MA232A—Euclidean and Non-Euclidean Geometry
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Section 6: Stereographic Projection

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6 Stereographic Projection

6.1 Stereographic Projection: A Coordinate Geometry Approach

Let a sphere in three-dimensional Euclidean space be given. A geometric construction known as *stereographic projection* gives rise to a one-to-one correspondence between the complement of a chosen point A on the sphere and the points of the plane Z through the centre C of that sphere perpendicular to the line AC. Specifically each point P on the sphere is mapped under stereographic projection to the point where the line PA intersects the plane Z.

Remark The ancient Greek mathematician Ptolemy wrote a work, the *Planisphere*, or *Planisphærium*, that describes stereographic projection and investigates its properties. No Greek text survives, but the work was translated into Arabic, and the work has survived through the medium of this Arabic translation.

For more information on Ptolemy's *Planiphere*, see the Wikipedia article on the *Planisphærium* at the following location:

`https://en.wikipedia.org/wiki/Planisphaerium`

A recent translation is the following:

> Nathan Sidoli and J.L. Berggren, *The Arabic version of Ptolemy's Planisphere or Flattening the Surface of the Sphere: Text, Translation, Commentary*, SCIAMVS 8 (2007), 37-139
> `http://individual.utoronto.ca/acephalous/Sidoli_Berggren_2007.pdf`

Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let (u, v, w) be a point of the unit sphere S^2 distinct from $(0, 0, -1)$. Then the unique line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ at the point (x, y) at which

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

It follows that stereographic projection from the point $(0, 0, -1)$ sends each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$ to the point $\psi(u, v, w)$

of $\mathbb{R}^2$, where $\psi\colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ is the mapping from $S^2 \setminus \{(0,0,-1)\}$ to $\mathbb{R}^2$ defined so that

$$\psi(u,v,w) = \left(\frac{u}{w+1}, \frac{v}{w+1}\right).$$

for all $(u,v,w) \in S^2 \setminus \{(0,0,-1)\}$.

Proposition 6.1 *Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that*

$$S^2 = \{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ be the sphereographic projection mapping defined such that

$$\psi(u,v,w) = \left(\frac{u}{w+1}, \frac{v}{w+1}\right)$$

for all points (u,v,w) of S^2. Then ψ is a bijective mapping whose inverse maps each point (x,y) of $\mathbb{R}^2$ to the corresponding point (u,v,w) of $S^2 \setminus \{(0,0,-1)\}$ determined by the equations

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \text{and} \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

Proof Let $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping defined so that

$$\lambda(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$$

for all points (x,y) of $\mathbb{R}^2$. Let (x,y) be an arbitrary point of $\mathbb{R}^2$. Then

$$(1-x^2-y^2)^2 = 1 + x^4 + y^2 + 2x^2y^2 - 2x^2 - 2y^2$$

and

$$\begin{aligned}(1+x^2+y^2)^2 &= 1 + x^4 + y^2 + 2x^2y^2 + 2x^2 + 2y^2 \\ &= 4x^2 + 4y^2 + (1-x^2-y^2)^2.\end{aligned}$$

It follows that if $(u,v,w) = \lambda(x,y)$ then

$$u^2 + v^2 + w^2 = \frac{4x^2 + 4y^2 + (1-x^2-y^2)^2}{(1+x^2+y^2)^2} = 1$$

for all real numbers x and y. Also if $u = 0$ and $v = 0$ then $x = 0$, $y = 0$ and $w = 1$. It follows that $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^3$ maps $\mathbb{R}^2$ into $S^2 \setminus \{(0,0,-1)\}$.

Moreover

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2} - 1,$$

and therefore

$$u = \frac{2x}{1 + x^2 + y^2} = (w + 1)x$$

and

$$v = \frac{2y}{1 + x^2 + y^2} = (w + 1)y$$

It follows that $(x, y) = \psi(u, v, w)$. Thus the $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is surjective.

Now let (u, v, w) be an element of S^2 distinct from $(0, 0, -1)$ Then u, v and w are real numbers for which $w \neq -1$ and $u^2 + v^2 + w^2 = 1$. Let $(x, y) = \psi(u, v, w)$, where ψ is the map from $S^2 \setminus \{(0, 0, -1)\}$ to $\mathbb{R}^2$ defined by stereographic projection from the point $(0, 0, -1)$. Then

$$x = \frac{u}{w + 1} \quad \text{and} \quad y = \frac{v}{w + 1},$$

and therefore

$$\begin{aligned} 1 + x^2 + y^2 &= \frac{(w + 1)^2 + u^2 + v^2}{(w + 1)^2} = \frac{u^2 + v^2 + w^2 + 2w + 1}{(w + 1)^2} \\ &= \frac{2w + 2)}{(w + 1)^2} = \frac{2}{w + 1}, \end{aligned}$$

It follows that

$$w + 1 = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (w + 1)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (w + 1)y = \frac{2y}{1 + x^2 + y^2}, \\ w &= \frac{2}{1 + x^2 + y^2} - 1 = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}. \end{aligned}$$

Thus $(u, v, w) = \lambda(x, y)$. We conclude therefore that $(u, v, w) = \lambda(\psi(u, v, w))$ for all $(u, v, w) \in S^2 \setminus \{(0, 0, -1)\}$. It follows directly from that that the mapping $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is injective.

We have now shown that the mapping $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ is both surjective and injective. It is therefore a bijective mapping establishing a one-to-one correspondence between points of $S^2 \setminus \{(0,0,-1)\}$ and points of $\mathbb{R}^2$. We have also shown that, for each point (u,v,w) of $S^2 \setminus \{(0,0,-1)\}$, if $(x,y) = \psi(u,v,w)$ then $(u,v,w) = \lambda(x,y)$ and therefore

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \text{and} \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

The result follows. ∎

6.2 Images of Circles under Sphereographic Projection

Let $(\ell.m.n)$ and (u,v,w) be points of the unit sphere S^2 in $\mathbb{R}^3$, where $\ell^2 + m^2 + n^2 = 1$ and $u^2 + v^2 + w^2 = 1$. Then

$$\ell u + mv + nw = \cos\theta,$$

where θ is the angle, at the centre of the sphere, between the line segments joining the centre to the given points. It follows that a subset C of S^2 is a circle on the sphere if and only if it takes the form

$$C = \{(u,v,w) \in S^2 : \ell u + mv + nw = c\},$$

where c, l, m and n are constants for which $\ell^2+m^2+n^2 = 1$ and $-1 < c < 1$.

Let $\psi\colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0,0,-1)$ onto the plane. It follows from Proposition 6.1 that $\psi(u,v,w) = (x,y)$ for all $(u,v,w) \in S^2$, where

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

Moreover

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \text{and} \quad v = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

Proposition 6.2 *Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that*

$$S^2 = \{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0,0,-1)$ onto the plane. Then the circles on S^2 that pass through the point $(0,0,-1)$ are in one-to-one correspondence under this stereographic projection mapping with straight lines in the plane. Specifically let ℓ, m and n be real constants satisfying the *conditions* $\ell^2 + m^2 + n^2 = 1$ *and* $-1 < n \leq 0$. *Then the circle on the unit sphere consisting of those points of the sphere whose Cartesian coordinates* u, v *and* w *satisfy the equation*

$$\ell u + mv + nw = -n$$

corresponds under stereographic projection to the line in $\mathbb{R}^2$ *consisting of those points of the plane whose Cartesian coordinates* x *and* y *satisfy the equation* $px + qy = k$, *where*

$$p = \frac{\ell}{\sqrt{\ell^2 + m^2}}, \quad q = \frac{m}{\sqrt{\ell^2 + m^2}} \quad \textit{and} \quad k = \sqrt{\frac{1}{\ell^2 + m^2} - 1}.$$

Also, given real constants p, q *and* k, *where* $p^2 + q^2 = 1$, *let*

$$\ell = \frac{p}{\sqrt{k^2 + 1}}, \quad m = \frac{q}{\sqrt{k^2 + 1}} \quad \textit{and} \quad n = -\frac{k}{\sqrt{k^2 + 1}}.$$

Then the line in $\mathbb{R}^2$ *expressed in Cartesian coordinates* x *and* y *by the equation* $px + qy = k$ *is the image under stereographic projection of the circle on the unit sphere where that sphere intersects the plane* $\ell u + mv + nw = -n$.

Proof Let C be a circle on S^2 that passes through the point $(0, 0, -1)$ Then

$$C = \{(u, v, w) \in S^2 : \ell u + mv + nw = -n\},$$

where ℓ, m and n are real constants satisfying the condition $\ell^2 + m^2 + n^2 = 1$ and $-1 < n \leq 0$. Let (x, y) be the image of a point (u, v, w) on the circle C under stereographic projection from the point $(0, 0, -1)$. Then

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}$$

(see Proposition 6.1), The equation $\ell u + mv + nw = -n$ satisfied by u, v and w then ensures that

$$\ell x + my = -n = \sqrt{1 - \ell^2 - m^2}.$$

Moreover every point on the line in $\mathbb{R}^2$ determined by this equation is the image under stereographic projection of some point on the circle C. Also the requirements that $\ell^2 + m^2 + n^2 = 1$ and $-1 < n \leq 0$ together ensure that $0 < \ell^2 + m^2 \leq 1$.

Setting $p = \ell/\sqrt{\ell^2 + m^2}$ and $q = m/\sqrt{\ell^2 + m^2}$, we see that the equation of the line can be written in the form

$$px + qy = k,$$

where $p^2 + q^2 = 1$ and

$$k = \sqrt{\frac{1}{\ell^2 + m^2} - 1}.$$

Now, given any line in the plane $\mathbb{R}^2$, there exist real numbers p, q and k, where $p^2 + q^2 = 1$, for which the equation of the line takes the form

$$px + qy = k.$$

Let

$$\ell = \frac{p}{\sqrt{k^2+1}}, \quad m = \frac{q}{\sqrt{k^2+1}} \quad \text{and} \quad n = \frac{-k}{\sqrt{k^2+1}}.$$

Then $\ell^2 + m^2 + n^2 = 1$ and $-1 < n \leq 0$. The line $px + qy = k$ is then the image under stereographic projection of the circle consisting of points on the unit sphere whose displacement vector from the centre of the sphere makes an angle θ the direction of the vector (l, m, n), where $\cos\theta = -n$. The result follows. ∎

Proposition 6.3 *Let S^2 be the unit sphere in $\mathbb{R}^3$, defined so that*

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let $\psi\colon S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ be the stereographic projection mapping that projects the complement of the point $(0, 0, -1)$ onto the plane. Then Then the circles on S^2 that do not pass through the point $(0, 0, -1)$ are in one-to-one correspondence under this stereographic projection mapping with circles in the Euclidean plane. Specifically the circle on the unit sphere consisting of those points of the sphere whose Cartesian coordinates u, v and w satisfy the equation $\ell u + mv + nw = c$, where $\ell^2 + m^2 + n^2 = 1$, $-1 < c < 1$ and $c \neq -n$ corresponds under stereographic projection to the circle in $\mathbb{R}^2$ consisting of those points of the plane whose Cartesian coordinates x and y satisfy the equation $(x - a)^2 + (y - b)^2 = r^2$, where

$$a = \frac{\ell}{c+n}, \quad b = \frac{m}{c+n} \quad \text{and} \quad r = \frac{\sqrt{1-c^2}}{|c+n|}.$$

Conversely, given real constants a, b and r, where $r > 0$, the circle in $\mathbb{R}^2$ of radius r centred on the point (a, b) is the image under stereographic projection of the circle on the unit sphere where that sphere intersects the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation

$$2au + 2bv + (1 + r^2 - a^2 - b^2)w = 1 - r^2 + a^2 + b^2.$$

Proof Let C be a circle on the unit sphere S^2 in $\mathbb{R}^3$ that does not pass through the point $(0,0,1)$. Then there exist real numbers ℓ, m, n and c satisfying the conditions $\ell^2+m^2+n^2=1$, $-1<c<1$ and $c\neq -n$ such that

$$C=\{(u,v,w)\in S^2 : \ell u+mv+nw=c\}.$$

Let (x,y) be the image of a point (u,v,w) on the circle C under stereographic projection from the point $(0,0,-1)$. Then

$$u=\frac{2x}{1+x^2+y^2},\quad v=\frac{2y}{1+x^2+y^2}\quad\text{and}\quad w=\frac{1-x^2-y^2}{1+x^2+y^2}$$

(see Proposition 6.1), and therefore

$$2\ell x+2my+n(1-x^2-y^2)=c(1+x^2+y^2).$$

Moreover every point on the curve in $\mathbb{R}^2$ determined by this equation is the image under stereographic projection of some point on the circle C.

Now $c+n\neq 0$. It follows that point of the plane lies on the curve

$$2\ell x+2my+n(1-x^2-y^2)=c(1+x^2+y^2)$$

if and only if

$$x^2+y^2-2ax-2by+s=0,$$

where

$$a=\frac{\ell}{c+n},\quad b=\frac{m}{c+n}\quad\text{and}\quad s=\frac{c-n}{c+n}.$$

The equation

$$x^2+y^2-2ax-2by+s=0$$

may be expressed in the form

$$(x-a)^2+(y-b)^2=r^2,$$

where

$$\begin{aligned} r^2 &= a^2+b^2-s=\frac{\ell^2+m^2+n^2-c^2}{(c+n)^2} \\ &= \frac{1-c^2}{(c+n)^2}. \end{aligned}$$

(We have used here the condition that $\ell^2+m^2+n^2=1$.) It follows that, under stereographic projection from the point $(0,0,-1)$ the image of the circle on the unit sphere along which the unit sphere intersects the plane

$$\ell u+mv+nw=c$$

(where $\ell^2 + m^2 + n^2 = 1$ and $-1 < c < 1$) is the circle of radius r about the point (a, b) of $\mathbb{R}^2$, where

$$a = \frac{\ell}{c+n}, \quad b = \frac{m}{c+n} \quad \text{and} \quad r = \frac{\sqrt{1-c^2}}{|c+n|}.$$

Now let a, b and r be real numbers, where $r > 0$. We determine which points (u, v, w) of the unit sphere $u^2 + v^2 + w^2 = 1$ are mapped by stereographic projection onto the circle of radius r centred on the point (a, b) of the Euclidean plane. Such points must satisfy the equation

$$\left(\frac{u}{w+1} - a\right)^2 + \left(\frac{v}{w+1} - b\right)^2 = r^2.$$

Expanding out, we find that

$$\frac{u^2+v^2}{(w+1)^2} - \frac{2au+2bv}{w+1} + a^2 + b^2 = r^2.$$

But $u^2 + v^2 = 1 - w^2 = (w+1)(1-w)$. It follows that

$$\frac{1-w-2au-2bv}{w+1} = r^2 - a^2 - b^2,$$

and therefore

$$2au + 2bv + (1 + r^2 - a^2 - b^2)w = 1 - r^2 + a^2 + b^2.$$

Now

$$(2a)^2 + (2b)^2 + (1 + r^2 - a^2 - b^2)^2 = K^2,$$

where

$$K = \sqrt{(1 + a^2 + b^2)^2 + 2(1 - a^2 - b^2)r^2 + r^4}.$$

The equation satisfied by the points on the unit sphere that map under projection to the circle of radius r about a point (a, b) of $\mathbb{R}^2$ therefore takes the form

$$\ell u + mv + nw = c,$$

where $\ell^2 + m^2 + n^2 = 1$, provided we take

$$\ell = \frac{2a}{K}, \quad m = \frac{2b}{K}, \quad n = \frac{1 + r^2 - a^2 - b^2}{K},$$

and

$$c = \frac{1 + a^2 + b^2 - r^2}{K}.$$

Moreover

$$c^2K^2 = (1 + a^2 + b^2)^2 - 2(1 + a^2 + b^2)r^2 + r^4 = K^2 - 4r^2,$$

and therefore $c^2 < 1$. Thus $-1 < c < 1$. The result follows. ∎

6.3 Stereographic Projection: A Vector Algebra Approach

Let S^2 denote the unit sphere in $\mathbb{R}^3$, defined so that

$$\{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| = 1\}.$$

let $\mathbf{Q}$ be a fixed element of S^2, let

$$\Pi_{\mathbf{Q}} = \{\mathbf{r} \in \mathbb{R}^3 : \mathbf{Q} \,.\, \mathbf{r} = 0\},$$

and let $T_{\mathbf{Q}}: \mathbb{R}^3 \to \Pi_{\mathbf{Q}}$ be the linear transformation characterized by the requirement that $T_{\mathbf{Q}}(\mathbf{p} + \lambda\mathbf{Q}) = \mathbf{p}$ for all $\mathbf{p} \in \Pi_{\mathbf{Q}}$ and $\lambda \in \mathbb{R}$. Then the point $\mathbf{Q}$ determines a stereographic projection mapping

$$\psi_{\mathbf{Q}}: S^2 \setminus \{\mathbf{Q}\} \to \Pi_{\mathbf{Q}},$$

where

$$\psi_{\mathbf{Q}}(\mathbf{r}) = \frac{1}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{r} - \frac{\mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{Q}.$$

Note that, for all elements $\mathbf{r}$ of S^2 distinct from $\mathbf{Q}$, the point $\psi_{\mathbf{Q}}(\mathbf{r})$ lies on the line passing through the points $\mathbf{r}$ and $\mathbf{Q}$. Moreover

$$\mathbf{Q} \,.\, \psi_{\mathbf{Q}}(\mathbf{r}) = \frac{1}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{Q} \,.\, \mathbf{r} - \frac{\mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}}\, \mathbf{Q} \,.\, \mathbf{Q} = 0,$$

because $\mathbf{Q} \,.\, \mathbf{Q} = 1$. It follows that $\psi_{\mathbf{Q}}(\mathbf{r}) \in \Pi_{\mathbf{Q}}$ for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$.

Now

$$\begin{aligned} |\psi_{\mathbf{Q}}(\mathbf{r})|^2 &= \frac{1}{(1 - \mathbf{Q} \,.\, \mathbf{r})^2} (\mathbf{r} - (\mathbf{Q} \,.\, \mathbf{r})\mathbf{Q}) \,.\, (\mathbf{r} - (\mathbf{Q} \,.\, \mathbf{r})\mathbf{Q}) \\ &= \frac{1}{(1 - \mathbf{Q} \,.\, \mathbf{r})^2} (1 - (\mathbf{Q} \,.\, \mathbf{r})^2) \\ &= \frac{1 + \mathbf{Q} \,.\, \mathbf{r}}{1 - \mathbf{Q} \,.\, \mathbf{r}} \end{aligned}$$

for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$, because $\mathbf{Q} \,.\, \mathbf{Q} = 1$ and $\mathbf{r} \,.\, \mathbf{r} = 1$. Then

$$|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1 = \mathbf{Q} \,.\, \mathbf{r}(1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2),$$

and therefore

$$\mathbf{Q} \,.\, \mathbf{r} = \frac{|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2},$$

Thus

$$1 - \mathbf{Q} \,.\, \mathbf{r} = \frac{2}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}.$$

It follows that

$$\begin{aligned} \mathbf{r} &= (1 - \mathbf{Q} \,.\, \mathbf{r})\, \psi_{\mathbf{Q}}(\mathbf{r}) + (\mathbf{Q} \,.\, \mathbf{r})\, \mathbf{Q} \\ &= \frac{2}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}\, \psi_{\mathbf{Q}}(\mathbf{r}) + \frac{|\psi_{\mathbf{Q}}(\mathbf{r})|^2 - 1}{1 + |\psi_{\mathbf{Q}}(\mathbf{r})|^2}\, \mathbf{Q}. \end{aligned}$$

Thus $\mathbf{r} = \lambda_{\mathbf{Q}}(\psi_{\mathbf{Q}}(\mathbf{r}))$ for all $\mathbf{r} \in S^2 \setminus \{\mathbf{Q}\}$, where

$$\lambda_{\mathbf{Q}}\colon \Pi_{\mathbf{Q}} \to \mathbb{R}^3$$

is the mapping from the plane $\Pi_{\mathbf{Q}}$ through the origin perpendicular to the vector $\mathbf{Q}$ to $\mathbb{R}^3$ defined such that

$$\lambda_{\mathbf{Q}}(\mathbf{p}) = \frac{2}{1 + |\mathbf{p}|^2}\, \mathbf{p} + \frac{|\mathbf{p}|^2 - 1}{1 + |\mathbf{p}|^2}\, \mathbf{Q}$$

for all $\mathbf{p} \in \Pi_{\mathbf{Q}}$.

6.4 The Angle-Preserving Properties of Stereographic Projection

Let S^2 denote the sphere of unit radius in $\mathbb{R}^3$ centred on the origin O, let Q be a point of S^2, and let Π_Q denote the plane through the origin O perpendicular to the line OQ. For each point P of the sphere S^2 that is distinct from Q, let $\psi(P)$ denote the image of P under stereographic projection from Q. Then, for each point P of S^2 distinct from the fixed point Q, $\psi(P)$ is the unique point of the plane Π_Q at which that plane intersects the line passing through both P and Q.

For each point P of the sphere S^2 let T_P denote the tangent plane to S^2 at the point P. Then, for each point P of S^2 the tangent plane T_P is the union of all lines passing through the point P that are perpendicular to the radius vector OP.

Proposition 6.4 *Let S^2 be the unit sphere centered at the origin O, let Q be a fixed point of S^2, let Π_Q be the plane passing through the origin O that is perpendicular to the line OQ, and let $\psi\colon S^2 \setminus \{0\} \to \Pi_Q$ be the mapping implementing stereographic projection from the point Q onto the plane Π_Q. Then the mapping ψ is an angle-preserving mapping from $S_2 \setminus \{0\}$ to Π_Q.*

Proof Let Q be a fixed point of the unit sphere S^2, let P be a point of S^2 distinct from the point Q, let Π_Q denote the plane through the origin perpendicular to the radius vector OQ, and let $\psi(P)$ be the image of P under stereographic projection from the point Q, so that $\psi(P)$ is the unique point of the plane Π_Q at which that plane intersects the line passing through both P and Q. Let L_1 and L_2 be distinct lines contained in the tangent plane T_P to the unit sphere S^2 at the point P, that intersect at the point P. Then $L_1 \subset T_P$, $L_2 \subset T_P$ and $L_1 \cap L_2 = \{P\}$. Let Π_1 denote the unique plane in $\mathbb{R}^3$ that contains both the line L_1 and the point Q, and let Π_2 denote the unique plane in $\mathbb{R}^3$ that contains both the line L_2 and the point Q. Let M_1 and M_2 denote the distinct lines in the tangent plane T_Q to the unit sphere at the point Q along which which the tangent plane T_Q intersects the planes Π_1 and Π_2. Then $M_1 = \Pi_1 \cap T_Q$, $M_2 = \Pi_2 \cap T_Q$ and $M_1 \cap M_2 = \{Q\}$. Let Λ denote the plane in $\mathbb{R}^3$ consisting of all points of $\mathbb{R}^3$ that are equidistant from the points P and Q, and let $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the reflection of the space $\mathbb{R}^3$ in the plane Λ. Then the plane Λ contains the centre O of the unit sphere S^2, located at the origin. Also the line segment PQ from P to Q is perpendicular to the plane Λ and is bisected by Λ. It follows that $\tau(P) = Q$. Also $\tau(\Pi_1) = \Pi_1$, because the plane contains a line, namely the line PQ, which is perpendicular to the plane Λ, and similarly $\tau(\Pi_2) = \Pi_2$. Also the mapping $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ preserves lengths and angles, and therefore $\tau(T_P) = T_Q$. It follows that

$$\tau(L_1) = \tau(T_P \cap \Pi_1) = \tau(T_P) \cap \tau(\Pi_1) = T_Q \cap \Pi_1 = M_1,$$

and similarly $\tau(L_2) = M_2$.

The angle-preserving property of the reflection τ therefore ensures that the angle between the lines L_1 and L_2 at their point P of intersection is equal to the angle between the lines M_1 and M_2 at their point Q of intersection.

Let N_1 and N_2 denote the lines along which the plane Π_Q through the origin O perpendicular to OQ cuts the planes Π_1 and Π_2 respectively. Then $N_1 = \Pi_Q \cap \Pi_1$ and $N_2 = \Pi_Q \cap \Pi_2$. The plane Π_Q is parallel to the tangent plane T_Q at the point Q. It follows that the lines N_1 and N_2 are parallel to the lines M_1 and M_2. Therefore the angle between the lines N_1 and N_2 at their point $\psi(P)$ of intersection is equal to the angle between the lines M_1 and M_2 at their point Q of intersection, and is therefore equal to the angle between the lines L_1 and L_2 at their point P of intersection.

Let C_1 and C_2 denote the circles on the unit sphere S^2 along which the unit sphere cuts the planes Π_1 and Pi_2 respectively. Then the line L_1 is tangent to C_1 at the point P, and similarly the line L_2 is tangent to C_2 at the point P. Now the point Q belongs to the both the planes Π_1 and Π_2. It follows from the definition of stereographic projection that $\psi(C_1) \subset \Pi_1$ and $\psi(C_2) \subset \Pi_2$.

But $\psi(C_1) \subset \Pi_Q$, $\psi(C_2) \subset \Pi_Q$, $\Pi_Q \cap \Pi_1 = N_1$ and $\Pi_Q \cap \Pi_2 = N_2$. It follows that $\psi(C_1) \subset N_1$ and $\psi(C_2) \subset N_2$. Moreover all points of the line N_1 are the images of points of $S^2 \cap \Pi_1$ under stereographic projection. It follows that $\psi(C_1) = N_1$. Similarly $\psi(C_2) = N_2$.

Now the angle between the circles C_1 and C_2 at the point P is equal to the angle between their tangent lines L_1 and L_2 at the point P. This angle has been shown to be equal to the angle between the lines N_1 and N_2. Therefore the stereographic projection mapping $\psi\colon S^2 \setminus \{Q\} \to \Pi_Q$ is angle-preserving, as required. ∎