MA232A—Euclidean and Non-Euclidean Geometry
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Section 2: Magnitude and Congruence

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2 Magnitude and Congruence

2.1 Magnitudes

Propositions in Euclid's *Elements* often express relationships satisfied by sums of magnitudes of the same species. The concept of *magnitude* (in Greek, μεγέθος, *megethos*) is introduced in the definitions commencing Book V of the *Elements*. Book V is concerned with the theory of proportion, determining whether, for magnitudes of a given species, a first magnitude bears to a second the same ratio, or a lesser or greater ratio, than a third magnitude to a fourth magnitude. Nevertheless the concept of comparisons between magnitudes of some given species clearly underlies the reasoning of the earlier books. This reasoning is underpinned by the *Common Notions* prefixed to Book I of the *Elements*.

Property EP–1 *Let some species of magnitude be given whose members can be compared one with another to determine whether or not the first is equal to the second. Suppose also that this relation of equality, which we denote by $\equiv$, conforms to the principles stated as Common Notions 1 and 4 of Book I of Euclid's Elements. (Thus we suppose that any magnitude of the species is equal to itself, and also that any magnitudes of the species that are equal to the same magnitude are also equal to one another.) Let α, β and γ be magnitudes belonging to the species. Then*

*(i) (*Reflexivity*) $\alpha \equiv \alpha$;*

*(ii) (*Symmetry*) if $\alpha \equiv \beta$ then $\beta \equiv \alpha$;*

*(iii) (*Transitivity*) if $\alpha \equiv \beta$ and $\beta \equiv \gamma$ then $\alpha \equiv \gamma$.*

Proof The relation $\alpha \equiv \alpha$ is an immediate consequence of Common Notion 4. Thus reflexivity holds.

Suppose that $\alpha \equiv \beta$. Now $\beta \equiv \beta$ by (i). Thus β and α are magnitudes that are both equal to the same thing, namely β. It follows from Common Notion 1 that they are equal to one another, and therefore $\beta \equiv \alpha$. This proves symmetry.

Finally suppose that $\alpha \equiv \beta$ and $\beta \equiv \gamma$. It follows from symmetry that $\gamma \equiv \beta$. Thus α and γ are both equal to β. It follows from Common Notion 1 that α and γ are equal to one another, and thus $\alpha \equiv \gamma$. This proves transitivity, completing the proof. ∎

Principle EP–2 (Addition of Magnitudes of the Same Species) *Given a list of magnitudes of the same species, the sum of the magnitudes in the list* *will be determined on specifying the magnitudes that occur in the list, and the number of times that those magnitudes occur in the list, but the sum of the magnitudes in the list will not depend on the order in which those elements are listed.*

Thus, for example, if α, β and γ are magnitudes of the same species then the sum of the list α, β, γ will be the same as the sums of the list β, γ, α, the list γ, α, β, the list γ, β, α, the list β, α, γ and the list α, γ, β.

But, given magnitudes α and β of the same species, the sum of the magnitudes in the list α, α, β will be greater than the sum of the magnitudes in the list α, β because the magnitude α occurs twice in the first list and once in the second list, whilst the magnitude β occurs once in both lists.

Many propositions in Euclid's *Elements* establish relationships that can be expressed in the form

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p = \beta_1 + \beta_2 + \cdots + \beta_q,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$ are magnitudes of the same species.

Other propositions might express an inequality, asserting for example that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p < \beta_1 + \beta_2 + \cdots + \beta_q,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$ are magnitudes of the same species.

It may be that an appropriate metaphor would be that of a balance used in weighing collections of objects. The magnitudes $\alpha_1, \alpha_2, \ldots, \alpha_p$ might be thought of as though they were weights, to be placed on the left hand side of the balance. The other magnitudes magnitudes $\beta_1, \beta_2, \ldots, \beta_q$ might be thought of as though they were weights to be placed on the right hand side of the balance. Then either the weights on the left hand side balance the weights on the right hand side, in which case equality holds, or else the weights on one side might overbalance those on the other side.

Now suppose that a certain species of magnitude is given, and that the magnitudes belonging to that species can be added together and that the resulting sums can be compared with one another. Suppose further that the results of such comparisons are always consistent with the five common notions stated in Book I of Euclid's *Elements*. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$ and $\gamma_1, \gamma_2, \ldots, \gamma_r$ and $\delta_1, \delta_2, \ldots, \delta_s$, be lists of magnitudes of the given species, where each list is a finite list of magnitudes containing at least one magnitude, and let the sums of the magnitudes in these four lists be denoted by

$$\sum_{i=1}^{p} \alpha_i, \quad \sum_{i=1}^{q} \beta_i, \quad \sum_{i=1}^{r} \gamma_i \quad \text{and} \quad \sum_{i=1}^{s} \delta_i$$

respectively.

Then Common Notion 2 can be cited to justify the proposition that if

$$\sum_{i=1}^{p} \alpha_i \equiv \sum_{i=1}^{r} \gamma_i \quad \text{and} \quad \sum_{i=1}^{q} \beta_i \equiv \sum_{i=1}^{s} \delta_i$$

then

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i \equiv \sum_{i=1}^{r} \gamma_i + \sum_{i=1}^{s} \delta_i.$$

Similarly Common Notion 3 can be cited to justify the proposition that if

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i \equiv \sum_{i=1}^{r} \gamma_i + \sum_{i=1}^{s} \delta_i \quad \text{and} \quad \sum_{i=1}^{q} \beta_i \equiv \sum_{i=1}^{s} \delta_i$$

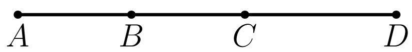
then

$$\sum_{i=1}^{p} \alpha_i \equiv \sum_{i=1}^{r} \gamma_i.$$

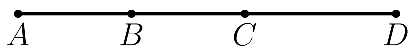
2.2 Addition of Line Segments and Angles

Principle EP–3 (Addition of Successive Line Segments) *If a line segment is partitioned by division points into subsegments, then the line segment is equal to the sum of the subsegments.*

Thus, for example, if a line segment $[AD]$ is partitioned by division points B and C into subsegments $[AB]$, $[BC]$ and $[CD]$, where B lies between A and C and C lies between B and D, the line segment $[AD]$ is equal to the sum of the subsegments $[AB]$, $[BC]$ and $[CD]$.



If we denote the *magnitudes* of the line segments $[AB]$, $[BC]$, $[CD]$ and $[AD]$ by $|AB|$, $|BC|$, $|CD|$ and $|AD|$ respectively, then the assertion that the line segment $|AD|$ is equal to the sum of the line segments $[AB]$, $[BC]$, $[CD]$



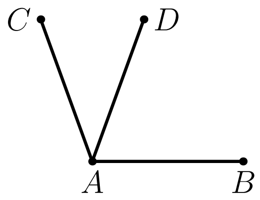
may be represented symbolically as follows:

$$|AD| = |AB| + |BC| + |CD|.$$

In a similar fashion we may denote the magnitude of an angle $\angle ABC$ at the point B using the notation $|\angle ABC|$.

Principle EP–4 (Addition of Rectilineal Angles) *Let A, B, C, and D be points in the plane, where the points A, B and C are not collinear, and where the point D lies in the interior of the angle $\angle BAC$. Then the angle $\angle BAC$ is equal to the sum of the angles BAD and DAC; thus, in symbols,*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

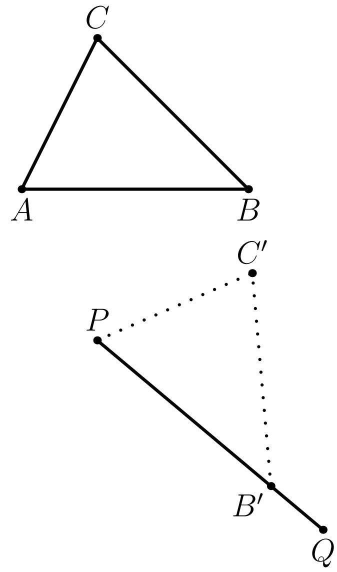


Moreover the angles BAD and DAC are parts of the angle $\angle BAC$, and therefore the angles BAD and DAC are both less than BAC; thus, in symbols

$$|\angle BAD| < |\angle BAC| \quad \textit{and} \quad |\angle DAC| < |\angle BAC|.$$

2.3 The Homogeneity and Isotropy of the Euclidean Plane

A few proofs in Euclid rely on the procedure of *applying* a geometrical figure such as a triangle to a given line segment. Let M be a geometric figure, such as a triangle, in a given plane, let A and B be distinct points forming part of the geometric figure M, and let P and Q be two other distinct points in that plane. Euclid presumes that the geometric figure M can be *applied* to the line segment $[PQ]$, moving the figure in the plane, *placing* it so as to obtain a geometrical figure M' in which the point A' corresponding to the point A of the original figure M coincides with the point P and the point B' of M' corresponding to the point B of M lies on the ray (or half-line) starting at the point P and passing through the point Q.



The diagram to the right depicts a situation in which a triangle $\triangle ABC$ is applied to a line segment $[PQ]$. The triangle $\triangle ABC$ is moved so that the vertex A of the triangle is placed on P and the side $[AB]$ is placed on the ray from P passing through the point Q.

In thus *applying* a geometrical figure to a given line segment, all the geometrical properties of the figure M are presumed to be preserved. In particular, line segments in the resultant applied figure M' are presumed to be equal to the corresponding line segments in the original figure, and similarly angles and areas in the resultant applied figure M' are presumed to be equal to the corresponding areas and angles in the original figure M.

Moreover, if C is a point of the original geometrical figure M that does not lie on the line through the points A and B, then Euclid presumes that the figure M can be applied to the line segment $[PQ]$ so as to obtain a geometrical figure M' in which the point C' corresponding to C lies on any chosen side of the line $[PQ]$.

The strategy of applying triangles to line segments is used in the proofs of Propositions 4 and 8 of Book I of Euclid's *Elements*. In Proposition 24 of Book III, a segment of a circle is applied to a line segment.

The method for applying a geometrical figure to a line segment is founded on the implicit presumption that the geometry of the Euclidean plane is *homogeneous* and *isotropic*. The *homogeneity* of the Euclidean plane requires that the geometrical properties of the plane specified with respect to one chosen point within the plane match up with the geometrical properties specified with respect to any other point of that plane. The *isotropy* of the Euclidean plane requires that geometrical properties specified with respect to one chosen direction from a given point of the plane match up with the geometrical properties specified with respect to any other direction from that given point.

The assertion that the Euclidean plane is *homogeneous* encapsulates the proposition that the geometry of the Euclidean plane appears the same at all points of the plane. Thus the geometry of the plane does not single out any particular point as having geometrical properties distinct from those of other points of the plane

Similarly the assertion that the Euclidean plane is *isotropic* encapsulates the proposition that the geometry of the Euclidean plane appears the same in all directions about a given point of the plane. Thus, at a given point of the plane, the geometry of the plane does not single out any particular direction at a given point as having geometrical properties distinct from those of other directions at that given point.

Consider a situation in which one is cycling from place to place. It seems in accord with everyday experience to presume that the spokes of the bicycle wheel do not change in length, and that the distances between points on the rim of the wheel, and the angles between successive spokes remain invariant as the bicycle moves from place to place, turning as it does so. Such aspects of experience make it natural to presume, or postulate, that geometrical figures can be moved around the Euclidean plane from one location to another without changing shape.

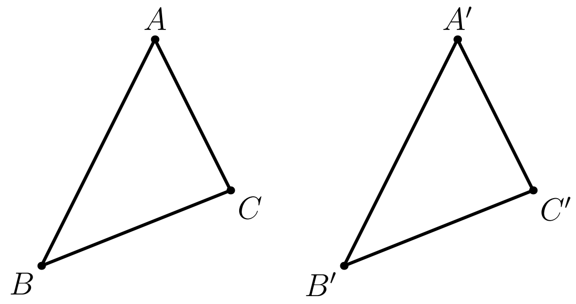
2.4 Congruence Rules

The results that Euclid obtains in Propositions 4 and 8 using the method of superposition in conjunction with Common Notion 4 are the *Side-Angle-Side* (SAS) and *Side-Side-Side* (SSS) *Congrence Rules* respectively.

Principle EP–5 (Characterization of Congruence for Triangles) *If two triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *are* congruent *then the sides* $[AB]$, $[AC]$ *and* $[BC]$ *of the first triangle are equal to the corresponding sides* $[A'B']$, $[A'C']$ *and* $[B'C']$ *respectively of the second triangle, and the angles of the first triangle at the vertices* A, B *and* C *are equal to the corresponding angles of the second triangle at* A', B' *and* C' *respectively.*

This characterization of congruence for triangles can be expressed in symbols, using the symbol $\equiv$ to denote the relation of equality (or congruence) of line segments and rectilineal angles, as follows:

> Triangles ABC and $A'B'C'$ are congruent if and only if $[AB] \equiv [A'B']$, $[AC] \equiv [A'C']$, $[BC] \equiv [B'C']$, $\angle BAC \equiv \angle B'A'C'$, $\angle CBA \equiv \angle C'B'A'$, and $\angle ACB \equiv \angle A'C'B'$.



Property EP–6 (The Side-Angle-Side (SAS) Congruence Rule) *If* ABC *and* $A'B'C'$ *are triangles, if the sides* $[AB]$ *and* $[AC]$ *of the first triangle are equal to the corresponding sides* $[A'B']$ *and* $[A'C']$ *respectively of the second triangle, and if the angle* $\angle BAC$ *at the vertex* A *of the first triangle is equal to the angle* $\angle B'A'C'$ *at the vertex* A' *of the second triangle (i.e., if* $[AB] \equiv [A'B']$*,* $[AC] \equiv [A'C']$ *and* $\angle BAC \equiv \angle B'A'C'$*), then the triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *are congruent (and therefore* $[BC] \equiv [B'C']$*,* $\angle CBA \equiv \angle C'B'A'$*, and* $\angle ACB \equiv \angle A'C'B'$*).*

The SAS Congruence Rule is established in Proposition 4 of Book I of Euclid's *Elements*.

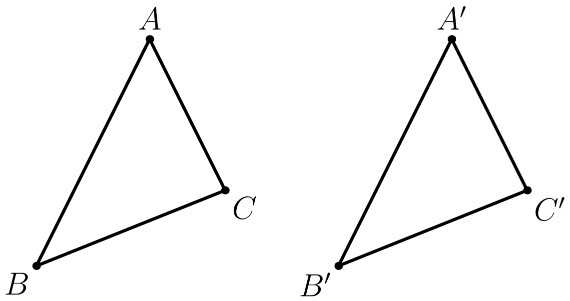
Property EP–7 (The Side-Side-Side (SSS) Congruence Rule) *If* ABC *and* $A'B'C'$ *are triangles, and if all the sides* $[AB]$*,* $[AC]$ *and* $[BC]$ *of the first triangle are equal to the corresponding sides* $[A'B']$*,* $[A'C']$ *and* $[B'C']$ *respectively of the second triangle (i.e., if* $[AB] \equiv [A'B']$*,* $[AC] \equiv [A'C']$ *and* $[BC] \equiv [B'C']$*), then the triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *are congruent (and therefore* $\angle BAC \equiv \angle B'A'C'$*,* $\angle CBA \equiv \angle C'B'A'$*, and* $\angle ACB \equiv \angle A'C'B'$*).*

The SSS Congruence Rule is established in Proposition 8 of Book I of Euclid's *Elements*.

The *Angle-Side-Angle* (ASA) and *Angle-Angle-Side* (AAS) *Congruence Rules* are proved by Euclid in Proposition 26 of Book I of the *Elements*. They may be stated as follows.

Property EP–8 (The Angle-Side-Angle (ASA) Congruence Rule) *If* ABC *and* $A'B'C'$ *are triangles, if the side* $[BC]$ *of the first triangle is equal to the corresponding side* $[B'C']$ *of the second triangle, and if the angles* ABC *and* ACB *at the vertices* B *and* C *of the first triangle are equal to the angles* $A'B'C'$ *and* $A'C'B'$ *at the vertices* B' *and* C' *respectively of the second triangle (i.e., if* $[BC] \equiv [B'C']$*,* $\angle ABC \equiv A'B'C'$ *and* $\angle ACB \equiv \angle A'C'B'$*), then the triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *are congruent (and therefore* $[AB] = [A'B']$*,* $[AC] \equiv [A'C']$ *and and* $\angle BAC \equiv \angle B'A'C'$*).*

Justification for the ASA Congruence Rule Let ABC and $A'B'C'$ be triangles. Suppose that $[BC] \equiv [B'C']$, $\angle ABC \equiv \angle A'B'C'$ and $\angle ACB \equiv A'C'B'$.



There exists a point A'' on the ray that starts at B' and passes through A' for which $[A''B'] \equiv [AB]$. (In the context of Euclid's geometry, this can be justified by Proposition 3 of Book I of the *Elements*.) Then $[A''B'] \equiv [AB]$, $[B'C'] \equiv [BC]$ and $\angle A''B'C' \equiv \angle ABC$. Applying the SAS Congruence Rule, we deduce that the triangles $\triangle A''B'C'$ and $\triangle ABC$ are congruent, and therefore $\angle A''C'B' \equiv \angle ACB$. But $\angle A'C'B' = \angle ACB$. It follows that the points A'', A' and C' are collinear. The points A'', A' and B' are also collinear. But the points A', B' and C' are not collinear. Therefore it must be the case that the points A'' and A' coincide. It then follows that the triangle $\triangle A'B'C'$ coincides with the triangle $\triangle A''B'C'$, and is therefore congruent to the triangle $\triangle ABC$, as required. ∎

Property EP–9 (The Angle-Angle-Side (AAS) Congruence Rule) *If ABC and $A'B'C'$ are triangles, if the side $[AB]$ of the first triangle is equal to the corresponding side $[A'B']$ of the second triangle, and if the angles ABC and ACB at the vertices B and C of the first triangle are equal to the angles $A'B'C'$ and $A'C'B'$ at the vertices B' and C' respectively of the second triangle (i.e., if $[AB] \equiv [A'B']$, $\angle ABC \equiv A'B'C'$ and $\angle ACB \equiv \angle A'C'B'$), then the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent (and therefore $[AC] = [A'C']$, $[BC] \equiv [B'C']$ and and $\angle BAC \equiv \angle B'A'C'$).*

The AAS Congruence Rule can be justified by a strategy analogous to that given above to justify the ASA Congruence Rule. Specifically there is a point C'' on the ray that starts at the point B' and passes though C' for which $C''B' \equiv CB$. Then $AB \equiv A'B'$, $BC \equiv B'C''$ and $\angle ABC \equiv \angle A'B'C''$. An application of the SAS Congruence Rule establishes that the triangles ABC and $A'B'C''$ are congruent. It follows that $\angle A'C''B' = \angle ACB$. But $\angle ACB \equiv A'C'B'$. It follows that $\angle A'C''B' \equiv A'C'B'$.

In order to complete this justification of the AAS Congruence Rule, one needs to show that the points C' and C'' must coincide. This can be done by making use of the result that an external angle of a triangle is always greater than either of the interior and opposite angles of that triangle. This result is obtained in Proposition 16 of Book I of Euclid's *Elements*. It ensures that the points C' and C'' must coincide, because if they did not coincide, the triangle $\triangle A'C'C''$ would have an external angle at one of the vertices C' and C'' equal to the internal angle at the other, and this would contradict Proposition 16 of Book I of Euclid's *Elements*.

The SAS Congruence Rule in fact encodes within itself the basic assumptions regarding *homogeneity* and *isotropy* that are assumed to be satisfied by the plane that is the object of investigation. Indeed let P and Q be points of a plane Π and let rays in that plane be chosen starting from the points P and Q. Let R be a point distinct from P that lies on the chosen ray starting at the point P, and let S be a point distinct from Q that lies on the chosen ray starting from the point Q. We also choose sides of these rays.

Now let A be any point of the plane Π. There then exists a well-defined map $\varphi\colon \Pi \to \Pi$ such that for all points A of Π, $\varphi(A) = A'$, where A' is determined as follows:

- if $A = P$ then $A' = Q$;
- if $A \neq P$ then the line segment $[QA']$ is equal to the line segment $[PA]$;
- if A lies on the ray starting at P and passing through R then A' lies on the ray starting at Q and passing through S;
- if A lies on the ray opposite R obtained on producing $[RP]$ beyond P then A' lies on the ray opposite S obtained on producing $[SQ]$ beyond Q;
- if A does not lie on the line through P and R then the angle $\angle A'QS$ is equal to the angle $\angle APR$;
- if A lies on the chosen side of $[PR]$ then A' lies on the chosen side of $[QS]$;
- if A lies on the side of $[PR]$ opposite to the chosen side then A' lies on the side of $[QS]$ opposite to the chosen side.

If standard assumptions are made (not in themselves requiring the plane Π to be either homogeneous or isotropic) concerning the nature of angles with vertices at the points P and Q, and if line segments can be produced beyond their endpoints to any required distance, and if, given any two points of the plane Π, there is a unique line segment joining those two points, then the construction just described should produce a well-defined map $\varphi\colon \Pi \to \Pi$ with the following properties:

- $\varphi(P) = Q$;
- if $A \neq P$ then the line segment from P to A is equal to the line segment from Q to $\varphi(A)$;
- φ maps lines through the point P to lines through the point Q;
- if A and B are points of the plane Π, and if the points A, B and P are distinct and not collinear, then the angle between the line segments joining P to A and B is equal to the angle between the line segments joining Q to $\varphi(A)$ and $\varphi(B)$.

If the SAS Congruence Rule is satisfied by triangles in the plane Π then the properties listed suffice to ensure that, for all points A and B of Π that constitute with P the vertices of a triangle in Π, that triangle with vertices P, A and B is congruent to the triangle with vertices Q, $\varphi(A)$ and $\varphi(B)$. It follows that the line segment joining the points A and B is equal to the line segment joining the points $\varphi(A)$ and $\varphi(B)$. This result also holds when A, B and P are collinear. It follows that $\varphi\colon \Pi \to \Pi$ is a *distance-preserving* map from the plane Π to itself. The fact that φ maps any triangle with a vertex at P onto a congruent triangle with a vertex at Q also ensures that $\varphi\colon \Pi \to \Pi$ is an *angle-preserving* map from the plane Π to itself.

The argument just presented shows that if the geometry of the plane Π satisfies the SAS congruence rule (in addition to other unspecified axioms or rules that ensure that the space is sufficiently ‘well-behaved’ in the immediate neighbourhood of a given point), then given any two points P and Q, and given any two directions represented by rays starting at P and Q, there exists a distance-preserving and angle-preserving map from the plane Π to itself which maps P onto Q, and also maps the chosen ray starting from the point P onto the chosen ray starting from the point Q. Therefore geometrical figures can be moved around and rotated in the plane Π without changing their shape or size.

This argument can be presented in more concrete terms as follows. Suppose that Alice is sitting at a desk in a school, facing north, with a piece of paper in front of her on which geometrical diagrams can be drawn. Suppose also that Bob is sitting at another desk in another classroom on the same floor (or indeed on a different floor) of that school, facing southeast, and that Bob also has a piece of paper in front of him on which geometrical diagrams can be drawn. Then the presumed validity of the SAS Congruence Rule should in theory enable one to match up positions on Alice's sheet with corresponding positions on Bob's sheet in a way that preserves both distances and angles so that, for every geometrical figure that can be drawn on Alice's sheet, there is a corresponding geometrical figure that could be drawn on Bob's sheet with the same geometrical properties as the figure on Alice's sheet.

2.5 Comparison between Flat and Spherical Geometry

Suppose that a fixed point is chosen in a flat Euclidean plane, and that two ants start walking away from this fixed point with speeds u and v respectively, in directions that make a right angle with one another. Then, at time t, the distance between the two ants will be $\sqrt{u^2 + v^2}t$. Therefore, at a given time t, the two ants, together with the chosen fixed point, constitute the vertices of a triangle with sides of length ut, vt and $\sqrt{u^2 + v^2}\,t$. Moreover the angles of this triangle remain constant as time progresses.

Such observations would not hold good were the ants to start walking away from a chosen point on a sphere in directions that initially make a some chosen fixed angle with one another. The great circle distance between the ants (i.e., the length of the arc of a great circle on the sphere joining the two ants) can be found using the formulae of spherical trigonometry. It would not increase linearly with time, and the angles of the spherical triangle determined by the two ants and the chosen fixed point would vary as time progresses.

Nevertheless the distance between the ants at a given time does not depend either on the fixed point chosen or on the initial directions chosen, provided that the ants walk at the same speeds in directions that initially make an angle with one another equal to the chosen fixed angle.

Thus the geometry of the sphere, like the geometry of a flat Euclidean plane, is both *homogeneous* and *isotropic*. Moreover appopriate analogues of the first fifteen propositions in Book I of Euclid's *Elements* are valid in spherical geometry, with straight lines replaced by arcs of great circles, provided that the lengths of such arcs (including the arcs that form the sides of spherical triangles) are less than the great circle distance between two poles of the sphere.

In particular the SAS Congruence Rule is valid in spherical geometry for spherical triangles whose sides are shorter than the great circle distance between two poles of the sphere.

2.6 Geodesics on Smooth Surfaces

Differential geometry developed in the nineteenth century following the publication by Gauss in 1828 of his important treatise *Disquisitiones generales circa Superficies curvas* (General investigations of closed surfaces).

When studying the geometry of a smooth surface in three-dimensional Euclidean space the analogues of the straight lines of the flat Euclidean plane are the *geodesics* on the surface: geodesics on a smooth surface are smooth curves on that surface characterized by the property that all sufficiently short segments of a geodesic minimize distance amongst all smooth curves in the surface that join its endpoints.

Example Consider the smooth surface in three-dimensional space defined by the equation

$$z = \frac{10}{1 + x^2 + y^2}.$$

Let $P = (1, 0, 5)$ and $Q = (-1, 0, 5)$. Then the points P and Q lie on the surface, and lie on the circle in the plane $z = 5$ of radius 1 about the point $(0, 0, 5)$. This circle lies on the surface. It follows that any length-minimizing curve on the surface from P to Q must have length not exceeding 4. It follows that a length-minimizing geodesic from P to Q cannot pass through the point $(0, 0, 10)$ and thus is not contained in the plane $y = 0$.

Given any geodesic on the surface joining the points P and Q, that geodesic can be reflected in the plane $y = 0$ to obtain another geodesic from P to Q. Now standard results in the theory of differential equations will guarantee the existance of at least one length-minimizing geodesic on the surface joining the points P and Q. But this length-minimizing geodesic cannot be the *unique* length-minimizing geodesic joining the points P and Q.

Consider two ants walking across a smooth surface at constant speed, following the paths of geodesics across the surface. If those ants had started from a single point in the Euclidean plane, walking at constant speeds along straight lines, setting out from that point at the same time, then the distance between the ants would increase linearly with the time elapsed since setting out.

On a *positively curved* smooth surface such as a sphere, or on a positively curved portion of a smooth surface, two ants setting out at the same time from a single point at constant speeds along distinct geodesics will at subsequent times be closer than the equivalent ants setting out at the same speeds along straight lines in the flat Euclidean plane that make the same angle with one another. Ants on *negatively curved* smooth surfaces would be further apart than the equivalent ants walking across the flat Euclidean plane.

2.7 Intersections of Lines and Circles

The following principle is a direct consequence of the definition of *parallel lines* in Book I of Euclid's *Elements*, combined with the principle that, given two distinct points, there exists at most one line passing through those points.

Property EP–10 *Given two lines in a single plane that are not parallel to one another, there exists a single point of the plane at which those two lines intersect one another.*

The following two principles concern intersections of circles with straight lines and with other circles.

Property EP–11 *Let a circle and a line segment be given, together with the centre of the circle. Suppose that one endpoint of the line segment lies closer to the centre of the circle than the points on the circle, and that the other endpoint of the line segment lies further away from the centre of the circle than the points on the circle. Then the circle and the line segment intersect one another.*

Proposition 2 of Book III of Euclid's *Elements* shows that if the endpoints of a line segment lie on a circle then the other points of the line segment lie in the interior of the circle. It follows immediately that a straight line cannot intersect a circle in more than two points.

Property EP–12 *Let two circles be given, together with the centre of the first of those circles. Suppose that there are points on the second circle that lie closer to the centre of the first circle than the points on the first circle, and that there are also points on the second circle that lie further away from the centre of the first circle than the points on the first circle. Then the two circles intersect one another.*

If two distinct circles in the plane intersect, then either they touch at a single point or else they cut one another at exactly two points: see Propositions 10 and 13 in Book III of Euclid's *Elements*.

We now explore links between these assumptions regarding intersections of straight lines and circles and the theory of *connectedness* in real analysis and topology that has developed over the past couple of centuries.

In the context of the mathematics in common use throughout the past century, the flat Euclidean plane can be identified with the space $\mathbb{R}^2$ of ordered pairs of real numbers. We then define a *path* in the plane to be a continuous function $\gamma\colon [0,1] \to \mathbb{R}^2$ mapping the the closed unit interval $[0,1]$ into $\mathbb{R}^2$, where

$$[0,1] = \{t \in \mathbb{R} : 0 \leq t \leq 1\}.$$

Such a path is a path from a point P to a point Q provided that $\gamma(0) = P$ and $\gamma(1) = Q$.

A subset V of $\mathbb{R}^2$ is said to be *open* in $\mathbb{R}^2$ if, given any point P of V, there exists some strictly positive real number δ such that all points lying within the circle of radius δ centred on the point P belong to the set V.

We now use results and methods developed in the latter part of the nineteenth century to show that, given any two non-empty disjoint open sets in the plane, any path that starts in one open set and ends in the other must pass through points that do not belong to either open set.

Property EP–13 *Let V and W be disjoint non-empty open sets in $\mathbb{R}^2$, and let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a path in $\mathbb{R}^2$ for which $\gamma(0) \in V$ and $\gamma(1) \in W$. Then there exists a real number s satisfying $0 < s < 1$ for which $\gamma(s) \notin V$ and $\gamma(s) \notin W$.*

Proof Let

$$S = \{t \in [0,1] : \gamma(t) \in V\},$$

and let $s = \sup S$ (so that the real number s is the least upper bound of the set S). Then $0 \leq s \leq 1$. We shall prove that $\gamma(s) \notin V$ and $\gamma(s) \notin W$.

Let r be a real number satisfying $0 \leq r \leq 1$. If $\gamma(r) \in V$ then it follows from the definitions of continuity and open sets that there exists some positive real number δ such that $\gamma(r) \in V$ for all real numbers t satisfying both $0 \leq t \leq 1$ and $r - \delta < t < r + \delta$. Similarly if $\gamma(r) \in W$ then there exists some positive real number δ such that $\gamma(r) \in W$ for all real numbers t satisfying both $0 \leq t \leq 1$ and $r - \delta < t < r + \delta$.

Suppose that $\gamma(r) \in V$. Then $r < 1$, because $\gamma(1) \in W$ and $V \cap W = \emptyset$. But then there exists a positive real number δ such that $r + \delta \leq 1$ and $\gamma(t) \in V$ for all real numbers t satisfying $t < r + \delta$. Then $t \in S$ for all real numbers t satisfing $r < t < r + \delta$, and therefore $r \neq \sup S$.

Next suppose that $\gamma(r) \in W$. Then $r > 0$, because $\gamma(0) \in V$ and $V \cap W = \emptyset$. But then there exists a positive real number δ such that $r - \delta \geq 0$ and $\gamma(t) \in W$ for all real numbers t satisfying $t > r - \delta$. Then $t \notin S$ for all real numbers t satisfing $r - \delta < t \leq r$, and therefore $r \neq \sup S$.

From these results, we conclude that if $s = \sup S$ then $\gamma(s) \notin V$ and $\gamma(s) \notin W$. Clearly $0 < s < 1$. The result follows. ∎

These results can be applied when V is the open set consisting of all points of the plane lying inside a given circle and W is the open set consisting of all points lying outside that given circle. It follows that if a path passes through points inside the circle, and also passes through points outside the circle, then the path must intersect the circle.

On page 235 of Vol. I of his translation of Euclid's Elements, Thomas L. Heath quotes formulations of a *Principle of Continuity* included by the 19th century German mathematician Wilhelm Killing in the second volume (page 43) of his treatise *Einführung in die Grundlagen der Geometrie*, published in 1893:

(a) Suppose a line belongs entirely to a figure which is divided into two parts; then, if the line has at least one point in common with each part, it must also meet the boundary between the parts; or

(b) If a point moves in a figure which is divided into two parts, and if it belongs at the beginning of the motion to one part and at the end of the motion to the other part, it must during the motion arrive at the boundary between the two parts.

We now consider the problem of determining points of intersections of circles from the point of view of the sort of *coordinate geometry* that became established in the 18th century.

Property EP–14 *Let C be the set of points in $\mathbb{R}^2$ lying on a circle of radius r about a point (a, b), and let D be the set of points lying on a circle of radius s about a point (c, d), so that*

$$\begin{aligned} C &= \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 = r^2\}, \\ D &= \{(x, y) \in \mathbb{R}^2 : (x-c)^2 + (y-d)^2 = s^2\}. \end{aligned}$$

Then any points where the circle C intersects the circle D lie on the line

$$2(c-a)x + 2(d-b)y = r^2 - s^2 - a^2 - b^2 + c^2 + d^2.$$

Also any points where the circle C intersects this line are also points where the two circles C and D intersect.

Proof Expanding out the equations for the circles we find that, at the points of intersection of the two circles, the Cartesian coordinates x and y satisfy the simultaneous equations.

$$\begin{aligned} x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2; \\ x^2 + y^2 - 2cx - 2dy + c^2 + d^2 = s^2. \end{aligned}$$

Subtracting one equation from the other and rearranging, we see that any points where the circles intersect must lie on the line

$$2(c-a)x + 2(d-b)y = r^2 - s^2 - a^2 - b^2 + c^2 + d^2.$$

Also subtracting this equation for the line to the equation for the first circle, we obtain the the equation for the first circle, and therefore any points at which the first circle intersects the line with the equation about also lie on the second circle. The result follows. ∎

Example Consider the special case of Property EP–14 where the first circle is a circle of radius r centred on the origin $(0,0)$, and the second circle is a circle of radius s centred on the point $(c,0)$. Then, on applying the result of Property EP–14 with $a = b = d = 0$, we find that any points where the first circle intersects the second circle must lie on the line

$$x = \frac{r^2 + c^2 - s^2}{2c}.$$

Conversely any points where the first circle intersects this line are also points where the first circle intersects the second circle. It follows that the two circles intersect if and only if this line passes through the interior of the first circle.

Now the line passes through the interior of the circle of radius r about the origin if and only if

$$-r < \frac{r^2 + c^2 - s^2}{2c} < r.$$

Thus the circles intersect in two points if and only if the inequalities

$$r^2 + c^2 - 2rc < s^2 \quad \text{and} \quad r^2 + c^2 + 2rc > s^2$$

are both satisfied, in which case the coordinates of the points of intersection are (u,v) and $(u,-v)$ where

$$u = \frac{r^2 + c^2 - s^2}{2c} \quad \text{and} \quad v = \sqrt{r^2 - u^2}.$$

Thus the circles intersect in two points if and only if both

$$(r-c)^2 < s^2 \quad \text{and} \quad (r+c)^2 > s^2,$$

and this is the case if and only if the three inequalities

$$s + c > r, \quad s + r > c \quad \text{and} \quad r + c > s$$

are simultaneously satisfied.

A subset L of the set $\mathbb{R}$ of real numbers is said to be a *subfield* of $\mathbb{R}$ if $0 \in L$, $1 \in L$, $x + y \in L$, $x - y \in L$ and $xy \in L$, $x/y \in L$ for all $x, y \in L$ for which $y \neq 0$.

Property EP–15 *Let L be a subfield of the field $\mathbb{R}$ of real numbers, and let (a, b), (c, d), (e, f) and (g, h) be points of $\mathbb{R}^2$, where the Cartesian components a, b, c, d, e, f, g and h belong to the subfield L of $\mathbb{R}$. Suppose that the circle centred on (a, b) and passing through the point (e, f) intersects the circle centred on (c, d) and passing through (g, h) at points (m, n) and (p, q). Then each of the real numbers m, n, p and q can be expressed in the form $u + \sqrt{v}$, where u and v belong to the subfield L of $\mathbb{R}$.*

Proof The determination of the point of intersection of the straight line joining (a, b) to (c, d) and the straight line joining (m, n) to (p, q) involves solving a pair of simultaneous linear equations in two real unknowns with coefficients in the subfield L of $\mathbb{R}$. The standard formulae for the solution of such simultaneous linear equations ensure that the Cartesian components of the point of intersection of these two straight lines belong to L. The determination of the points of intersection themselves then finding roots of quadratic polynomials with coefficients in L. The result follows. ∎

Let $\mathcal{C}$ be the collection consisting of all subfields L of the field of real numbers with the property that $\sqrt{x} \in L$ for all $x \in L$ satisfying $x \geq 0$, and let $\mathbb{K}$ be the intersection of all subfields of $\mathbb{R}$ that belong to the collection $\mathcal{C}$. Then $\mathbb{K}$ is itself a subfield of $\mathbb{R}$. It is the field of *constructible numbers.*

The field $\mathbb{K}$ of constructible numbers may be characterized as the smallest subfield L of the field of real numbers that satisfies the following property: $\sqrt{x} \in L$ for all $x \in L$ satisfying $x \geq 0$.

The following result follows from Property EP–15

Property EP–16 *Let (a, b), (c, d), (e, f) and (g, h) be points of $\mathbb{R}^2$, where the Cartesian components a, b, c, d, e, f, g and h belong to the field $\mathbb{K}$ of constructible numbers. Suppose that the circle centred on (a, b) and passing through the point (e, f) intersects the circle centred on (c, d) and passing through (g, h). Then the Cartesian components of the points of intersection belong to the field $\mathbb{K}$ of constructible numbers.*

Let $P = (\cos \frac{\pi}{3}, \sin \frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $Q = (\cos \frac{\pi}{9}, \sin \frac{\pi}{9})$. Then P and Q are points on the unit circle centred on the origin in $\mathbb{R}^2$, the line joining the point P to the origin make an angle of $\frac{\pi}{3}$ radians (60°) with the positive x-axis, and the line joining the point Q to the origin make an angle of $\frac{\pi}{9}$ radians (20°) with the positive x-axis. Moreover the Cartesian components of the point P

both belong to the field $\mathbb{K}$ of constructible numbers. However techniques of abstract algebra involving the theory of algebraic field extensions, the Tower Law, and basic results concerning splitting fields of polynomials can be used to show that the Cartesian components of the point Q do not belong to the field $\mathbb{K}$ of constructible numbers.

Now Property EP–16 can be used to show that if some point of the flat Euclidean plane can be obtained from some given collection of points by means of a ruler and compass construction of the sort that appears frequently in Euclid's *Elements*, and if the Cartesian components of the given points all belong to the field $\mathbb{K}$ of constructible numbers, then the Cartesian components of the point constructed from them also belongs to the field of constructible numbers.

It follows from the results just described that there cannot exist any ruler and compass construction of the type employed in Euclid's *Elements* that provides a geometric construction for trisecting an arbitrary angle in the Euclidean plane.

Suppose that one has a complete set of axioms for planar Euclidean geometry, including not only that axioms, postulates and common notions set out by Euclid but also those implicit in the propositions contained in the first six books of Euclid's *Elements*. Once such a complete set of axioms has been compiled, the propositions of the first six books of Euclid's *Elements* should follow by strict application of principles of pure logic that codify the rules employed by mathematicians for deducing propositions by logical deduction from sets of axioms.

We can then consider *models* for the axioms of planar Euclidean geometry. By definition, these are mathematical structures that satisfy the necessary axioms. One model for plane Euclidean geometry is the Cartesian plane $\mathbb{R}^2$ whose elements are represented as ordered pairs of real numbers, and *points*, *straight lines* and *circles* are defined in the usual fashion.

Another model is provided by the set $\mathbb{K}^2$ of ordered pairs of constructible numbers. In this model one essentially disregards all 'points' other than those that can be constructed from the reference points $(0, 0)$ and $(1, 0)$ by ruler and compass constructions in accordance with the usual rules.

Whilst concepts of 'continuity', 'completeness' and 'connectedness' developed in the 19th century and ubiquitous in the fields of mathematical analysis and topology from that time onwards might be imported into a set of axioms for 'Euclidean' geometry, some might see disadvantages in such an approach. Consider for example the *Peano space-filling curve*: a continuous path parameterized by the unit interval that passes through every point of the closed unit square in the plane. If, for example, one adopts axioms that ensure that any line within the 'Euclidean plane' is a complete metric space, then this might well have the effect of populating all models of those axioms with ‘monsters’ such as the Peano space-filling curve.