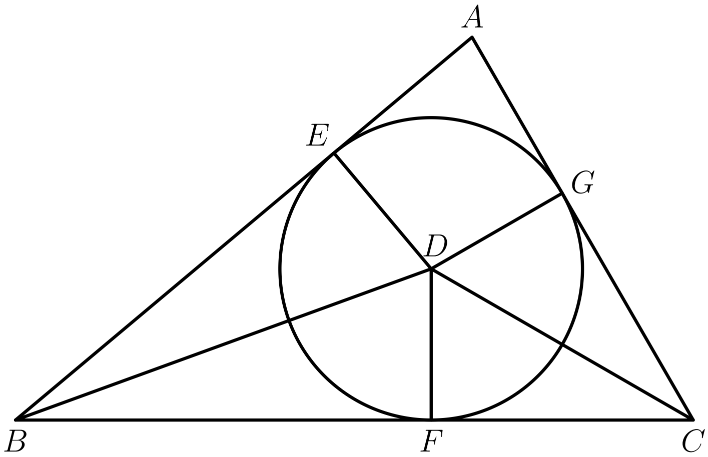
Q.E.F.

Proposition 4

In a given triangle to inscribe a circle.

Let ABC be the given triangle; thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB be bisected by the straight lines BD, CD [I. 9], and let these meet one another at the point D; from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA.



Now, since the angle ABD is equal to the angle CBD, and the right angle BED is also equal to the right angle BFD, EBD, FBD are two triangles having the two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles; therefore they will also have the remaining sides equal to the remaining sides; therefore DE is equal to DF.

For the same reason DG is also equal to DF. Therefore the three straight lines DE, DF, DG are equal to one another; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right. For if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle: which was proved absurd [III. 16]; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA; therefore it will touch them, and will be the circle inscribed in the triangle ABC [IV. Def. 5]. Let it be inscribed, as FGE. Therefore, in the given triangle ABC the circle EFG has been inscribed.

Q.E.F.

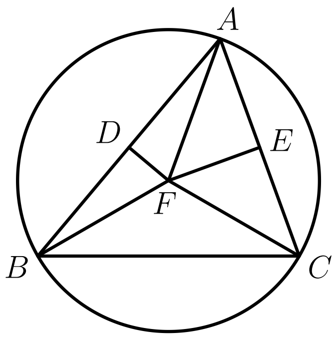
Proposition 5

About a given triangle to circumscribe a circle.

Let ABC be the given triangle; thus it is required to circumscribe a circle about the given triangle ABC.

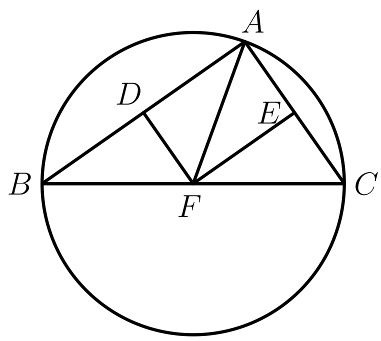
Let the straight lines AB, AC be bisected at the points D, E [I. 10], and from the points D, E let DE, DF be drawn at right angles to AB, AC; they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.



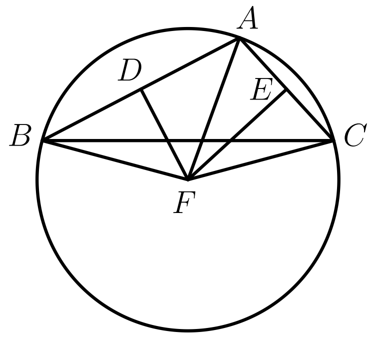
Then, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base FB [I. 4]. Similarly we can prove that CF is also equal to AF; so that FB is also equal to FC; therefore the three straight lines FA, FB, FC are equal to one another, Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC. Let it be circumscribed, as ABC.

Next, let DE, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.



Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.



Then again, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base BF [I. 4]. Similarly we can prove that CF is also equal to AF; so that BF is also equal to FC; therefore the circle described with centre F and distance on of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

Therefore about the given triangle a circle has been circumscribed.

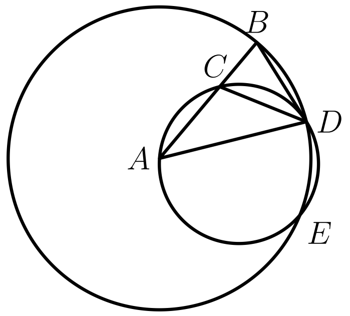
Q.E.F.

And it is manifest that, when the centre of the circle falls within the triange, the angle BAC, being in a segment greater than the semicircle, is less than a right angle; when the centre falls on the straight line BC, the angle BAC, being in a semicircle, is right; and when the centre of the circle falls outseide the triangle, the angle BAC, being in a segment less than a semicircle, is greater than a right angle [III. 31].

Proposition 10

To construct an isosceles triangle havng each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA [II. 11]; with centre A and distance AB let the circle BDE be described, and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE [IV. 1]. Let AD, DC be joined, and let the circle ACD be circumscribed about the triangle ACD [IV. 5].



Then, since the rectangle AB, BC is equal to the square on AC, and AC is equal to BD, Therefore the rectangle AB, BC is equal to the square on BD.

And, since a point B has been taken outside the circle ACD, and from B the two straight lines BA, BD have fallen on the circle ACD, and one of them cuts it, while the other falls on it, and the rectangle AB, BC is equal to the square on BD, therefore BD touches the circle ACD [III. 37]. Since, then, BD touches it, and DC is drawn across from the contact at D, therefore the angle BDC is equal to the angle DAC in the alternate segment of the circle [III. 32]. Since, then, the angle BDC is equal to the angle DAC, let the angle CDA be added to each; therefore the whole angle BDA is equal to the two angles CDA, DAC. But the exterior angle BCD is equal to the angles CDA, DAC; therefore the angle BDA is also equal to the angle BCD. But the angle BDA is equal to the angle CBD, since the side AD is also equal to AB [I. 5]; so that the angle DBA is also equal to the angle BCD. Therefore the three angles BDA, DBA, BCD are equal to one another. And, since the angle DBC is equal to the angle BCD, the side BD is also equal to the side DC [I. 6]. But BD is by hypothesis equal to CA; therefore CA is also equal to CD, so that the angle CDA is also equal to the angle DAC

[I. 5]; therefore the angles CDA, DAC are double of the angle DAC. But the angle BCD is equal to the angles CDA, DAC; therefore the angle BCD is also double of the angle CAD. But the angle BCD is equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is also double of the angle DAB. Therefore the isosceles triangle ABD has been constructed having each of the angles at the base DB double of the remaining one.

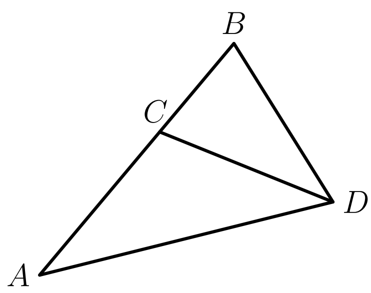
Q.E.F.

NOTE (DRW) ON EUCLID'S *Elements*, BOOK IV, PROPOSITION 10

This proposition describes and justifies the construction and basic properties of a so-called *golden triangle*. The golden triangle $\triangle ABD$ is an isosceles triangle, with equal sides $[AB]$ and $[AD]$, in which $[BD]$ is equal in length to $[AC]$, where C is the point on the side $[AB]$ determined so that

$$|AB| \times |BC| = |AC|^2.$$

Euclid proves that, in such a triangle, the two equal angles at vertices B and D are double the angle at the remaining vertex A.



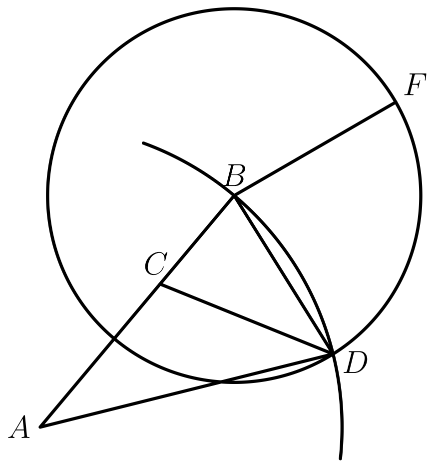
Now Book V of Euclid's *Elements of Geometry* develops a general theory of ratio and proportion, that can be used when comparing "magnitudes" of the same "species". When a point C cuts a line segment $[AB]$ so as to satisfy the condition stated above, the ratio of $|AB|$ to $|AC|$ is equal to the ratio of $|AC|$ to $|BC|$. Adopting a more "modern" approach to geometry, within which ratios of lengths of line segments could be represented by positive real numbers, one would write

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|BC|} = \varphi,$$

where $\varphi = \frac{1}{2}(\sqrt{5}+1) \approx 1.61803\ldots$. The use of the "modern" language of real numbers is of course an anachronism in the context of a discussion of Euclid's approach to synthetic geometry. And the language of ratio, similarity and proportion first appears in the *Elements* in Books V and VI. According to the definitions commencing Book VI, if a point C on a line segment $[AB]$ divides that line segment into two subsegments in accordance with the condition stated above, then the line segment $[AB]$ is said to be "cut in extreme and mean ratio". This is the case, according to Euclid, when "as the whole line is to the greater segment, so is the greater to the less" (T.L. Heath, *The Thirteen Books of Euclid's Elements*, Volume 2, p.188). From the nineteenth century onwards, the term *golden ratio* has come into common use to refer to Euclid's "extreme and mean ratio".

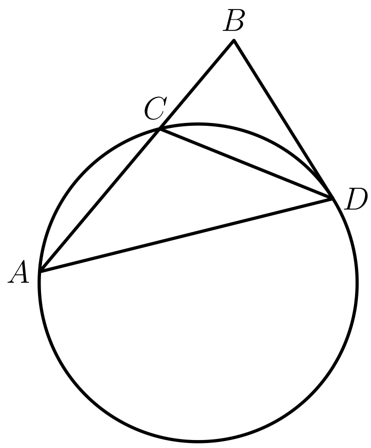
Euclid begins the discussion of Proposition 10 of Book IV by setting out the construction of the "golden triangle".

A circle may be drawn centred on the point A and passing through the point B. A point D may then be constructed on the circumference of this circle so that $[BD]$ is equal in length to $[AC]$ (*Elements*, IV, 1). This construction may be described in more detail as follows.



A point F can be constructed so as to ensure that the line segment $[BF]$ joining the point F to the point B is equal in length to the line segment $[AC]$ (*Elements*, I, 2). Then the circle centred on the point B and passing through the point F will intersect the circle centred on the point A and passing through the point B, because the line segment $[BF]$ is shorter than the line segment $[AC]$ and is thus shorter than a diameter of the circle centred on the point A and passing through the point B. Let D be a point of intersection of these two circles. Then $|AD| = |AB|$ and $|BD| = |BF| = |AC|$. This construction is a particular instance of that for constructing a triangle with sides equal in length to three given line segments, where those line segments satisfy the condition that the sum of any two of them is greater than the third (*Elements*, I, 22).

Having constructed the golden triangle $\triangle ABD$, together with the point C on the side $[AB]$ for which $|AB| = |AD|$, $|AC| = |BD|$ and $|AB| \times |BC| = |AC|^2$, it is necessary to show that the angles of this golden triangle at vertices B and D are double the angle of this triangle at the vertex A. An important step in the proof of this result is that of showing that the angles $\angle BAD$ and $\angle BDC$ are equal. Now the theory of ratio and proportion is the subject of Book V of Euclid's *Elements of Geometry*, and various applications of this theory to plane geometry, including the theory of similar triangles, are presented in Book VI of the *Elements*. Had those theories been available for use in proving the propositions contained in Book IV, then one could have argued that the two triangles $\triangle ABD$ and $\triangle DBC$ are similar, because $|\angle ABD| = |\angle DBC|$ and $|AB|$ is to $|BD|$ as $|DB|$ is to $|BC|$, and therefore the angle $\angle BAD$ of the first triangle at A is equal to the angle $\angle CDB$ of the second triangle at D (*Elements*, VI, 6). But, because the theory of ratio and proportion, and the theory of similar triangles founded on it, is the subject of the books following Book IV, Euclid presents an alternative proof of Proposition 10 of Book IV, founded on propositions established in Books I, II and III.



A circle can be circumscribed about the triangle $\triangle ACD$ (*Elements*, IV, 5). The point B lies outside this triangle, and moreover

$$|AB| \times |BC| = |AC|^2 = |BD|^2.$$

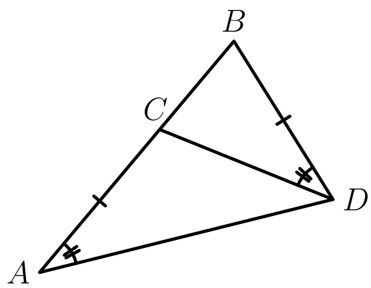
Therefore the line segment $[BD]$ touches the circle (i.e., is tangent to the circle) at the point D (*Elements*, III, 37). The angle $\angle BDC$ formed the tangent $[DB]$ and the chord $[DC]$ is therefore equal to the angle $\angle CAD$ in the alternate segment $DCAD$ cut off by the chord $[DC]$ (*Elements*, III, 32). Thus

$$|\angle BAD| = |\angle BDC|.$$

The proof of the result that the angles of the golden triangle $\triangle ABD$ at B and D are double the angle of this triangle at A can now be completed using well-known propositions from Book I of Euclid's *Elements of Geometry*.

The line segments and angles depicted in figure below satisfy the following conditions:

$$|AB| = |AD|, \quad |AC| = |BD|, \quad |\angle CAD| = |\angle CDB|.$$



Now the external angle $\triangle DCB$ of the triangle $\triangle ACD$ is equal to the sum of the two opposite (or remote) interior angles of this triangle at D and A (*Elements*, I, 32). It follows that

$$|\angle DCB| = |\angle ADC| + |\angle CAD| = |\angle ADC| + |\angle CDB| = |\angle ADB|.$$

But $|\angle ADB| = |\angle ABD|$, because $\triangle ADB$ is an isosceles triangle with equal sides $[AB]$ and $[AD]$ (*Elements*, I, 5). Now $\angle ABD = \angle CBD$, because the point C lies between the points A and B. Thus

$$|\angle DCB| = |\angle DBC| = |\angle BDA|.$$

It follows that $\triangle DCB$ is an isosceles triangle with

$$|DC| = |DB| = |AC|$$

(*Elements*, I, 6). But then $\triangle CAD$ is also an isosceles triangle, and therefore

$$|\angle CDB| = |\angle BAD| = |\angle CAD| = |\angle CDA|$$

(*Elements*, I, 5), and therefore

$$|\angle ABD| = |\angle ADB| = |\angle CDB| + |\angle CDA| = 2 \times |\angle BAD|.$$

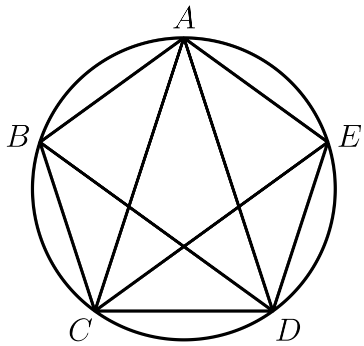
This completes the proof that the equal angles of the golden triangle $\triangle ABD$ at the vertices B and D are double the angle of that triangle at the vertex A.

Proposition 11

In a given circle to inscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle; thus it is required to inscribe in the circle $ABCDE$ an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G, H double of the angle at F [IV. 10]; let there be inscribed in the circle $ABCDE$ the triangle ACD equiangular with the triangle FGH, so that the angle CAD is equal to the angle at F, and the angles at G, H respectively equal to the angles ACD, CDA [IV. 2]; therefore each of the angles ACD, CDA is also double of the angle ACD. Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB [I. 9], and let AB, BC, DE, EA be joined.



Then, since each of the angles ACD, CDA is double of the angle CAD, and they have been bisected by the straight lines CE, DB, therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. But equal angles stand on equal circumferences [III. 26]; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another. But equal circumferences are subtended by equal straight lines [III. 29]; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; therefore the pentagon $ABCDE$ is equilateral.

I say next that it is also equiangular.

For since the circumference AB is equal to the circumference DE, let BCD be adde to each; therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$. And the angle AED stands on the circumference $ABCD$, and the angle BAE on the circumference $EDCB$; therefore the angle BAE is also equal to the angle AED [III. 27]. For the same reason each of the angles ABC, BCD, CDE is also equal to each of the angles BAE, AED; therefore the pentagon $ABCDE$ is equiangular. But it was also proved equilateral; therefore in a given circle an equilateral and equiangular pentagon has been inscribed.

Q.E.F.