MA232A—Euclidean and Non-Euclidean Geometry
School of Mathematics, Trinity College
Michaelmas Term 2017
Section 1: Incidence, Linear Ordering and Pasch's Axiom

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Contents

1 Incidence, Linear Ordering and Pasch's Axiom

1.1 Axioms of Incidence

Incidence Geometry is concerned with geometric systems consisting of collections of points and lines, together with a relation of *incidence* that determines, for any point and any line, whether the point and line are *incident*.

In what follows, we denote points by roman capital letters A, B, C, etc., and we denote lines by and lines by lower-case roman letters l, m, n, etc.

If, in a system of incidence geometry, a point A and a line l are incident, then this relation may be indicated through a variety of phrases in common use: the point A may be said to *lie on* the line l; the point A may be said to *belong to* the line l; the line l may be said to pass through the point A.

Axiom AFPG-1 *There is exactly one line passing through any two given points.*

We denote the unique line through two points A and B by AB.

Definition If three or more points lie on a single line, we say that they are **collinear**.

Axiom AFPG-2 *Any line in a plane passes through at least two distinct points of that plane.*

The following proposition is essentially merely a restatement of the uniqueness requirement incorporated in Axiom (AFPG-1).

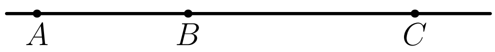
Proposition 1.1 *Any two distinct lines have at most one point of intersection.*

Proof Let l and m be distinct lines. Suppose that there were to exist distinct points A and B that were points of intersection of the lines l and m. Then both the lines l and m would coincide with the unique line AB passing through the points A and B (see Axiom (AFPG-1)). Therefore the lines l and m would coincide with one another, contradicting the requirement that these lines be distinct. The result follows. ∎

1.2 Axioms of Linear Ordering

We suppose that there is a relation of *betweenness* satisfied by points on a line. This relation is a *ternary relation* that determines, given three points A, B and C, whether or not the point B lies *between* the points A and C. The following axioms capture formally various basic properties of such a "betweenness" relation.

Axiom AFPG-3 *If A, B and C are points, and if B lies between A and C then the points A, B, C are distinct.*

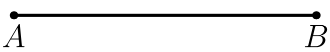


Axiom AFPG-4 *If A, B and C are points, and if B lies between A and C, then B lies between C and A.*

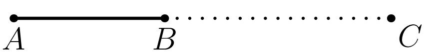
Axiom AFPG-5 *If A, B and C are distinct points, then at most one of those points lies between the other two.*

Axiom AFPG-6 *If A, B and C are distinct points, then those points are collinear if and only if one of those points lies between the other two.*

Definition Let A and B be distinct points. The *line segment* $[AB]$ with *endpoints* A and B consists of the endpoints A and B together with all points that lie between A and B.

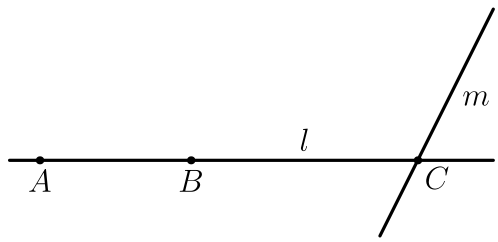


Axiom AFPG-7 *Given distinct points A and B, there exists a point C distinct from A and B such that the point B lies between A and C.*



In the situation described in the statement of Axiom (AFPG-7), where, given two distinct points A and B, a point C has been introduced for which A, B and C are collinear and B lies between A and C, then we say that the line segment $[AB]$ has been *produced* beyond B to the point C.

Proposition 1.2 *Let A, B and C be distinct points on a line l, where the point B lies between A and C, and let m be a line distinct from the line l. Suppose that the lines l and m intersect at the point C. Then the line m does not meet the line segment $[AB]$.*



Proof The lines l and m intersect at no more than one point (Proposition 1.1). It follows that the lines l and m cannot intersect at any point other than the point C. The point B lies between A and C, and therefore the point C does not lie between A and B. (Axiom (AFPG-5)). Nor does the point C coincide with either of the points A and B. It follows that the line m cannot intersect the line l at any point lying on the line segment $[AB]$. The result follows. ∎

Definition Let A and B be distinct points. The *ray* $[AB$ *starting at* the point A and passing through the point B consists of the points A and B together with all points C for which either C lies between A and B or else B lies between A and C.

Proposition 1.3 *Let A and B be distinct points, and let C be a point. Then the point C belongs to the ray $[AB$ if and only if the points A, B and C are collinear but the point A does not lie between B and C.*

Proof Suppose that C belongs to the ray $[AB$. If C coincides with A or B then the three points A, B and C are collinear. Suppose therefore that the three points A, B and C are distinct. Then either B lies between A and C or else C lies between A and B, and therefore the three points are collinear (Axiom (AFPG-6)). Also at most one of the points A, B and C can lie between the other two (Axiom (AFPG-5)), and, by the definition of rays, that point must be B or C. It follows that the point A does not lie between the points B and C.

Conversely suppose that the three points A, B and C are collinear, but the point A does not lie between B and C. The points A and B are distinct. Therefore if the points A, B and C are not distinct then the point C must coincide with one or other of the points A and B, and therefore, in this case, the point C lies on the ray $[AB$. If the points A, B and C are not distinct then one of them must lie between the other two (Axiom (AFPG-6)). But the point A does not lie between B and C. Therefore either B lies between A and C or else C lies between A and B, and therefore, in this case also, the point C lies on the ray $[AB$. This completes the proof. ∎

1.3 The Ordering of Points on a Single Line

Line Ordering Axiom AFLIN-A *There are at least two points on any line.*

The above axiom corresponds to Axiom AFPG-2

Line Ordering Axiom AFLIN-B *If A, B and C are points, and if B lies between A and C then the points A, B, C are distinct.*

Line Ordering Axiom AFLIN-C *If A, B and C are points, and if B lies between A and C, then B lies between C and A.*

The above two axioms merely reproduce the content of Axioms (AFPG-3) and (AFPG-4).

Line Ordering Axiom AFLIN-D *If A, B and C are distinct points on a line, then exactly one of those points lies between the other two.*

This axiom merely replaces the combination of Axioms (AFPG-6) and (AFPG-5) when studying betweenness relations satisfied by points that all lie on a single line.

Line Ordering Axiom AFLIN-E *Given distinct points A and B on a line, there exists a point C on that line distinct from A and B such that the point B lies between A and C.*

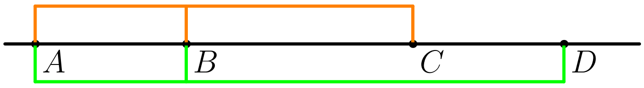
Line Ordering Axiom AFLIN-F *Given any two distinct points A and B on a line, there exists a point on that line distinct from A and B that lies between A and B.*

For a proof that the points on a line in a plane satisfy Line Ordering Axiom AFLIN-F provided that the points and lines in that plane satisfy Axioms AFPG-1 to AFPG-7 above, together with Axiom (AFPG-8) and Pasch's Axiom (Axiom (AFPG-9)), stated below, see Proposition 1.7 below.

Line Ordering Axiom AFLIN-G *If A, B, C and D are distinct points, and if the point B lies between A and C and the point C lies between A and D, then the point C lies between B and D and the point B lies between A and D.*

For a proof that the points on a line in a plane satisfy Line Ordering Axiom AFLIN-G provided that the points and lines in that plane satisfy Axioms AFPG-1 to AFPG-7 above, together with Axiom (AFPG-8) and Pasch's Axiom (Axiom (AFPG-9)), stated below, see Proposition 1.9 below.

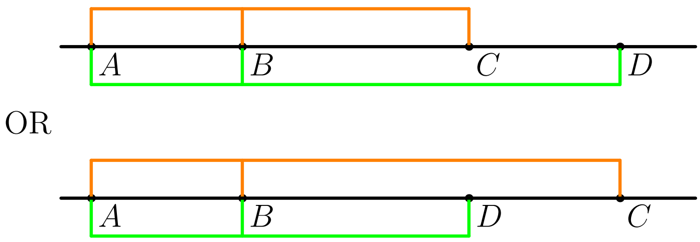
Line Ordering Axiom AFLIN-H *If A, B, C and D are distinct points, and if the point B lies between A and C and also lies between A and D, then the point B does not lie between C and D,*



For a proof that the points on a line in a plane satisfy Line Ordering Axiom AFLIN-H provided that the points and lines in that plane satisfy Axioms AFPG-1 to AFPG-7 above, together with Axiom (AFPG-8) and Pasch's Axiom (Axiom (AFPG-9)), stated below, see Proposition 1.10 below.

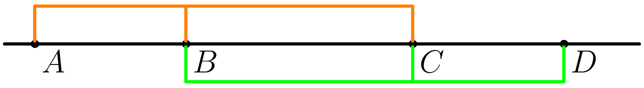
We refer to axioms (AFLIN-A) to (AFLIN-H) as the *Line Ordering System*

Proposition 1.4 *In a system of points on a line that satisfies the Line Ordering Axioms, given distinct points A, B, C and D on the line, if the point B lies between A and C and also lies between A and D, then either the point C lies between B and D or else the point D lies between B and C.*



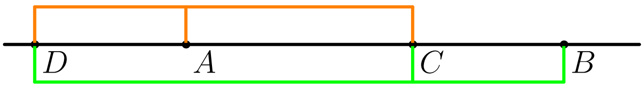
Proof Line Ordering Axiom (AFLIN-H) ensures that the point B does not lie between C and D. But the points C and D both lie on the line AB (Axioms (AFPG-1) and (AFPG-6)), and therefore the points B, C and D are collinear. It then follows from Axioms (AFPG-5) and (AFPG-6) that either C lies between B and D or else D lies between B and C. ∎

Proposition 1.5 *In a system of points on a line that satisfies the Line Ordering Axioms, given distinct points A, B, C, D on a line, if the point B lies between A and C and the point C lies between B and D then the points B and C both lie between A and D.*



Proof Line Ordering Axiom (AFLIN-D) ensures that exactly one of the three points A, C and D lies between the other two.

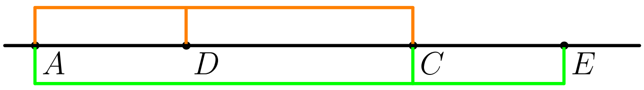
Suppose that the point A were to lie between C and D. The conditions of the proposition require that the point C lies between B and D. It then follows from Axiom (AFLIN-C) that the point C lies between D and B. Line Ordering Axiom (AFLIN-G) would then ensure that the point C lay between A and B.



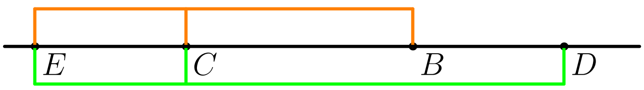
But this would contradict Axiom (AFLIN-D) because the conditions of the proposition require that B lies between A and C. We conclude therefore that the point A cannot lie between C and D.

Suppose that the point D were to lie between A and C. The conditions of the proposition would then ensure that both B and D would lie between A and C. Line Ordering Axiom (AFLIN-E) ensures the existence of a point E such that the point C lies between A and E. Line Ordering Axiom (AFLIN-G) would then ensure that the point C lay both between B and E and also between D and E.

AND



Line Ordering Axiom (AFLIN-H) would then ensure that the point C did not lie between B and D, contradicting the requirements of the proposition.



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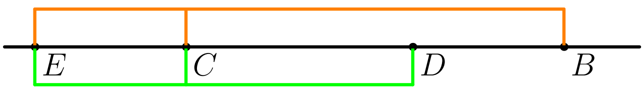
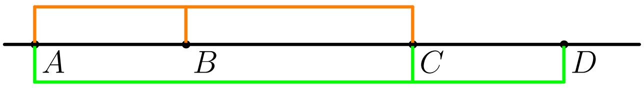


Therefore the point D cannot lie between A and C.

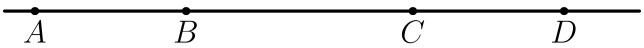
However one of the three points A, C, D must lie between the other two. We have shown that A cannot lie between C and D, and also that D cannot lie between A and C. It follows therefore that the point C must lie between A and D.

We have just shown that the point C lies between A and D. The conditions of the proposition require that B lies between A and C. Line Ordering Axiom (AFLIN-G) therefore ensures that the point B lies between A and D.



This completes the proof. ∎

Proposition 1.6 *In a system of points on a line that satisfies the Line Ordering Axioms, given four distinct points on a line, those points can be labelled as A, B, C and D so as to ensure that the point B lies between A and C and between A and D and the point C lies between A and D and between B and D.*



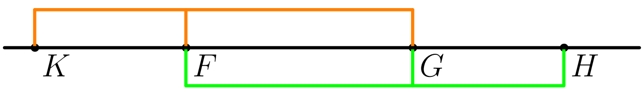
Proof First select three of those points. One of those those three must lie between the other two. Label the three points as F, G and H so as to ensure that the point G lies between the points F and H.

Let K label the point not selected. Then one of the three points F, H and K lies between the other two. If H lies between F and K then we can relabel F and H as H and F respectively so as to ensure that G still lies between F and H and F lies between H and K. After relabelling F and H in this fashion, if necessary, we now have four points labelled F, G, H and K that satisfy exactly one of the two following conditions:—

(i) G and K both lie between F and H;

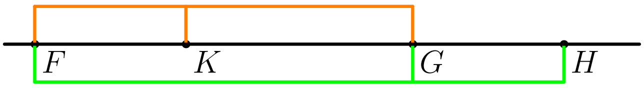
(ii) G lies between F and H, and F lies between H and K.

First consider the case where F lies between K and G. The points have been labelled so that G lies between F and H. It therefore follows from Proposition 1.5 that F and G both lie between K and H.



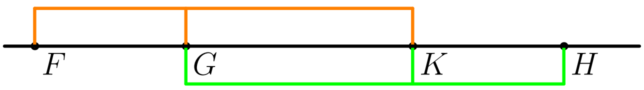
In this case the required conditions are satisfied on labelling K, F, G and H as A, B, C and D respectively.

Next consider the case when K lies between F and G. The points have been labelled so that G lies between F and H. It therefore follows from Line Ordering Axiom AFLIN-G that K lies between F and H and G lies between K and H.



In this case the required conditions are satisfied on labelling F, K, G and H as A, B, C and D respectively. Finally consider the case when G lies between F and K The points have been labelled so that G lies between F and H. It therefore follows from Proposition 1.4 that either K lies between G and H or else H lies between F and K.

Now were it the case that H lies between F and K then Axiom (AFLIN-D) would ensure that F would not lie between H and K, excluding case (ii) above, and K would not lie between F and H, excluding case (i) above. Therefore the manner in which vertices have been labelled excludes the possibility that H lies between F and K, and therefore K lies between G and H. But also G lies between F and K. It therefore follows from Proposition 1.5 that K lies between F and H.



In this case the required conditions are satisfied on labelling F, G, K and H as A, B, C and D respectively. This completes the proof. ∎

1.4 Planar Geometry and Pasch's Axiom

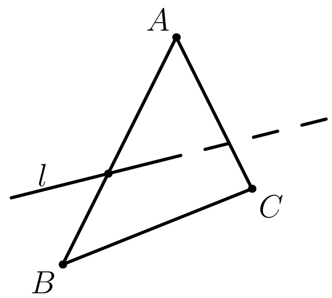
In what follows, we restrict attention to systems of *planar geometry*, in which all points and lines are considered to belong to some given plane.

Axiom AFPG-8 *The points of a plane do not all lie on a single line.*

Remark Most of the axioms found in formal theories of planar geometry are typically either satisfied by the system of points lying on a single line or else are inapplicable to collections of non-collinear points. Therefore an axiom such as Axiom (AFPG-8) is required in order to distinguish planar geometry from axiomatic systems describing one-dimensional geometry.

The following axiom, formulated by Moritz Pasch (1943–1930), encapsulates, in a statement concerning the intersections of lines and triangles, a basic property of flat planes. The introduction of this axiom can be justified on the basis that any line in a plane that does not pass through any of the vertices of a given triangle should separate the plane into two sides, and that, as a result, if the line intersects any edge of the triangle, then two vertices of the triangle should lie on one side of the line, and the remaining vertex on the other.

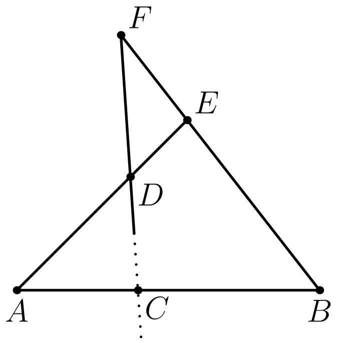
Axiom AFPG-9 (Pasch's Axiom) *Let a triangle in a plane be given. Suppose that a line in the same plane does not pass through any of the vertices of this triangle but does meet at least one side of the triangle. Then the line meets at least two sides of the triangle.*



In what follows in this account of axiomatic planar geometry, unless explicitly stated to the contrary, it will be assumed that, when considering properties of points and lines in a single given plane, the systems of points and lines within that given plane satisfy Axioms (AFPG-1) to (AFPG-9). Propositions that can be shown to be logical consequences of those axioms will therefore establish, by logical reasoning from stated axioms, basic properties of systems of points and lines that are confined to a single plane.

Proposition 1.7 *Let A and B be distinct points of a plane. Then there exists a point C of the plane distinct from A and B that lies between A and B.*

Proof The points of the plane do not all lie on a single line (Axiom (AFPG-8)). Therefore exists a point D of the plane l that does not lie on the line AB. The line segment $[AD]$ can then be produced to some point E so that the point D lies between the points A and E (Axiom (AFPG-7)). The line segment $[AE]$ can then be produced to a point F so that the point E lies between the points B and F (Axiom (AFPG-7) again).



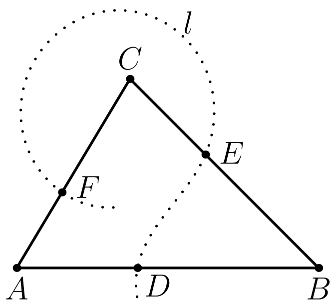
Consider the unique line FD that passes through the points F and D. This line intersects the unique line passing through the points B and E at the point F, and the point E lies between B and F. Therefore line FD does not meet the side $[BE]$ of the triangle $\triangle ABE$ (Proposition 1.2). In particular, the line $[FD]$ does not pass through either the point E or the point B. If the line FD were to pass through the point A, then it would pass through the point E, because A, D and E are collinear. But the line FD does not pass through the point E. Therefore this line does not pass through the point A. We have now shown that the line FD does not pass through any of the vertices of the triangle $\triangle ABE$, and also that this line does not meet the side $[BE]$ of that triangle. But this line meets the side $[AE]$ of the triangle $\triangle ABE$ at the point D. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line $[BE]$ meets the third side $[AB]$ of the triangle $\triangle ABE$ at a point between A and B. Let C be the point of intersection of the lines FD and AB. Then the points A, B and C are distinct, and the point C lies between A and B, as required. ∎

Proposition 1.8 *Let a triangle in a plane be given. Suppose that a line in the same plane does not pass through any of the vertices of this triangle but does meet at least one side of the triangle. Then the line meets exactly two sides of the triangle.*

Proof Let l be a line in the given plane that does not pass through any of the vertices of a given triangle. Pasch's Axiom (Axiom (AFPG-9)) ensures that the line meets at least two sides of this triangle. It remains to prove that the line cannot meet all three sides of the triangle.

Suppose to the contrary that the line l were to meet all three sides of the triangle. Then the three points at which the line l met the sides of the triangle could be ordered and labelled as D, E, F so that the point E lay between D and F. The sides of the given triangle containing the points D and F would meet at a vertex of the triangle: let this vertex be labelled as A.

Let the other vertices then be labelled as B and C so as to ensure that the point D lies on the side $[AB]$ and the point F on the side $[AC]$ of the triangle $\triangle ABC$. The point E would then line on the side $[BC]$ of this triangle.



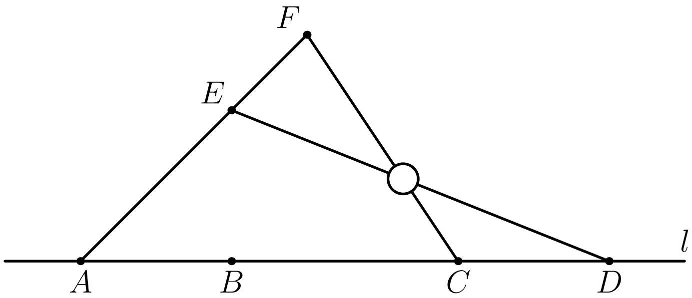
Now consider the points of intersection (if any) of the line BC with the sides of the triangle $\triangle ADF$. The point D would lie between the point A and the point B of intersection of the lines BC and AD. Therefore the line BC could not meet the side $[AD]$ of the triangle $\triangle ADF$ (see Proposition 1.2). Similarly the line BC could not meet the side $[AF]$ of the triangle $\triangle ADF$. The line BC would however meet the side $[DF]$ of this triangle, because it would pass through a point E lying between D and F. Thus the line l would meet exactly one side of the triangle $\triangle ADF$. But this would contradict Pasch's Axiom (Axiom (AFPG-9)), which requires that a line not passing through any vertex of a triangle must meet at least two of its sides. It follows that, in a theory of plane geometry satisfying Axioms (AFPG-1) to (AFPG-9), it is impossible for a line not passing through any vertex of a triangle to meet all three sides of that triangle. The result follows. ∎

1.5 Linear Ordering on Lines in Planes

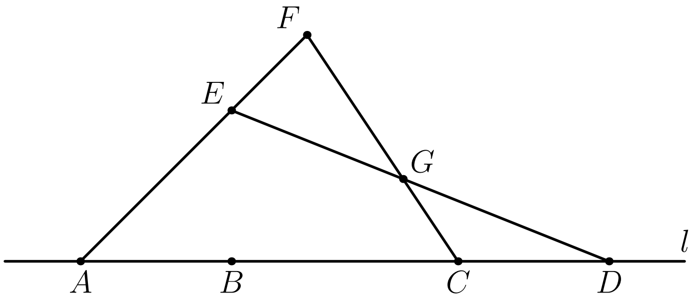
Proposition 1.9 *Let A, B, C and D be distinct points in a given plane. Suppose that the point B lies between A and C and that the point C lies between A and D. Then the point C lies between B and D and the point B lies between A and D.*

Proof There exists a unique line l that passes through the points A and C (Axiom (AFPG-1)). The points A, B and C are collinear, because B lies between A and C (Axiom (AFPG-6)). Therefore the point B lies on the line l. Also the points A, C and D are collinear, because the point C lies between A and D (Axiom (AFPG-6)). Therefore the point D also lies on the line l. We conclude therefore that the points A, B, C and D all lie on the line l.

The given plane also contains points that do not lie on the line l (Axiom (AFPG-8)). Let E be a point of the given plane that does not lie on the line l. The line segment $[AE]$ may then be produced to a point F so that the point E lies between A and F (Axiom (AFPG-7)). The line CF then intersects the line passing through the points A and E at the point F, and the point E lies between A and F. Therefore the line CF does not meet the side $[AE]$ of the triangle $\triangle ADE$. (Proposition 1.2).

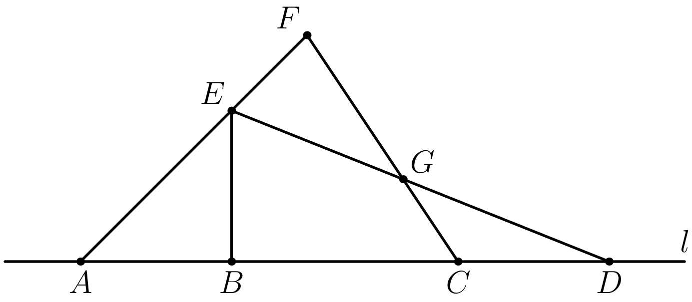


However the line CF does intersect the line segment $[AD]$ at the point C, because the unique point C at which the line CF meets the line l through the points A, B, C and D is required by the conditions of the proposition to lie between A and D. The line CF thus meets the side $[AD]$ of the triangle $\triangle ADE$, but does not meet the side $[AE]$ of that triangle. Now Pasch's Axiom (Axiom (AFPG-9)) requires that the line CF meet at least two sides of the triangle $\triangle ADE$. Therefore the line CF must meet the side $[DE]$ of the triangle $\triangle ADE$. Let the line CF meet the line DE at the point G. Then the point G lies between D and E.



Next we observe that the line DE does not meet the side $[AC]$ of the triangle $\triangle ACF$, because the vertex C of that triangle lies between the vertex A and the point D at which the line DE intersects the line l (Proposition 1.2). But it meets the side $[AF]$ of this triangle at the point E, because E lies between A and F. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line DE meets the side $[CF]$ of the triangle $\triangle ACF$, and therefore the point G where the lines DE and CF intersect must lie between C and F.

We have now shown that the lines DE and CF intersect at a point G that lies between C and F and also lies between D and E.

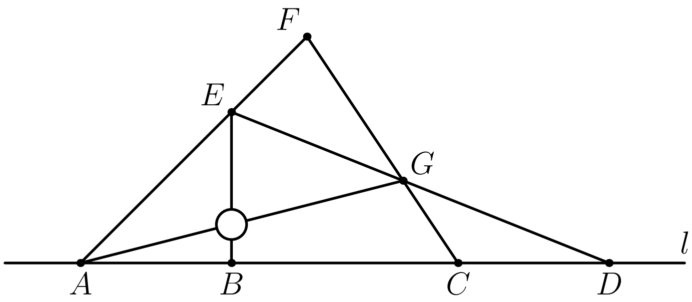


Next we consider the relationship of the line CF to the triangle $\triangle ABE$. The vertex B of that triangle lies between the vertex A and the point C where the line CF intersects the line l through A and B. It follows that the line CF does not meet the side $[AB]$ of the $\triangle ABE$ (Proposition 1.2). Similarly the vertex E of that triangle lies between the vertex A and the point F at which the line CF meets the line through A and E. It follows that the line CF does not meet the side $[AE]$ of the $\triangle ABE$. Pasch's Axiom (Axiom (AFPG-9)) requires that any line that does not pass through any vertex of a triangle but meets at least one side must meet at least two sides of that triangle. It follows that the line CF cannot meet the side $[BE]$ of the triangle $\triangle ABE$.

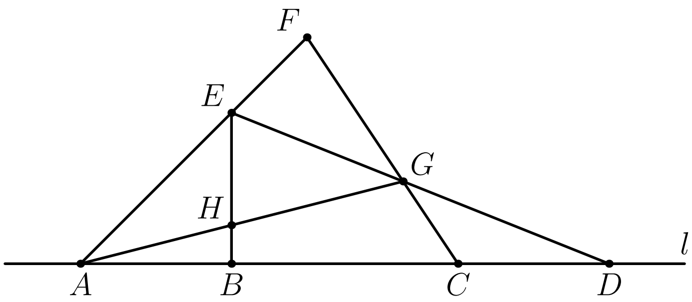
But the line segment $[BE]$ is also a side of the triangle $\triangle BDE$, and the line CF meets the side DE of this triangle at the point G. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line CF must meet the remaining side $[BD]$ of this triangle. Therefore the point C must lie between the points B and D. This is one of the results that we set out to prove.

It remains to show that the point B lies between the points A and D. First we show that the line BE does not meet any edge of the the triangle $\triangle CDG$. Now the line BE intersects the line through the points C and D at the point B, and we have just shown that the point C lies between the points B and D. Therefore the line BE does not meet the edge $[CD]$ of the triangle $\triangle CDG$ (Proposition 1.2). Also the point G lies between the points E and D, and therefore the line BE does not meet the edge $[DG]$ of the triangle $\triangle CDG$. The line BE does not pass through any vertex of this triangle. It therefore follows from Pasch's Axiom (Axiom (AFPG-9)) that the line $[BE]$ does not meet the edge $[CG]$ of the triangle $\triangle CDG$, because if the line were to meet at least one edge, then it would have to meet at least two edges of the triangle.

Now $[CG]$ is also an edge of the triangle $\triangle ACG$, and the line BE meets the edge $[AC]$ of this triangle at the point B, because the conditions of the proposition require the point B to lie between the points A and C.



It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line BE must also meet the edge $[AG]$ of the triangle $\triangle ACG$. Let H be the point where the lines BE and AG intersect. Then the point H lies between A and G.

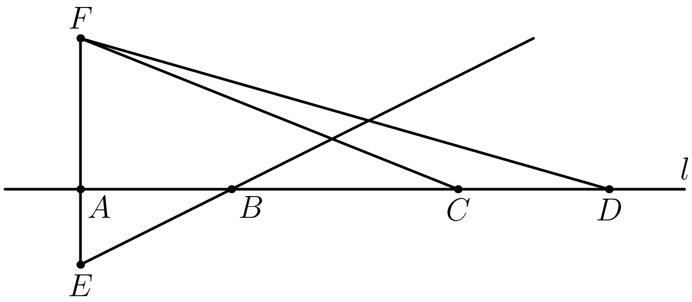


Now we have already noted that the line BE does not meet the line segment $[DG]$. Thus the line $[BE]$ meets the side $[AG]$ of the triangle $\triangle ADG$, but does not meet the side $[DG]$ of this triangle. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line BE meets the side $[AD]$ of the triangle $\triangle ADG$. But the line BE meets the unique line l through the points A and D at the point B. Therefore the point B lies between the points A and D. This completes the proof. ∎

Proposition 1.10 *Let A, B, C and D be distinct points in a given plane. Suppose that the point B lies between A and C and also lies between A and D. Then the point B does not lie between C and D,*

Proof Let l be the unique line AB passing through the points A and B. Then the points C and D lie on the line l, because A, B and C are collinear and A, B and D are collinear (Axiom (AFPG-6)). We conclude therefore that the points A, B, C and D all lie on the line l.

The given plane also contains points that do not lie on the line l (Axiom (AFPG-8)). Let E be a point of the given plane that does not lie on the line l. The line segment $[EA]$ may then be produced to a point F so that the point A lies between E and F (Axiom (AFPG-7)).

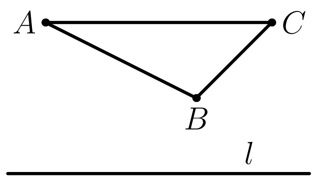


Now the line EB does not pass through any of the points F, A. C or D. Also the line EB does not meet the line segment $[AF]$ because the the point A lies between the point F and the point E at which the lines EB and AF intersect (Proposition 1.2). But the line EB meets the side $[AC]$ of the triangle $\triangle ACF$ at the point B, because the conditions of the proposition require the point B to lie between the points A and C. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line EB must meet the side $[CF]$ of the triangle $\triangle ACF$. Similarly the line EB must meet the side $[DF]$ of the triangle $\triangle ADF$, because the point B is also required to lie between the points A and D. It is a consequence of Pasch's Axiom that a line that does not pass through any vertex of a triangle cannot meet all three sides of that triangle (Proposition 1.8). We conclude therefore that the line EB cannot meet the side $[CD]$ of the triangle $\triangle CDF$. It follows that the point B cannot lie between the points C and D. This completes the proof. ∎

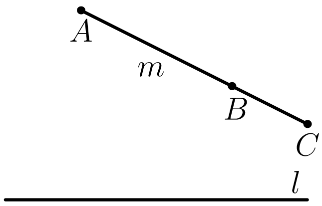
1.6 The Sides of a Line in a Plane

Proposition 1.11 *Let l be a line in a given plane, and let A, B and C be three points in the same plane, none of which lie on the line l. Suppose that l does not intersect either of the line segments $[AB]$ and $[BC]$. Then the line l does not intersect the line segment $[AC]$.*

Proof In the case where the points A, B and C are not collinear, those points are the vertices of a triangle $\triangle ABC$. Were the line l to meet at least one side of that triangle, then Pasch's Axiom (Axiom (AFPG-9)) would ensure that the line l would meet at least two sides of the triangle. But, in this case, the line does not meet the sides $[AB]$ and $[AC]$ of this triangle, and therefore it could meet at most one side of the triangle. Therefore the line l cannot meet any side of that triangle, and, in therefore the line l does not intersect the line segment $[AC]$.

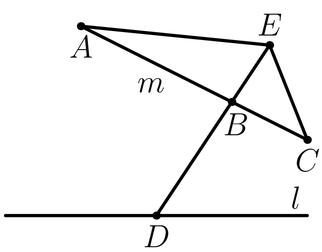


It remains to prove the result in the case when there exists a line m that passes through all three of the points A, B and C. Let m denote the line passing through the points A, B and C.



The lines l and m have at most one point of intersection (Proposition 1.1). But the line l passes through at least two points of the plane (Axiom (AFPG-2)). Therefore there is a point D of the plane that lies on the line l but does not lie on the line m. The line segment $[DB]$ can then be produced beyond B to a point E such that D, B and E are collinear and B lies between D and E (see Axiom (AFPG-7)).

Now the lines DE and m have at most one point of intersection (Proposition 1.1), and that point is the point B, which is distinct from E. Therefore the point E does not lie on the line m. It follows that the points A, B, E are not collinear, and similarly the points C, B, E are not collinear.



Now the line l intersects the line DE at the point D, and the point B lies between D and E. Therefore the line l does not meet the line segment $[BE]$ (Proposition 1.2).

Now the conditions of the proposition ensure that the line l does not intersect the line segment $[AB]$. It follows that the line l does not intersect at least two sides of the triangle ABE. It therefore follows from Pasch's Axiom (Axiom (AFPG-9)) that the line l does not meet any side of the triangle ABE.

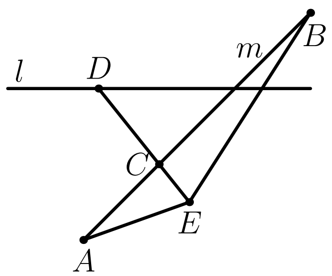
Again, the line l does not intersect the side $[BC]$ of the triangle CBE. Moreover we have already shown that the line l does not meet the side $[BE]$ of this triangle. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line l does not meet any side of the triangle CBE.

We have now shown that the line l cannot meet either of the sides $[AE]$ and $[EC]$ of the triangle ACE. Therefore, applying Pasch's Axiom again, we conclude that the line l cannot meet any side of the triangle ACE. In particular the line l does not meet the line segment $[AC]$. This completes the proof. ∎

Proposition 1.12 *Let l be a line in a given plane, and let A, B and C be three points in the same plane, none of which lie on the line l. Suppose that l intersects both of the line segments $[AB]$ and $[BC]$. Then the line l does not intersect the line segment $[AC]$.*

Proof If the points A, B and C are not collinear then these points are the vertices of a triangle. The result in this case then follows directly from Proposition 1.8.

It only remains to consider the case when the points A, B and C are collinear and the line l does not pass through any of the points A, B or C but meets both $[AB]$ and $[BC]$. In this case let m denote the line that passes through the points A, B and C. This line intersects the line l, and moreover there is exactly one point of intersection (Proposition 1.1). Therefore a point D can be chosen on the line l that does not lie on the line m. The line segment $[DC]$ can then be produced to a point E so that the point C lies between D and E (see Axiom (AFPG-7)). Then the points B, C and E are not collinear, because $[BC]$ is a segment of the line m and the point E does not lie on the line m. Thus B, C and E are the vertices of a triangle.



The point D lies on the line l, and the point C lies between D and E. Therefore the line l does not meet the side $[CE]$ of the triangle $\triangle BCE$ (Proposition 1.2). But the line l intersects the side $[BC]$ of this triangle. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line l intersects the side $[BE]$ of the triangle $\triangle BCE$.

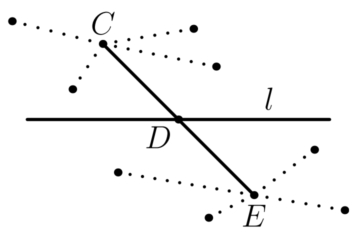
The points A, B and E are not collinear, and are thus the vertices of a triangle. The line l intersects the sides $[AB]$ and $[BE]$ of this triangle. The line l cannot intersect all three sides of the triangle (Proposition 1.8). Therefore the line l cannot intersect the side $[AE]$ of the triangle $\triangle ABE$.

The points A, C and E are not collinear, and are thus the vertices of a triangle. We have shown that the line l cannot intersect the sides $[CE]$ and $[AE]$ of this triangle. But Pasch's Axiom (Axiom (AFPG-9)) ensures that if the line l were to meet at least one side of this triangle, then it would meet at least two sides of the triangle. It follows that the line l cannot meet the third side $[AC]$ of the triangle $\triangle ACE$. The result follows. ∎

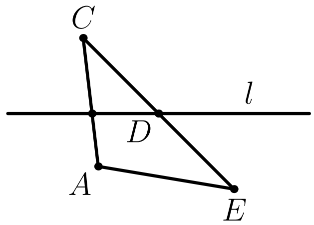
Proposition 1.13 (Plane Separation Property) *Let l be a line in a given plane. Then the line separates those points in the given plane that do not lie on the line l into two classes known as* sides*: two distinct points A and B belonging to the given plane but not lying on the line l belong to the same side of the line l if and only if the line segment $[AB]$ does not intersect the line l.*

Proof It is not possible for all the points of the plane to lie on the line l, for that would contradict Axiom (AFPG-8). Therefore there must exist some point C that does not lie on the line l. There must also exist some point D that lies on the line l (Axiom (AFPG-2)). The line segment $[CD]$ could then be produced to a point E so that the points C, D and E are collinear and D lies between C and E (Axiom (AFPG-7).

Let us define the *side* of the line represented by the point C to consist of all points of the plane that can be joined to C by a line segment that does not intersect the line l. Similarly let us define the the *side* of the line represented by the point E to consist of all points of the plane that can be joined to E by a line segment that does not intersect the line l.



Let A be a point of the plane that does not lie on the line l. If the point A does not belong to the side of the line represented by the point C then the line segment $[AC]$ must intersect the line l. Now the line segment $[CE]$ intersects the line l at the point D. It follows from Proposition 1.12, that the line segment $[AE]$ does not intersect the line l, and therefore the point A lies on the side of the line represented by the point E.



Thus every point of the given plane that does not lie on the line l must belong to one or other of the sides of the line represented by the points C and E.

Now if the point A were to belong to both sides then neither of the line segments $[AC]$ and $[AE]$ would intersect the line l, and therefore, by Proposition 1.11, the line segment $[CE]$ could not intersect the line l. But the line segment $[CE]$ intersects the line l at the point D. We conclude therefore that every point of the given plane that does not lie on the line l must belong to exactly one of the sides of the line represented by the points C and E, and thus the line l does indeed separate the given plane into two sides.

Let A and B be points of the given plane that do not lie on the line l. Suppose that the line segment joining the points A and B does not intersect the line l. If A lies on the side of the line l containing the point C then l does not intersect either of the line segments $[CA]$ and $[AB]$. It therefore follows from Proposition 1.11 that the line l does not intersect the line segment $[CB]$, and thus the point B also lies on the side of the line l that contains the point C. Similarly if A lies on the side of the line l that contains the point E then the point B also lies on the side of l that contains the point E. We conclude therefore that if the line l does not meet the line segment $[AB]$ then the points A and B lie on the same side of the line l.

On the other hand, suppose that the line l meets the line segment $[AB]$. If the point A lies on the side of l that contains the point C then A lies on the opposite side of l to the point E. Therefore l meets the line segments $[AE]$ and $[AB]$. It then follows from Proposition 1.12 that l does not meet the line segment $[BE]$ and therefore the point B lies on the same side of the line l as the point E, and thus lies on the opposite side of the line l to the point A. Similarly if A lies on the same side of l as the point E then B lies on the same side of l as the point C, and thus lies on the opposite side of the line l to the point A. This completes the proof. ∎

Proposition 1.14 *Let l be a line in a given plane, and let A, B and C be distinct points of that plane that are collinear and do not lie on the line l. Suppose also that the point B lies between the points A and C. If the line l meets the line segment $[AB]$ or the line segment $[BC]$ then it also meets the line segment $[AC]$.*

Proof Let D be the point where the line l intersects the line that passes through the points A, B and C. If the point D coincides with A, B or C then the result follows immediately. We may therefore suppose that the points A, B, C and D are distinct.

If the line l intersects $[AB]$ then D lies between A and B. Also B lies between A and C. It therefore follows from Proposition 1.9 that D lies between A and C, and therefore the line l meets the line segment $[AC]$. On replacing A, B, C by C, B, A respectively, it follows that if the line l meets $[BC]$ then it also meets $[AC]$. This completes the proof. ∎

Proposition 1.15 *Let l be a line in a given plane, and let A and C be points that do not lie on the line l but lie on the same side of the line l. Let B be a point of the plane that lies between the points A and C. Then the point B lies on the same side of the line l as the points A and C.*

Proof It follows from Proposition 1.14 that if the line l were to meet the line segment $[AB]$ then it would also meet the line segment $[AC]$. But the line l does not meet the line segment $[AC]$ because the points A and C lie on the same side of the line l. Therefore the line l does not meet the line segment $[AB]$, and therefore the points A and B lie on the same side of the line l. Similarly the points B and C lie on the same side of the line l. ∎

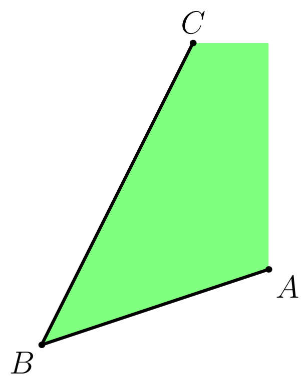
Proposition 1.16 *Let l be a line in a plane, let A be a point lying on the line l and let B be a point that does not lie on the line l. Then all points lying on the ray $[AB$ distinct from the starting point A lie on the same side of the line l as the point B.*

Proof Let C be a point lying on the ray $[AB$ that is distinct from A and B. Then either C lies between A and B or else B lies between A and C. In both cases it follows from Proposition 1.2 that the line segment $[BC]$ does not meet the line l. Therefore the point C lies on the same side of the line l as the point B. The result follows. ∎

1.7 Interiors of Angles and Triangles

Definition Let A, B and C be non-collinear points in a given plane. We define the *interior* of the *angle* $\angle ABC$ to consist of all points of the plane that do not lie on either of the lines AB or BC but lie on the same side of BC as the point A and on the same side of AB as the point C.

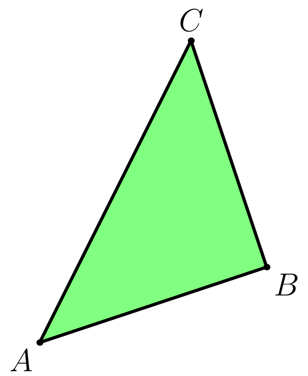
Depicting points of the plane in the usual fashion, the interior of an angle $\angle ABC$ formed by three collinear points is represented by the shaded region in the diagram below.



Proposition 1.17 *Let A, B and C be non-collinear points in a given plane, let D and E be points that lie in the interior of the angle $\angle ABC$, and let F be a point of the plane that lies between the points D and E. Then F lies in the interior of the angle $\angle ABC$.*

Proof It follows from Proposition 1.15 that the point F lies on the same side of the line AB as the points D and E. Moreover the points D and E lie on the same side of the line AB as the point C, because they lie in the interior of the angle $\angle ABC$. Therefore the point F lies on the same side of the line AB as the point C. Similarly the point F lies on the same side of the line BC as the point A. Therefore the point F lies in the interior of the angle, as required. ∎

Definition Let A, B and C be non-collinear points in a given plane. We define the *interior* of the *triangle* $\triangle ABC$ to consist of all points of the plane that do not lie on any of the lines that pass through a pair of vertices of the triangle but lie on the same side of BC as the point A, on the same side of AC as the point B, and on the same side of AB as the point C.



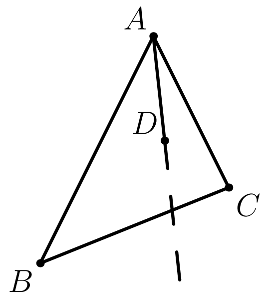
Proposition 1.18 *Let A, B and C be non-collinear points in a given plane, let D and E be points that lie in the interior of the triangle $\triangle ABC$, and let*

F be a point of the plane that lies between the points D and E. Then F lies in the interior of the triangle $\triangle ABC$.

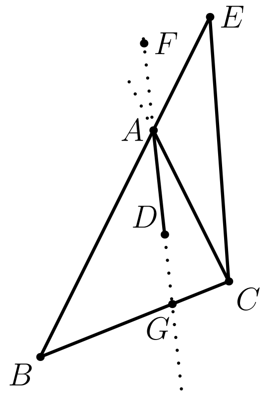
Proof It follows from the definition of the interiors of angles and triangle that a point lies in the interior of the triangle $\triangle ABC$ if and only if it lies in the interiors of both the angle $\angle ABC$ and the angle $\angle ACB$. The result therefore follows directly on applying Proposition 1.17.

1.8 The Crossbar Theorem

Theorem 1.19 (Crossbar Theorem) *Let A, B and C be vertices of a triangle in a plane, and let D be a point of the interior of the triangle $\triangle ABC$. Then the ray $[AD$ starting at the vertex A and passing through the interior point D meets the side $[BC]$ of the triangle $\triangle ABC$.*



Proof Produce the side $[AB]$ of the triangle beyond A to a point E, so that the vertex A of the triangle lies between B and E. Then join E and C.



Now the points of the line segment $[EC]$ distinct from C lie on the same side of the line AC as the point E. (This follows from a straightforward application of Proposition 1.2.) Also the points of the ray $[AD$ distinct from

A lie on the same side of AC as the point D (see Proposition 1.16). It therefore follows from the definition of interiors of angles that all points of the ray $[AD$ distinct from A lie on the same side of AC as the point B. But E and B lie on opposite sides of the line AC. Therefore the points of the line segment $[EC]$ distinct from C lie on the opposite side of AC to the points of the ray $[AD$ distinct from A. Therefore the ray $[AD$ cannot intersect the line segment $[EC]$.

Let F be a point of the line AD that lies on the opposite side of A to the point D. Then points of the ray $[AF$ distinct from A lie on the same side of the line AB as the point F (Proposition 1.16). and therefore lie on the opposite side of the line AB to the point D. Also the points of the line segment $[EC]$ distinct from E lie on the same side of AB as the point C, and thus lie on the same side of the line AB as the point D. Therefore the points of the ray $[AF$ distinct from A lie on the opposite side of AB to the points of the line segment $[EC]$ distinct from E. Therefore the ray $[AF$ cannot intersect the line segment $[EC]$.

Now every point of the line AD lies on one or other of the rays $[AD$ and $[AF$. It follows that the line AD cannot meet the side $[EC]$ of the triangle $\triangle EBC$. But the line AD meets the side $[BE]$ of this triangle at A. It follows from Pasch's Axiom (Axiom (AFPG-9)) that the line AD must meet the third side $[BC]$ of this triangle. Let G be the point where AD meets $[BC]$. Then G lies between B and C, and therefore the line AB does not meet the line segment $[GC]$ (Proposition 1.2). Therefore the point G must lie on the same side of the line AB as the point C. The point D also lies on the same side of AB as the point C, because D lies in the interior of the triangle. Therefore the points D and G lie on the same side of the line AB, and therefore the point G lies on the ray $[AD$, as required. ∎

Remark The essential idea underlying the proof of the Crossbar Theorem given above is that the lines AB and AC partition those points of the plane not lying on either line into four distinct regions. The points of the ray $[AD$ distinct from the starting point A belong to the region consisting of points that lie on the same side of AB as the point C and on the same side of AC as the point B. Points that lie in the interior of the line segment $[EC]$ lie in a second distinct region, namely the region consisting of points that lie on the same side of AB as the point C but lie on the opposite side of AC to the point B. Also points of the ray $[AF$ distinct from the starting point A belong to a third distinct region, namely the region consisting of points that lie on the opposite side of the line AB to the point C and on the opposite side of the line AC to the point B. These regions into which the plane is subdivided following removal of the lines AB and AC are disjoint. Therefore neither of the rays $[AD$ and $[AF$ can meet the line segment $[EC]$, and therefore the line AD cannot meet the line $[EC]$. The proof is then completed by applying Pasch's Axiom to the triangle $\triangle EBC$.

Remark The work of Moritz Pasch (1943–1930) on the foundations of synthetic geometry was extensively developed in the late nineteenth and early twentieth centuries. Results such as *Pasch's Axiom* and the *Crossbar Theorem* encode, in precise language, assumptions that generations of previous mathematicians had essentially taken for granted.

In the later twentieth century mathematicians and computer scientists developed computer programs that generated proofs of theorems of geometry. One does not expect a computer program to reflect on spatial intuition that, for example, incorporate tacitly into proofs the assumption that a straight line that enters the interior of a triangle, by passing through a vertex or crossing an edge, must exit the triangle either by passing through a vertex or crossing an edge. Thus the recognition and formalization of the propositions of "geometrical intuition" represented by Pasch's Axiom and the Crossbar Theorem was an essential prerequisite for the development of computer programs for generating proofs of theorems of synthetic geometry.