The Pentagram

David R. Wilkins

Relevant reference:

Kurt von Fritz, "The Discovery of Incommensurability by Hippasus of Metapontum", *Annals of Mathematics*, Second Series, Vol. 46, No. 2 (1945), pp. 242–264.
Stable URL: `http://www.jstor.org/stable/1969021`

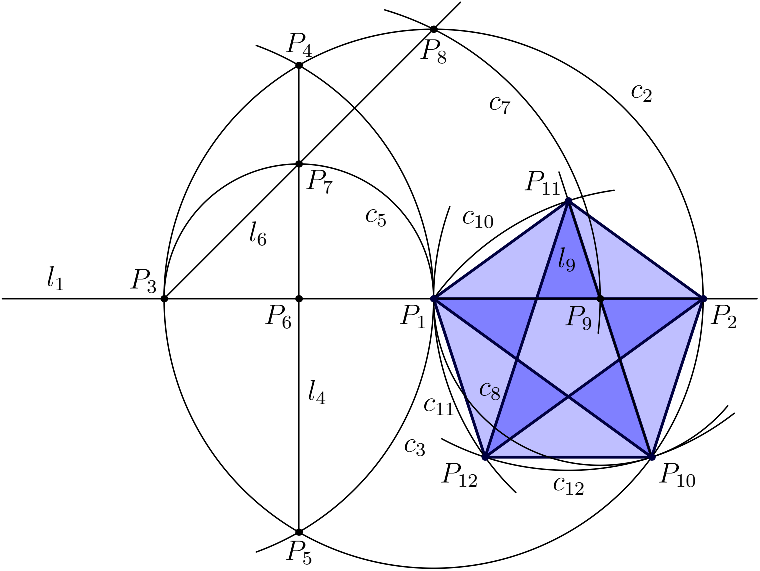
Note on Citations

In what follows, a citation expressed as (*Elements*, I, 47), for example, refers to Proposition 47 in Book I of Euclid's *Elements of Geometry*, and a citation expressed as (*PPG*, Theorem 14), for example, refers to Theorem 14 of *Geometry for post-primary school mathematics*, as specified in the Irish Leaving Certificate Mathematics Syllabus (for examination from 2015). (The abbreviation *PPG* here signifies *post-primary geometry*.)

1.1. Statement of the Problem

We discuss below a method for constructing regular pentagons using straightedge and compass. This is based on relevant propositions in Euclid, including in particular the construction of the “golden ratio” (*Elements*, II, 11), the construction of the “golden triangle” (*Elements*, IV, 10), and the construction of the regular pentagon from the “golden triangle” (*Elements*, IV, 11).

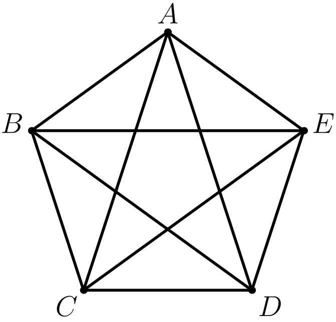
1. The Pentagram (continued)



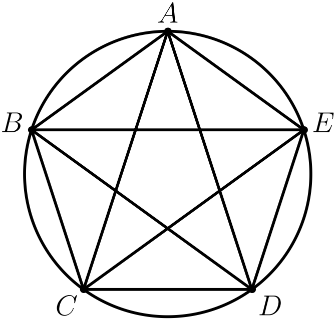
1.2. Angles between the Lines of Regular Pentagrams and Pentagons

Let us suppose that regular pentagons exist. Suppose we are given a regular pentagram in a regular pentagon with vertices A, B, C, D and E in cyclic order as depicted.

Problem: can we use the Alternate Angles Theorem to prove that the lines BE and CD are parallel?

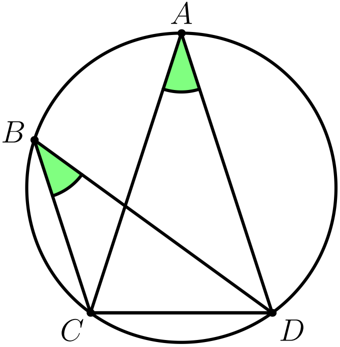


We will need to use the fact that the pentagon is regular. Regular polygons can be inscribed within circles. We suppose therefore that the points A, B, C, D and E lie on a circle, as depicted.



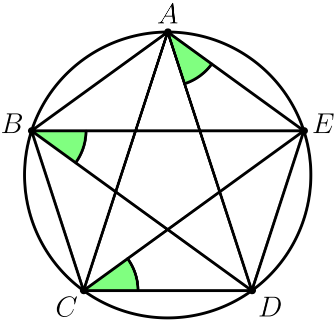
A well-known theorem of Euclidean Geometry (*Elements*, III, 21) (*PPG*, Corollary 2 following Theorem 19) may be stated as follows:

All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, B, C and D lie on a circle, and both A and B are on the same side of the line CD, then $\angle CAD = \angle CBD$.



We now apply this theorem to investigate the angles made by the lines of the regular pentagon and pentagram with one another. First we note that the angles $\angle DAE$ $\angle DBE$ and $\angle DCE$ all stand on the same base $[DE]$, and the points A, B and C lie on the same side of $[DE]$. Therefore

$$\angle DAE = \angle DBE = \angle DCE.$$



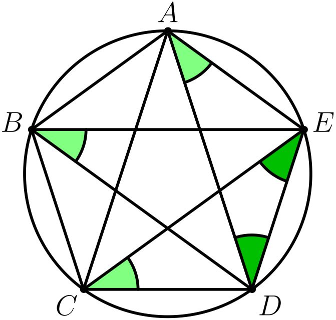
Next we note that

$$\angle DAE = \angle ADE$$

and

$$\angle DCE = \angle DEC$$

by the Isosceles Triangle Theorem (*Elements*, I, v) (*PPG*, Theorem 2).

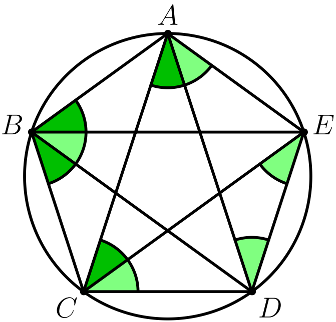


It then follows that the angles on the bases $[AE]$ and $[CD]$ are equal, and therefore

$$\angle ABE = \angle ACE = \angle ADE$$

and

$$\angle CAD = \angle CBD = \angle CED.$$



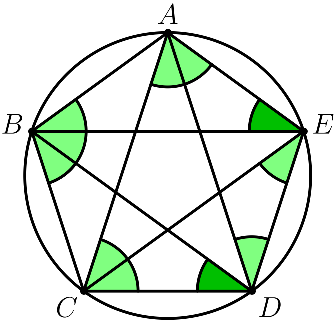
Next we note that

$$\angle ABE = \angle AEB$$

and

$$\angle CBD = \angle BDC$$

by the Isosceles Triangle Theorem (*Elements*, I, v) (*PPG*, Theorem 2).

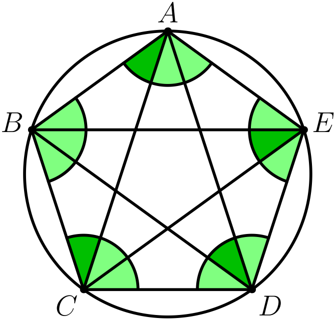


It then follows that the angles on the bases $[AB]$ and $[BC]$ are equal, and therefore

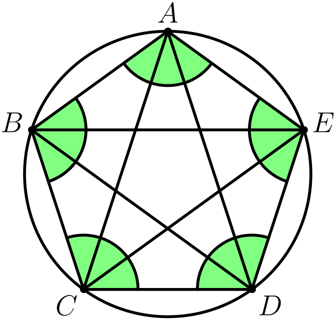
$$\angle ACB = \angle ADB = \angle AEB$$

and

$$\angle BAC = \angle BEC = \angle BDC.$$

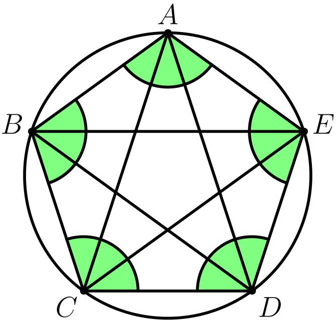


It follows that all the angles (shaded in green in the accompanying figure) of the small triangles that lie at the vertices of the pentagon $ABCDE$ (and are not divided by lines of the pentagram) are equal to one another.

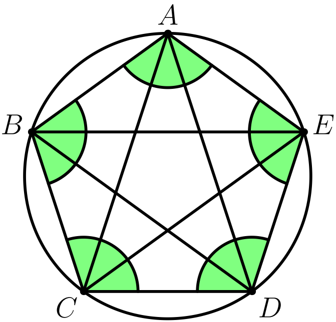


It now follows from the Alternate Angles Theorem (*Elements*, I, 27) (*PPG*, Theorem 3) that

the lines BE and CD are parallel to one another,
the lines CA and DE are parallel to one another,
the lines DB and EA are parallel to one another,
the lines EC and AB are parallel to one another,
the lines AD and BC are parallel to one another.



The results so far obtained yield important guidance to aid the quest for a construction of the regular pentagon. Consider the triangle $\triangle ACD$. This is an isosceles triangle whose vertices are three of the vertices of the regular pentagon (assuming that this regular pentagon actually exists). The two equal angles $\angle ACD$ and $\angle ADC$ at the base of the triangle $\triangle ACD$ are double the angle $\angle CAD$ at the apex of the triangle. Moreover the lines bisecting the two equal base angles of the triangle pass through the remaining two vertices B and E of the pentagon.

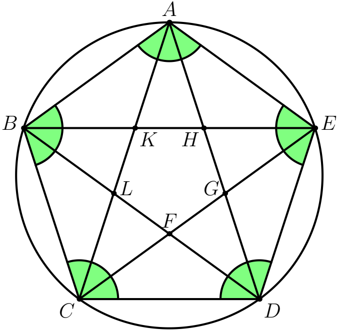


Thus if we can discover a straightedge and compass construction that yields an isosceles triangle $\triangle ACD$ in which the two equal angles $\angle ACD$ and $\angle ADC$ at the base are twice the remaining angle $\angle CAD$, then we ought to be able to construct the remaining two vertices B and E of a regular pentagon by bisecting the angles at the base, and locating the points B and E on the resultant rays from C and D whose distance from C and D respectively is equal to the length of the two equal edges of the isosceles triangle $\triangle ACD$.

This is in fact the construction that Euclid uses in Proposition 11 of Book IV of the *Elements of Geometry* to construct a regular pentagon inscribed in a circle, given an isosceles triangle $\triangle ACD$ inscribed within that circle whose base angles are double the remaining angle.

We now label the vertices of the inner pentagon formed at the intersections of the pentagram lines as F, G, H, K and L, as shown, and we consider the angles made by the triangles in the figure at these vertices.

The results already obtained show that the quadrilateral $ABFE$ is a parallelogram. Opposite sides and angles of parallelograms are equal (*Elements*, I, 34) (*PPG*, Theorem 9).

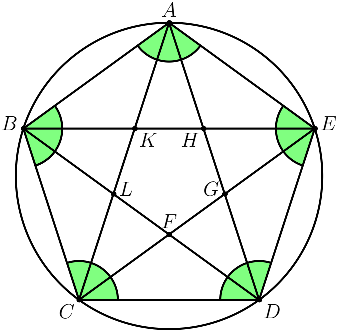


It follows that

$$\angle BFE = \angle BAE,$$

and therefore

$$|\angle BFE| = 3 \times |\angle BAC|.$$

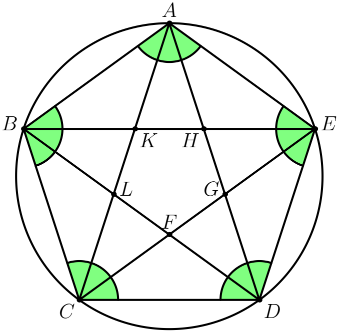


Analogous equalities hold at the other four vertices of the inner pentagon. Thus

$$|\angle BFE| = |\angle CGA| = |\angle DHB| = |\angle EKC| = |\angle ALD| = 3\times|\angle BAC|.$$

Next we note that $\angle CFD = \angle BFE$, because these angles are vertical angles at the vertex F (*Elements*, I, 15) (*PPG*, Theorem 1), and therefore

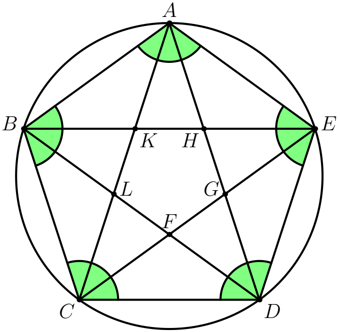
$$|\angle CFD| = 3 \times |\angle BAC|.$$



Analogous equalities hold at the other four vertices of the inner pentagon. Thus

$$|\angle CFD| = |\angle DGE| = |\angle EHA| = |\angle AKB| = |\angle BLC| = 3\times|\angle BAC|.$$

We now consider the remaining angles at the vertices of the inner pentagon. These angles are exterior angles of other isosceles triangles. Thus, for example, $\angle BKC$ is an exterior angle of the isosceles triangle $\triangle KAB$ and is therefore equal to twice the interior opposite (or remote) angle $\angle BAC$ (*Elements*, I, 32) (*PPG*, Theorem 6),

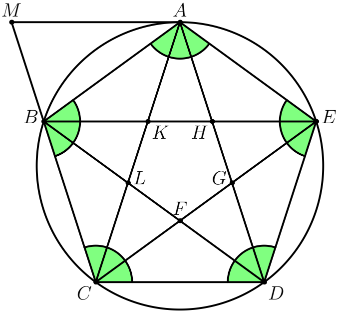


Analogous equalities hold at the other four vertices of the inner pentagon. Thus

$$
\begin{aligned}
|\angle BKC| &= |\angle CLD| = |\angle DFE| = |\angle EGA| = |\angle AHB| = |\angle BLA| \\
&= |\angle CFB| = |\angle DGC| = |\angle DFE| = |\angle EGA| = 2 \times |\angle BAC|.
\end{aligned}
$$

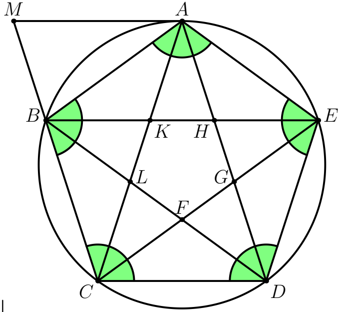
We have now shown that every angle of every triangle in the figure is equal either to the angle $\angle BAC$ itself, or to twice that angle, or to three times that angle. Moreover the interior angles of the outer and inner pentagons are equal to three times the angle $\angle BAC$.

We can locate even more angles that are equal to or are multiples of the angle $\angle BAC$. Let the tangent line to the circle at the point A intersect the line through the vertices B and C of the pentagon at the point M.



The perpendicular bisector of the line segment $[BE]$ passes through the centre of the circle (*Elements*, III, 1) (*PPG*, Theorem 21). This perpendicular bisector also passes through the point A, because the triangle $\triangle ABE$ is an isosceles triangle. The line AM is tangent to the circle, and is therefore perpendicular to the perpendicular bisector of $[BE]$. It follows that the line AM is parallel to the line BE, and is therefore also parallel to the line CD.

Now the lines BC and AD are parallel. It follows that the quadrilateral $CDAM$ is a parallelogram. Now opposite sides and angles of a parallelogram are equal (*Elements*, I, 34) (*PPG*, Theorem 9). It follows that



$$|\angle AMB| = |\angle ADC| = 2\times|\angle BAC|.$$

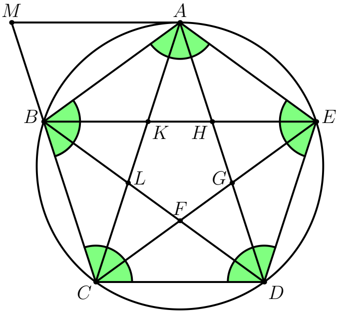
and $|MC| = |AD|$.

Moreover

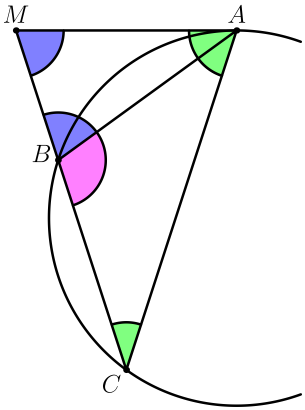
$$|\angle MAB| = |\angle ABE| = |\angle BAC|$$

because $\angle MAB$ and $\angle ABE$ are alternate angles formed by a transversal AB of the parallel lines AM and BE (*Elements*, I, 29) (*PPG*, Theorem 3). Similarly

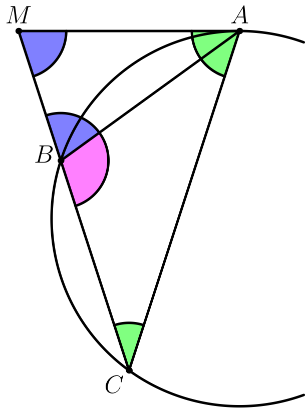
$$|\angle MBA| = |\angle BAD| = 2\times|\angle BAC|$$



Consider now the large triangle on the left of the figure. In this figure the three small angles $\angle CAB$, $\angle BCA$ and $\angle BAM$ are equal to one another. Also the three medium angles $\angle CAM$, $\angle CMA$ and $\angle ABM$ are equal to one another, and a medium angle is the double of a small angle such as $\angle CAB$. The large angle $\angle ABC$ is the triple of a small angle such as $\angle CAB$.

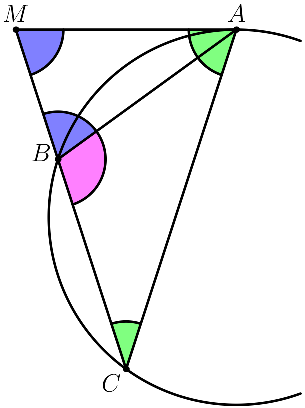


The equality of the angles $\angle CAM$ and $\angle CMA$ ensures that the triangle $\angle CAM$ is an isosceles triangle. Moreover the the two equal angles of this triangle are each double the remaining angle. The triangle $\angle BCA$ is also an isosceles triangle, as is the triangle $\triangle BAM$.

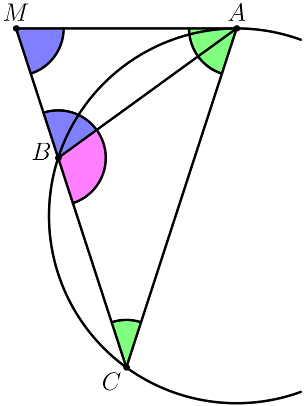


In order to proceed further, we presuppose that we have established a viable theory of proportion and similarity, such as is to be found in Books V and VI of Euclid's *Elements of Geometry*, or which follows from Theorems 12 and 13 of *Geometry for post-primary school mathematics* (which those theorems are presumed valid in their fullest generality).

For purposes of analysis, we suppose that line segments have lengths whose ratio to some pre-established unit of length is a real number. Ratios of line segments are then represented by real numbers. In the isosceles triangle $\triangle BCA$, the side $[CA]$ is a side of the pentagram that we have been investigating, and the side $[CB]$ is a side of the regular pentagon. Let φ denote the ratio of these two sides. Then φ is the ratio of the sides of the pentagram to the sides of the regular pentagon.

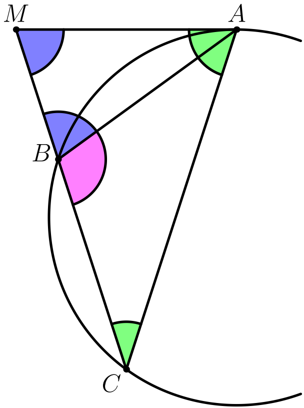


Now the angles of the triangle $\triangle CAM$ at C, A and M are respectively equal to the angles of the triangle $\triangle AMB$ at A, B and M. It follows that $\triangle CAM$ and $\triangle ABM$ are similar triangles. Assuming the standard theorems concerning similar triangles, it follows that $[CA]$ is to $[AB]$ as $[AM]$ is to $[BM]$.



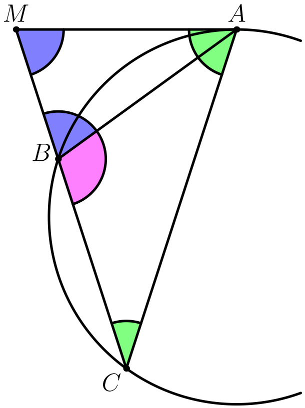
Now $|CA| = |CM|$ and $|AM| = |AB|$, because $\triangle CAM$ and $\triangle ABM$ are isosceles triangles. It follows that

$$\varphi = \frac{|CM|}{|AB|} = \frac{|AB|}{|BM|} = \frac{|CB|}{|BM|}.$$



Now $|CA| = |CM|$ and $|AM| = |AB|$, because $\triangle CAM$ and $\triangle ABM$ are isosceles triangles. It follows that

$$\varphi = \frac{|CM|}{|CB|} = \frac{|CM|}{|AB|} = \frac{|AB|}{|BM|} = \frac{|CB|}{|BM|}.$$

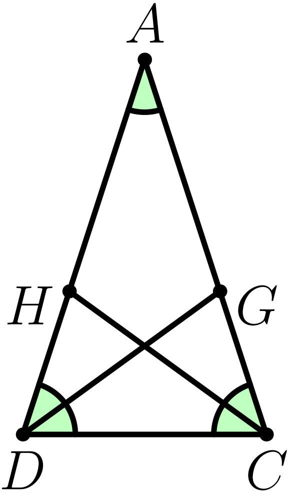


From this analysis of the properties of the regular pentagons and their associated pentagons is that a construction of a regular pentagon using straightedge and compass is feasable if and only if there is a construction, using ruler and compass, that, given a line segment $[CM]$, constructs a point B on that line segment with the property that

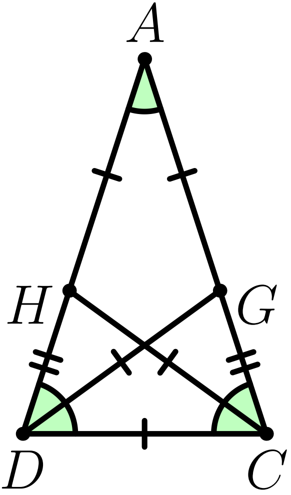
$$\frac{|CM|}{|CB|} = \frac{|CB|}{|BM|}.$$

in Euclid's terminology (as translated by T.L. Heath), such a point B is said to cut the line segment *in extreme and mean ratio*. From the middle of the nineteenth century onwards, the ratio of $|CM|$ to $|CB|$ is come to be referred to as the *golden ratio*.

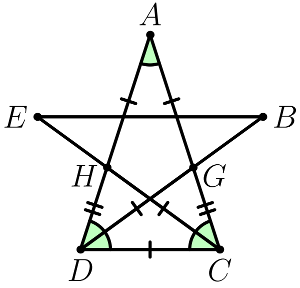
A *golden triangle* is an isosceles triangle in which the angles at the base are double the angle at the vertex. Let $\triangle ACD$ be a golden triangle with equal angles at C and D, let the bisector of the angle C intersect the side $[AB]$ at G, and let the bisector of the angle B intersect the side $[AC]$ at H. The angles $\angle CAB$, $\angle ACH$, $\angle HCD$, $\angle CDG$, and $\angle GDA$ are then all equal to one another.



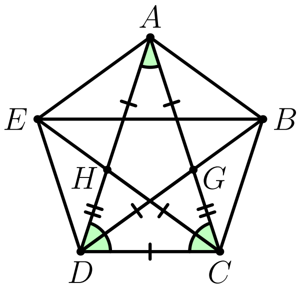
The existence of so many equal angles gives rise to several isosceles triangles within the figure. If two angles of a triangle are equal to one another, then the sides opposite those angles are also equal to one another (*Elements*, I, 6). Applying this result in conjunction with standard congruence rules We can establish that the line segments $[AH]$, $[HC]$, $[CD]$, $[DG]$ and $[GA]$ are equal to one another. The equality of the line segments $[AC]$ and $[AD]$ then ensures that the line segments $[GC]$ and $[HD]$ are equal to one another.



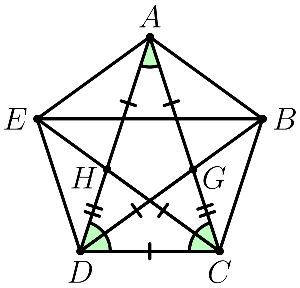
Produce the line segment $[DG]$ beyond G to a point B for which $[DB]$ and $[AD]$ are equal. Similarly produce the line segment $[CH]$ beyond H to a point E for which $[CE]$ and $[AC]$ are equal. Then join the points B and E by a line segment BE.



Now complete the pentagram by joining D to E, E to A, A to B, and B to C by line segments. Applying the SAS Congruence Rule (*Elements*, I, 4), it follows easily that the sides $[CD]$, $[EA]$ and $[AB]$ are equal to one another. The SAS Congruence Rule also ensures that the triangles $\triangle BDC$ $\triangle ADG$, $\triangle ACH$ and $\triangle ECD$ are congruent.



The triangle $\triangle ADG$ is an isosceles triangle. Therefore all four triangles $\triangle BDC$ $\triangle ADG$, $\triangle ACH$ and $\triangle ECD$ are isosceles triangles. It follows that the sides $[BC]$ and $[DE]$ are both equal to the line segment $[DG]$, and are therefore equal to the other sides of the pentagon $ABCDE$. Moreover congruence rules ensure that the angles of this pentagon are equal to three times the angle $\angle DAC$. Therefore the pentagon $ABCDE$ is a regular pentagon.



1.4. Golden Triangles, Proportion and the Golden Ratio

One can construct a golden triangle, given a line segment together with a point on that line segment which cuts the segment in the *golden ratio*. The golden ratio can be defined, using the language of ratio and proportion, as follows:—

Definition

Let G be a point on a line segment $[AC]$. Then G divides the line segment in the *golden ratio* if $[AC]$ is to $[AG]$ as $[AG]$ is to $[GC]$.

In the symbolic notation often used in synthetic geometry to specify proportions:

$$AC : AG :: AG : GC$$

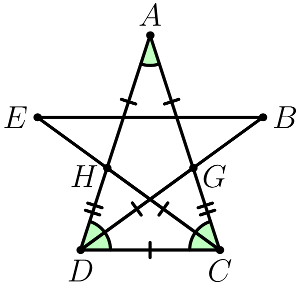
This golden proportion holds if and only if

$$|AC| \times |GC| = |AG|^2.$$

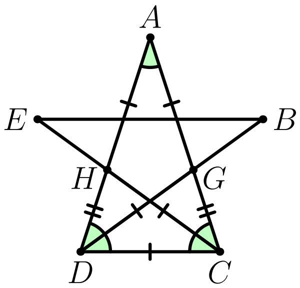
Thus the point G cuts the line segment $|AC|$ in the golden ratio if and only if a rectangle with sides meeting at a corner equal to $[AC]$ and $[CG]$ is equal in area to a square with side $[AG]$.

Let $\triangle ACD$ be a golden triangle with equal sides $[AC]$ and $[AC]$, and let G be the point at which the bisector of the angle $\angle ADC$ at D meets the side $[AC]$.

The angles of the triangle $\triangle ACD$ at A, C and D are equal to the angles of the triangle $[DC]G$ at D, C and G respectively.



Assuming the validity of a theory of proportion and similarity (such as that developed in Books V and VI of Euclid's *Elements of Geometry*, it follows that the sides of the triangle $\triangle ACD$ are proportional to the corresponding sides of the triangle $\triangle DCG$. Thus $[AC]$ is to $[CD]$ as $[CD]$ is to $[CG]$. But the sides $[CD]$ and $[CG]$ are equal. It follows that $[AC]$ is to $[AG]$ as $[AG]$ is to $[GC]$. Thus the point G cuts the line segment $[AC]$ in the golden ratio.

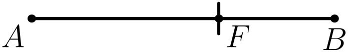


1.5. Construction of the Golden Ratio

The term *golden ratio* (or *golden section*) refers to the division of a line segment $[AB]$ into two subsegments $[AF]$ and $[FB]$ by a point F lying between A and B in such a way as to ensure that

$$\frac{|AB|}{|AF|} = \frac{|AF|}{|FB|}.$$

In words, the ratio of the whole to the greater segment is equal to the ratio of the greater segment to the lesser segment.

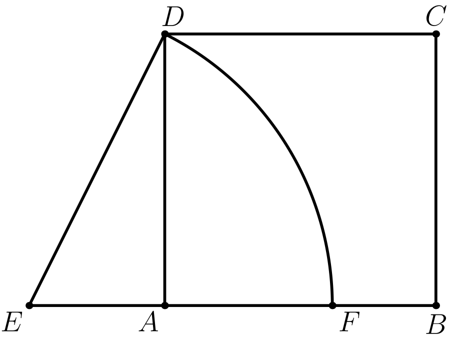


The terms *golden ratio* or *golden section* are not to be found in Euclid. In Book VI of the *Elements of Geometry* (as translated by T.L. Heath), Euclid defined what is meant by saying that a line segment is *cut in extreme and mean ratio* as follows: "as the whole line is to the greater segment, so is the greater to the less.

It is widely reported that the term *golden section* (in German, *Goldener Schnitt*) was introduced by Martin Ohm (1792–1872) in 1835, a mathematical textbook entitled *Die reine Elementar-Mathematik, weniger abstrakt, sondern mehr anschaulich* ("The pure elementary mathematics, less abstract, but more intuitive").

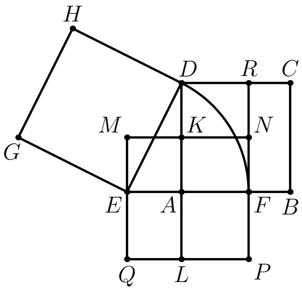
Proposition 1.1

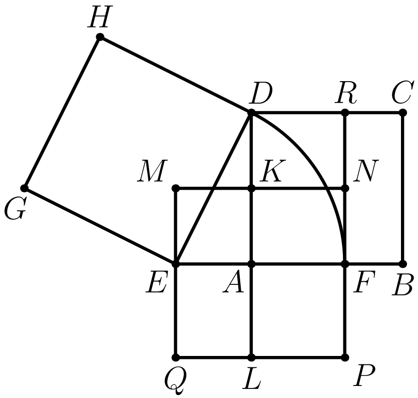
Let $[AB]$ be a line segment, let $ABCD$ be a square constructed on the side $[AB]$, let E be a point on the ray from B through A, determined so that A lies between E and B and $|AB| = 2 \times |AE|$, and let F be the point on $|AB|$ lying between A and B for which $|EF| = |ED|$. Then F cuts the line segment $[AB]$ in the golden ratio.



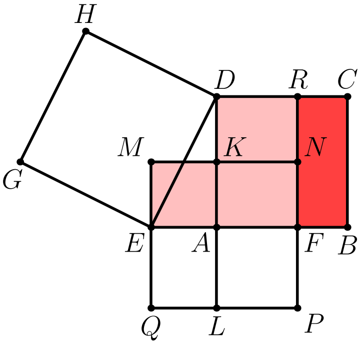
Proof

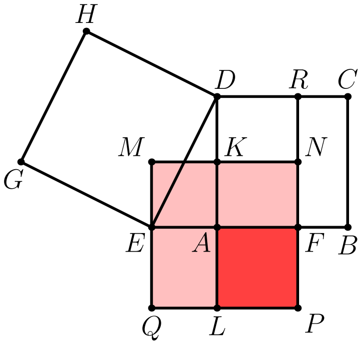
Let the point K bisect the side $[AD]$ of the square $ABCD$, and let the square $QPNM$ be constructed so that the sides $[QP]$ and $[MN]$ of $QPNM$ are parallel to the sides $[AB]$ and $[DC]$ of $ABCD$, the sides $[QM]$ and $[PN]$ of $QPNM$ are parallel to the sides $[AD]$ and $[BC]$ of $ABCD$, the point A lies inside the square $QPNM$, the point F lies on the side $[PN]$ and the point K lies on the side $[MN]$.





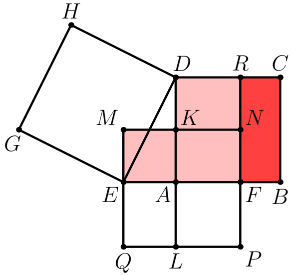
Pythagoras's Theorem (*Elements*, I, 47) (*PPG*, Theorem 14) ensures that $|ED|^2 = |EA|^2 + |AD|^2$. But $|ED| = |EF| = |QP|$. It follows that

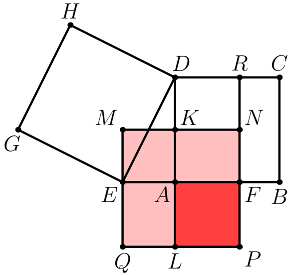




Thus, applying Pythagoras's Theorem,

$$\begin{aligned}
\operatorname{area}(EBCDKM) &= \operatorname{area}(EAKM) + \operatorname{area}(ABCD) \\
&= |EA|^2 + |AD|^2 = |ED|^2 \\
&= |EF|^2 = |QP|^2 = \operatorname{area}(QPNM).
\end{aligned}$$



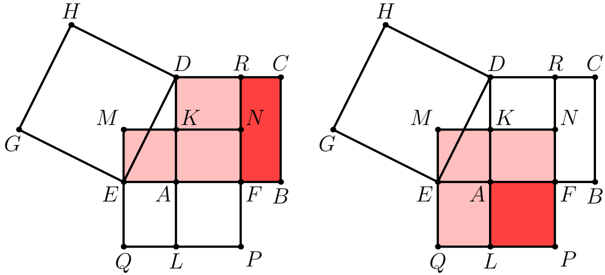


But

$$
\begin{aligned}
&\text{area}(EBCDKM) \\
&\qquad = \text{area}(FBCR) + \text{area}(EFRDKM) \\
&\qquad = \text{area}(FBCR) + \text{area}(EAKM) + 2 \times \text{area}(AFNK)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{area}(QPNM) &= \text{area}(LPFA) + \text{area}(QLAFNM) \\
&= \text{area}(LPFA) + \text{area}(EAKM) + 2 \times \text{area}(AFNK)
\end{aligned}
$$

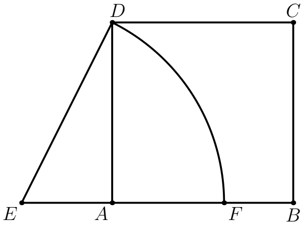


It follows that

$$\begin{aligned}
&\text{area}(FBCR) + \text{area}(EAKM) + 2 \times \text{area}(AFNK) \\
&\quad = \text{area}(EBCDKM) = \text{area}(QPNM) \\
&\quad = \text{area}(LPFA) + \text{area}(EAKM) + 2 \times \text{area}(AFNK),
\end{aligned}$$

and thus

$$|AB| \times |FB| = \text{area}(FBCR) = \text{area}(LPFA) = |AF|^2.$$



The identity

$$|AB| \times |FB| = |AF|^2$$

expresses the result that a rectangle whose sides are of length $|AB|$ and $|FB|$ is equal in area to a square whose sides are of length $|AF|$ (*Elements*, II, 11).

The equality of areas expressed in the formula

$$|AB| \times |FB| = |AF|^2$$

gives rise to a corresponding proportion (*Elements*, VI, 16): $[AB]$ is to $[AF]$ as $[AF]$ to $[FB]$. In traditional notation, this proportion would be expressed thus:

$$AB : AF :: AF : FB.$$

Expressed as ratios of lengths (where lengths are expressed as real numbers relative to some chosen unit of length), the proportion can be written thus:

$$\frac{|AB|}{|AF|} = \frac{|AF|}{|FB|}.$$

Thus the point F has cut the line segment $[AB]$ in the golden ratio, as required. $\blacksquare$

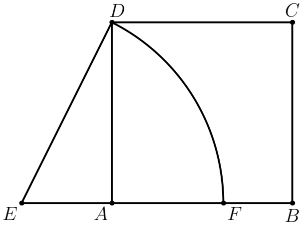
Remark

The proof of Proposition 1.1 has been adapted from the proof of Proposition 11 in Book II of Euclid's *Elements of Geometry*.

Corollary 1.2

let A and B be distinct points, and let F cut the line segment $[AB]$ in the golden ratio. Let $a = |AB|$ and $b = |AF|$. Then

$$\frac{a}{b} = \frac{\sqrt{5}+1}{2}.$$



Proof

In the figure, $|EA| = \frac{1}{2}a$, and therefore, by Pythagoras's Theorem,

$$b = |EF| - \tfrac{1}{2}a = |ED| - \tfrac{1}{2}a = \sqrt{\frac{1}{4}a^2 + a^2} - \tfrac{1}{2}a = \tfrac{1}{2}(\sqrt{5} - 1)a.$$

Thus

$$\frac{b}{a} = \frac{\sqrt{5}-1}{2}.$$

Now

$$\frac{\sqrt{5}+1}{2} \times \frac{\sqrt{5}-1}{2} = \frac{5-1}{4} = 1.$$

It follows that

$$\frac{a}{b} = \frac{\sqrt{5}+1}{2},$$

as required. $\blacksquare$

The result of Corollary 1.2 may also be obtained algebraically. Let $a = \varphi b$. The definition of the golden ratio requires that

$$\frac{a}{b} = \frac{b}{a - b}.$$

Therefore

$$\varphi = \frac{1}{\varphi - 1}.$$

Multiplying out, we find that

$$\varphi^2 - \varphi - 1 = 0.$$

The roots of the polynomial $x^2 - x - 1$ are $\frac{1}{2}(1 + \sqrt{5})$ and $\frac{1}{2}(1 - \sqrt{5})$. The first of these is positive, and the second negative. Now $\varphi > 0$. It follows that $\varphi = \frac{1}{2}(\sqrt{5} + 1)$. Using the fact that the product of the roots of the quadratic polynomial $x^2 - x - 1$ is equal to -1, we see also that $\varphi^{-1} = \frac{1}{2}(\sqrt{5} - 1)$.

Remark

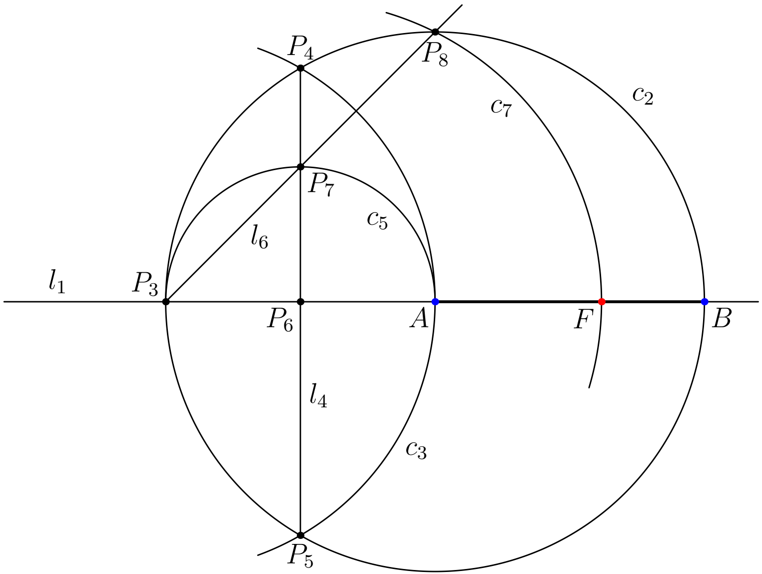
The Greek letter φ (phi) is often used to denote the golden ratio. This notation was introduced by Mark Barr (1871–1950). It is reported that he chose this letter from the Greek alphabet to commemorate the sculptor Phidias. Sir Theodore Andrea Cook (1867–1928) (*The curves of life; being an account of spiral formations and their application to growth in nature, to science and to art; with special reference to the manuscripts of Leonardo da Vinci* (1928), wrote as follows (page 420):

Mr. Mark Barr suggested to Mr. Schooling that this ratio should be called the φ proportion for reasons given below. [. . . .] The symbol φ given to this proportion was chosen partly because it has a familiar sound [. . .] and partly because it is the first letter of the name of Pheidias, in whose sculpture this proportion is seen to prevail when the distances between salient points are measured. [. . .]

URL: `https://archive.org/details/cu31924028937179`

The following figure exhibits a straightedge and compass construction for cutting a line segment $[AB]$ in the golden ratio. We start with points A and B (setting $P_1 = A$ and $P_2 = B$ and $l_1 = AB$). Let c_2 be the circle centred on A and passing through the point B. This circle cuts the ray from B through A at a point P_3. The point A then lies between B and P_3 and is equidistant from B and P_3. Let c_3 be the circle centred on P_3 and passing through A. Then the circles c_2 and c_3 intersect at two points P_4 and P_5. The line through these two points is the perpendicular bisector of the line segment $[P_3A]$, and cuts that line segment a point P_6 midway between P_3 and A.

The circle c_5 centred on P_6 and passing through A then intersects the line P_4P_5 at a point P_7. The ray starting at P_3 and passing through P_7 then intersects the circle c_2 at a point P_8. The line segment $[AP_8]$ is then perpendicular to and equal in length to the line segment $[AB]$. The circle c_7 centred on P_6 and passing through P_8 then intersects the line AB at a point F between A and B. It follows from Proposition 1.1 that the point F cuts the line segment $[AB]$ in the golden ratio.



P_4
P_8
c_2
c_7
P_7
c_5
l_6
l_1
P_3
P_6
A
F
B
l_4
c_3
P_5

1.6. Some Circle Theorems

Proposition 1.3

Let a circle of radius r be taken in a plane, centred on some chosen origin of Cartesian coordinates, and let let D be a point of the plane whose distance from the origin is greater than r. Either suppose that a ray from D cuts the circle in two distinct points points A and C, where C lies between A and D, or else suppose that the ray starting at D is tangent to the circle at the point A and that, in this case, the point C coincides with A. Then

$$|DA| \times |DC| = |DO|^2 - r^2.$$

Proof

We give a proof using coordinate geometry. We suppose, without loss of generality, that a system of Cartesian coordinates have been chosen whose origin is the centre of the circle, and we suppose also that the coordinate system has been oriented so that the line passing through the points A, C and D is parallel to the x-axis, with the point D on that side of the y-axis on which the x-coordinate is positive.

Let $D = (u, v)$ where $u > 0$. Because the line $y = v$ is required to cut the circle, it must be the case that $-r \leq v \leq r$. Then $A = (-w, v)$ and $C = (w, v)$, where $w^2 + v^2 = r^2$. It follows that

$$\begin{aligned} |DA| \times |DC| &= (u + w)(u - w) = u^2 - w^2 \\ &= (u^2 + v^2) - (w^2 + v^2) \\ &= u^2 + v^2 - r^2 \\ &= |DO|^2 - r^2, \end{aligned}$$

as required. ■

Corollary 1.4

Let a circle of radius r be taken in a plane, centred on some chosen origin of Cartesian coordinates, and let let D be a point of the plane whose distance from the origin is greater than r. Suppose that a ray from D cuts the circle in two distinct points A and C, where C lies between A and D, and that B is some point that lies on the circle. Then

$$|DA| \times |DC| = |DB|^2$$

if and only if the line DB is tangent to the circle at B.

Proof

Suppose that the line DB is tangent to the circle at B. It then follows from Proposition 1.3 that

$$|DA| \times |DC| = |DO|^2 - r^2 = |DB|^2.$$

Conversely suppose that $|DA| \times |DC| = |DB|^2$. Then

$$|DB|^2 = |DO|^2 - r^2,$$

where $r^2 = |BO|^2$. It follows that

$$|DO|^2 = |DB|^2 + |BO|^2,$$

and therefore the angle at B in the triangle $\triangle DBO$ is a right angle (*Elements*, I, 48) (*PPG*, Theorem 15). It follows from this that the line DB is tangent to the given circle at B (*Elements*, III, 16) (*PPG*, Theorem 15). This completes the proof. ∎

Remark

Proposition 1.3 and Corollary 1.4 correspond closely to Propositions 37 and 38 in Book III of Euclid's *Elements of Geometry*. The use of coordinate geometry in the proofs given here, though of course anachronistic in the context of Euclid's geometry, might be regarded as being, in essence, reasonably close in spirit to the underlying arguments in Euclid's proofs, which rely on Pythagoras's Theorem (*Elements*, I, 47) and the "geometric algebra" developed in Book II of Euclid's *Elements*, specifically (*Elements*, II, 6).