Euclid's *Elements of Geometry* Extracts from Book V (T.L. Heath's Edition)

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DEFINITIONS

1 A magnitude is a **part** of a magnitude, the less of the greater, when it measures the greater.

NOTE BY D.R. WILKINS. Following T.L. Heath, we denote magnitudes by small italic letters a, b, c etc., and we denote positive integers by small italic letters m, n, p etc. from m onwards.

According to the definition in *Euclid*, Book V, a magnitude a is a **part** of a magnitude b of the same kind if and only if there exists some natural number n such that b = n.a. This is a more restricted notion of "part" than that employed in Book I. According to the definition adopted in Book V, a line segment AB is a "part" of a line segment AC if and only if AC can be subdivided into line segments of equal length, one of which is AB.

2 The greater is a **multiple** of the less when it is measured by the less.

NOTE BY D.R. WILKINS. Thus a magnitude b is a multiple of a magnitude a of the same kind if and only if there exists some natural number n such that b = n.a. (Thus b is a multiple of a, and is "measured" by a if b is twice a, thrice a, four times a, etc.

3 A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.

NOTE BY D.R. WILKINS. Where magnitudes have a ratio to one another, the ratio of a magnitude a to a magnitude b is denoted by a : b.

4 Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.

NOTE BY D.R. WILKINS. Magnitudes a and b of the same kind have a ratio to one another if and only if there exist positive integers m and n such that a < m.b and b < n.a.

5 Magnitudes are said to be **in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

NOTE BY D.R. WILKINS. This definition encapsulates the famous definition of *proportion* attributed to Eudoxus of Cnidus (408–355 B.C.), a Greek astronomer and mathematician who was a contemporary of Plato. Suppose we are given four magnitudes a, b, c and d, where a and b have a ratio to one another and c and d have a ratio to one another. We say that "a is to b as c is to d", that "a and b are in the same ratio as c and d", or that "d is a fourth proportional to a, b and c", and write a : b :: c : d if and only if the following three conditions are satisfied:—

- (i) m.a > n.b for all pairs m, n of positive integers for which m.c > n.d;
- (ii) m.a = n.b for all pairs m, n of positive integers for which m.c = n.d;
- (iii) m.a < n.b for all pairs m, n of positive integers for which m.c < n.d.

The magnitudes a and b may for example both represent lengths, or both represent angles, or both represent areas, and the magnitudes c and d may both represent lengths, or both represent angles, or both represent areas.

Suppose that quantities a and b have a ratio that, according to modern practice, is represented by some real number x, and that quantities c and d have a ratio that, according to modern practice, is represented by some real number y, so that a = x.b and c = y.d. Then m.a > n.b if and only if mx > n, in which case x > n/m. Similarly m.a = n.b if and only if mx = n, in which case x = n/m, and m.a < n.b if and only if mx < n, in which case x < n/m. It follows that if ratios a : b and c : d are represented by real numbers x and y then a is to b as c is to d (a : b :: c : d) if and only if the following three conditions are satisfied:—

- (i) a positive rational number q satisfies q < x if and only if q < y;
- (ii) a positive rational number q satisfies q = x if and only if q = y;

(iii) a positive rational number q satisfies q > x if and only if q > y;

Now any positive real number x is completely determined by the set of positive rational numbers q that satisfy q < x. It follows that if a, b, c and d are magnitudes, where a has a ratio to b and c has a ratio to d, and if a : b and c : d are represented by positive real numbers x and y respectively, according to modern mathematical practice, then a : b :: c : dif and only if x = y.

6 Let magnitudes which have the same ratio be called **proportional**.

NOTE BY D.R. WILKINS. Proportional magnitudes a, b, cand d are those where a is to b as c is to d.

7 When, of the equimultiples, the multiple of the first exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

NOTE BY D.R. WILKINS. Let a, b, c and d be magnitudes, where a has a ratio to b and c has a ratio to d. Then a has a greater ratio to b than c has to d, according to the definition of Euclid, if and only if there exists a pair m, n of positive integers such that m.a > n.b but $m.c \neq n.d$. Thus if the ratios a : b and c : d are represented, according to modern mathematical practice, by real numbers x and y respectively, so that a = x.b and c = y.d, then a has a greater ratio to bthan c has to d, according to the definition of Euclid, if and only if there exists a pair m, n of positive integers such that $y \leq n/m < x$. Such a pair positive integers exists if and only if x > y. In symbols, if the ratio a : b is represented by the real number x, and if c : d is represented by the real number y then a : b > c : d if and only if x > y.

- 8 A proportion in three terms is the least possible.
- 9 When three magnitudes are proportional, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

NOTE BY D.R. WILKINS. A proportion in three terms will be determined by magnitudes a, b and c that have ratios to each other and satisfy a : b :: b : c, so that a is to b as b is to c. In this situation, the ratio that a has to c is the duplicate of the ratio which a has to b. Suppose that, in this situation, the ratio which a has to b, and which b in turn has to c, is represented by the real number x. Then the duplicate ratio, being the ratio that a has to c, is represented by the real number x^2 .

10 When four magnitudes are (continuously) proportional, the first is said to have to the fourth the **triplicate** of that which it has to the second, and so on continually, whatever be the proportion.

NOTE BY D.R. WILKINS. Thus if we have four magnitudes a, b, c and d that have ratios to each other, and if a:b::b:c and b:c::c:d, then the ratio that a has to d is the triplicate of the ratio that a has to b. In this situation the ratios a:b, b:c and c:d are all represented in modern mathematical practice by some real number x, where a = x.b, b = x.c and c = x.d. Then $a = x^3.d$, and thus the ratio a:d is represented by the real number x^3 .

11 The term **corresponding magnitudes** is used of antecedents in relation to antecedents, and of consequents in relation to consequents.

NOTE BY D.R. WILKINS. Let a, b, c and d be magnitudes, where a has a ratio to b and c has a ratio to d. The magnitudes a and c are referred to as the *antecedents* and the magnitudes b and d are referred to as the *consequents*. Then a and c are corresponding magnitudes, and b and d are corresponding magnitudes.

12 Alternate ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

NOTE BY D.R. WILKINS. Let a, b, c and d magnitudes that have ratios to each other, and let a be to b as c is to d. It follows from Proposition 16 of Book V of Euclid's *Elements of Geometry*, that a is then to c as b is to d. Thus if a : b :: c : dthen a : c :: b : d. The ratios a : c and b : d are the alternate ratios corresponding to the ratios a : b and c : d.

13 **Inverse ratio** means taking the consequent as antecedent in relation to the antecedent as consequent.

NOTE BY D.R. WILKINS. Let a, b, c and d magnitudes that have ratios to each other, and let a be to b as c is to d. It follows from the Porism following Proposition 7 of Book V of Euclid's *Elements of Geometry* that b is then to a as dis to c. Thus if a:b::c:d then b:a:d:c. The ratios b:aand d: c are the inverse ratios corresponding to the ratios a:b and c:d.

- 14 Composition of a ratio means taking the antecedent together with the consequent as one in relation to the consequent by itself.
- NOTE BY D.R. WILKINS. Given magnitudes a and b, where a has a ratio to b and is greater than b the separation of the ratio a: b is represented by the ratio a - b: b. It follows from Proposition 18 of Book V of Euclid's Elements of Geometry that if a, b, c and d are magnitudes and if a:b::c:d then a+b:b::c+d:d.

Separation of a ratio means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

NOTE BY D.R. WILKINS. Given magnitudes a and b, where a has a ratio to b and is greater than b the separation of the ratio a:b is represented by the ratio a-b:b. It follows from Proposition 17 of Book V of Euclid's Elements of Geometry that if a, b, c and d are magnitudes satisfying a > b and c > d, and if a : b :: c : d then a - b : b :: c - d : d.

16 Conversion of a ratio means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

> NOTE BY D.R. WILKINS. Given magnitudes a and b, where a has a ratio to b and is greater than b the conversion of the ratio a: b is represented by the ratio a: a - b. It follows from the Porism following Proposition 17 of Book V of Euclid's *Elements of Geometry* that if a, b, c and d are magnitudes satisfying a > b and c > d, and if a : b :: c : dthen a - b : b :: c - d : d.

17 A ratio **ex aequali** arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes. Or

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in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

NOTE BY D.R. WILKINS. Let $a_1, a_2, a_3, \ldots, a_n$ and b_1, b_2, \ldots, b_n be magnitudes that have ratios to each other and suppose that b_i is to b_{i-1} as a_i is to a_{i-1} for $i = 2, 3, \ldots, n$. In this situation a_1 is to a_n as b_1 is to b_n , and this ratio of a_1 to a_n , or b_1 to b_n , is the ratio that arises *ex aequali* in this situation.

Suppose that, as in modern mathematical practice, we represent the ratio of a_i to a_{i+1} b_i to b_{i+1} by the real number x_i for i = 1, 2, ..., n-1, so that $a_i = x_i . a_{i+1}$ and $b_i = x_i . b_{i+1}$ for i = 1, 2, ..., n. Then

$$a_1 = x_1 x_2 \cdots x_{n-1} a_n$$
 and $b_1 = x_1 x_2 \cdots x_{n-1} b_n$.

Thus the real number $x_1x_2\cdots x_{n-1}$ represents the ratio ex aequali that a_1 has to a_n , and that b_1 has to b_n .

18 A **perturbed proportion** arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

> NOTE BY D.R. WILKINS. Suppose that we have six magnitudes a, b, c, d, e and f that have ratios to one another. A perturbed proportion arises when a : b :: e : f and b : c ::d : e. Let the ratios a : b and b : c be represented, as in modern mathematical practice, by real numbers x and y, so that a = x.b and b = y.c. Then, where there is perturbed proportion, the ratios amongst d, e and f will satisfy c = y.dand e = x.f. In that case we have a = xy.c and c = yx.d. But xy = yx, and therefore a : c :: d : f. Proposition 23 of Book V of Euclid's *Elements of Geometry* proves this result using the theory of proportion based on Definition 5 of Book V.

If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will be of all.

Let any number of magnitudes whatever AB, CD be respectively equimultiples of any magnitudes E, F equal in multitude;

I say that, whatever multiple AB is of E, that multiple will AB, CD also be of E, F.



For, since AB is the same multiple of E that CD is of F, as many magnitudes as there are in AB equal to E, so also are there in CD equal to F.

Let AB be divided into the magnitudes AG, GB equal to E,

and CD in CH, HD equal to F;

then the multitude of the magnitudes AG, GB will be equal to the multitude of magnitudes in CH, HD.

Now since AG is equal to E and CH to F,

therefore AG is equal to E, and AG, CH to E, F.

For the same reason

GB is equal to E, and GB, HD to E, F;

therefore, as many magnitudes as there are in AB equal to E, so many are there in AB, CD equal to E, F;

therefore, whatever multiple AB is of E, that multiple will AB, CD also be fo E, F.

Therefore etc.

Q.E.D.

NOTE BY T.L. HEATH. De Morgan remarks of V. 1–6 that they are "simple propositions of concrete arithmetic, covered in language which makes them unintelligible to modern ears. The first states no more than that *ten* acres and *ten* roods make *ten* times as much as one acre and one rood." One aim therefore of notes on these as well as the other propositions of Book V. should be to make their purport clearer to the learner by setting them side by side with the same truths expressed in the much shorter and more familiar modern (algebraical) notation. In doing so, we shall express magnitudes by

the first letters of the alphabet a, b, c etc., adopting small instead of capital letters so as to avoid confusion with Euclid's lettering; and we shal luse the small letters, m, n, p etc. to represent integral numbers. Thus ma will always mean m times a or the m^{th} multiple of a (counting 1.a as the first, 2.a as the second multiple, and so on).

Prop 1 then asserts that, if ma, mb, mc etc. be any equimultiples of a, b, c etc., then

$$ma + mb + mc + \dots = m(a + b + c + \dots).$$

If a first magnitude be the same multiple of a second that a third is of a forth, and a fifth also be the same multiple of the second that a sixth is of the fouth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Let a first magnitude, AB, be the same multiple of a second, C, that a third, DE, is of a fourth, F, and let a fifth BG be alos the same multiple of the second, C that the a sixth EH, is of the fourth F;

I say that the sum of the first and fifth, AG, will be the same multiple of the second, C, that the sum of the third and fifth, DH, is of the fourth, F.



For, since AB is the same multiple of C that DE is of F, therefore, as many magnitudes as there are in AB equal to C, so many also are there in DE equal to F.

For the same reason also,

as many as there are in BF equal to C, so many are there also in EH equal to F;

therefore as many as there are in the whole of AG equal to C, so many also are there in the whole DH equal to F.

Therefore, whatever multiple AG is of C, that multiple also is DH of F. Therefore the sum of the first and fifth, AG, is the same multiple of the second C, that the sum of the third and sixth, DH is of the fourth, F.

Therefore etc.

Q.E.D.

NOTE BY T.L. HEATH. To find the corresponding formula for the result of this proposition, we may suppose a to be the "second" magnitude and b the "fourth". If now the "first magnitude" is ma, the "third" is, by hypothesis, mb, and, if the "fifth" magnitude is na, the "sixth" is nb. The proposition then asserts that ma + na is the same multiple of a that mb + nb is of b.

More generally, if pa, qa, \ldots and pb, qb, \ldots be any further equimultiples of a, b respectively, $ma + na + pa + qa + \cdots$ is the same multiple of a that $mb + nb + pb + qb + \cdots$ is of b. This extension is stated in Simson's corollary to V. 2 thus:

"From this it is plain that, if any number of magnitudes AB, BG, GH be multiples of another C; and as many DE, EK, KL be the same multiples of F, each of each; the whole of the first, viz. AH, is the same multiple of C that the whole of the last, viz. DL is of F."

The course of the proof, which separates m into its units and also n into its units, practically tells us that the multiple of a arrived t by adding the *two* multiples is the (m + n)th multiple; or practically we are shown that

$$ma + na = (m+n)a,$$

or, more generally, that

$$ma + na + pa + \dots = (m + n + p + \dots)a.$$

If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also ex aequali the magnitudes will be equimultiples respectively, the one of the second and the other of the fourth.

[Euclid's proof omitted]

NOTE BY D.R. WILKINS. Let a, b, c, d, e and f be magnitudes that have ratios to one another. Suppose that there exist positive integers m and nsuch that c = n.a, d = n.b, e = m.c and f = m.d. Then Proposition 3 claims that e and f are equimultiples of a and b respectively. This is clearly the case, because e = (mn).a and f = (mn).b. Here a and b are the "second" and "fourth" magnitudes respectively in the statement of the proposition, c and d are the "first" and "third" respectively, and e and f are the "equimultiples taken of the first and third".

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whetever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

[Euclid's proof omitted]

NOTE BY D.R. WILKINS. Let a, b, c and d be magnitudes for which a is to b as c is to d. The claim is that, given any positive integers p and q, p.a is to q.b as p.c is to q.d. Presented symbolically, if a : b :: c : d then p.a : q.b :: p.c : q.d.

To see this, let m and n be positive integers. If m.(p.a) > n.(q.b) then (mp).a > (nq).b. It then follows from proportionality that (mp).c > (nq).d and therefore m.(p.c) > n.(q.d). Similarly if m.(p.a) = n.(q.b) then m.(p.c) = n.(q.d), and if m.(p.a) < n.(q.b) then m.(p.c) < n.(q.d). It then follows from the definition of proportionality that p.a is to q.b as p.c is to q.d, as required.

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

For let the magnitude AB be the same multiple of the magnitude CD that the part AE subtracted is of the part CF subtracted;

I say that the remainder EB is also the same multiple of the remainder FD that the whole AB is of the whole CD.

For, whatever multiple AE is of CF, let EB be made that multiple of CG.

Then, since AE is the same multiple of CF that EB is of GC,

therefore AE is the same multiple of CF that AB is of GF.

Therefore AB is the same multiple of each of the magnitudes GF, CD; therefore GF is equal to CD.

Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD.

And since AE is the same multiple of CF that EB is of GC, and GC is equal to DF,

therefore AE is the same multiple of CF that EB is of FD. But by hypothesis,

AE is the same multiple of CF that AB is of CD;

therefore EB is the same multiple of DF that AB is of CD.

The remainder EB will be the same multiple of the remainder FD that the whole AB is of the whole CD.

Therefore etc.

Q.E.D.

NOTE BY D.R. WILKINS. Let a represent the magnitude of the whole, b represent the magnitude of the part that is subtracted from the whole, and let n be a positive integer. The statement of the Proposition essentially says that

$$n.a - n.b = n(a - b).$$

In the proof, and the accompanying figure, the magnitude a is represented by the line segment CD, and the magnitude b of the part subtracted by the line segment CF, where D is a point in the interior of the line segment CD. The hypotheses of the theorem require that there exist some positive integer n such that $AB = n \times CD$ and $AE = n \times CF$. Note that n = 3 in the representative example depicted in the accompanying figure. However there are no tick marks in the figure that indicate that AB is three times the length of CD.

Euclid's proof then requires the construction of a point D such that $EB = n \times GC$, where this point G is located on the line CD produced past C (so that G lies on the opposite side of C to the points F and D). Commentators have noted that Euclid has not, up to this point, supplied a construction to establish the existence or location of such a point G. Then the magnitude AB = AE + EB, where $AE = n \times CF$ and $EB = n \times GC$. It then follows from Proposition 1 of Book 5 that $AB = n \times (GC + CF)$ and thus $AB = n \times GF$. But $AB = n \times CD$. It follows that the lines GF and CD must be equal. Now it follows from Common Notion 3 of Book I of Euclid that "if equals be subtracted from equals, the remainders are equal". It follows that if we subtract the common subsegment CF from the equal line segments GF and CD, the remainders GC and FD are equal. Therefore FD = GC = g, and thus $EB = n \times FD$. This completes the proof.

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

[Euclid's proof omitted]

NOTE BY D.R. WILKINS. Let a and b be magnitudes, and let m and n be positive integers, where m < n. The proposition essentially asserts that n.a - m.a is the same multiple of a that n.b - m.b is of b. In both cases, the multiples n.a - m.a and n.b - m.b are obviously the (n - m)th multiples of the magnitudes a and b respectively.

Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

[Euclid's proof omitted]

NOTE BY D.R. WILKINS. Let a and b be equal magnitudes, and let c be some other magnitude. Suppose that a and b have a ratio to c. The Proposition claims that then a is to c as b is to c (a : c :: b : c), and that c is to a as c is to b (c : a :: c : b).

Let m and n be integers. Then m.a is equal to m.b. This is a consequence of the Common Notion 2 in Book I of Euclid, which states that "If equals be added to equals, the wholes are equal." It follows that m.a > n.c if and only if m.b > n.c, m.a = n.c if and only if m.b = n.c and m.a < n.c if and only if m.b < n.c. It follows from the definition of proportion [V. Def. 5] that a is to c as b is to c. It also follows that c is to a.

PORISM FOLLOWING PROPOSITION 7

From this it will be manifest that, if any magnitudes are proportional, they will also be proportional inversely.

NOTE BY D.R. WILKINS. Let a, b, c and d where a has a ratio to b and c has a ratio to d. The claim is that if a is to b as c is to d (i.e., a : b :: c; d) then b is to a as d is to c (i.e., b : a :: d : c). This is not a consequence of Proposition 7 itself, but rather follows from the method used to prove that proposition.

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Let AB, C be unequal magnitudes, and let AB be greater; let D be another chance magnitude;

I say that AB has to D a greater ratio than C has to D, and D has to C a greater ratio than it has to AB.

For, since AB is greater than C, let BE be made equal to C;

then the less of the magnitudes AE, EB, if multiplied, will sometime be greater than D. [v. Def. 4]



 $[Case \ 1.]$

First, let AE be less than EB; let AE be multiplied, and let FG be a multiple of it which is greater than D;

then, whatever multiple FG is of AE, let GH be made the same multiple of EB and K of C;

and let L be taken double of D, M triple of it, and successive multiples increasing by one, until what is taken is a multiple of D and the first that is greater than K. Let it be taken, and let it be N which is quadruple of Dand the first multiple of it that is greater than K.

Then, since K is less than N first,

therefore K is not less than M.

And, since FG is the same multiple of AE that GH is of EB,

therefore FG is the same multiple of AE that FH is of AB. [V. 1]

But FG is the same multiple of AE that K is of C;

therefore FH is the same multiple of AB that K is of C. therefore FH, K are equimultiples of AB, C.

Again, since GH is the same multiple of EB that K is of C, and EB is equal to C,

therefore GH is equal to K.

But K is not less than M;

therefore neither is GH less than M.

And FG is greater than D;

therefore the whole FH is greater than D, M together.

But D, M together are equal to N, inasmuch as M is the triple of D, and M, D together are the quadruple of D, while N is also quadruple of D; whence M, D together are equal to N.

But FH is greater than M, D;

therefore FH is in excess of N,

while K is not in excess of N,

And FH, K are equimultiples of AB, C, while N is another, chance, multiple of D;

therefore AB has to D a greater ratio than C has to D. [V. Def. 7] I say next that, D has to C a greater ratio than D has to AB.

For with the same construction, we can prove similarly that N is in excess of K whele N is not in excess of FH. And N is a multiple of D, while FH, K are other, chance, equalmultiples of AB, C;

therefore D has to C a greater ratio than D has to AB. [5. Def. 7]

 $[Case \ 2.]$

Again, let AE be greater than EB.

Then the less, EB, if multipled, will sometime be greater than D. [V. Def. 4]

Let it be multiplied, and let GH be a multiple of EB and greater than D;

and whatever multiple GH is of EB, let FG be made the same multiple of AE, and K of C.

Then we can prove similarly that FH, K are equimultiples of AB, C;

and similarly let N be taken a multiple of D but the first that is greater than FG,

so that FG is again not less than M.

But GH is greater than D;

therefore the whole FH is excess of D, M, that is of N.



Now K is not in excess of N, inasmuch as FG also, which is greater than GH, that is, than K, is not in excess of N.

And in the same manner, by following the above argument, we complete the demonstration.

Therefore, etc.

Q.E.D.

NOTE BY D.R. WILKINS. Let a, b and c be magnitudes that have a ratio to one another, where a > b. The proposition asserts that a : c > b : c and c : a < c : b.

Now because a > b, we can subdivide a as the sum of two magnitudes, where one of those magnitudes is equal to b. Let f be the other magnitude. Then a = b + f.

Now there exists some positive integer m that is large enough to ensure that both m.f > c and m.b > c. Let n be the smallest positive integer for which n.c > m.b. Then $(n-1).c \le m.b$. Now the choice of m also ensures that c < m.f. It follows that m.b < n.c < m.b + m.f. But b + f = a. It therefore follows from Proposition 1 of Book 5 that m.b < n.c < m.a. It then follows from Definition 7 of Book 5 that, the ratio of a to c is greater than the ratio of b to c.

On comparing the proof above with that given in Heath's translation of Euclid, we see that, of the geometric magnitudes referred to in Heath's translation, in the case when AE is less than EB, AB has magnitude a, C and EB have magnitude b, AE has magnitude f, FG has magnitude m.f where m = 2 in the case represented in the figure, GH and K have magnitude m.b, FH has magnitude m.f + m.b, L has magnitude 2.c, M has magnitude (n-1).c and N has magnitude n.c, where n = 4 in the particular case employed in the proof. Proposition 1 of Book 5 is used to deduce that FH and K are equimultiples of AB and C with magnitudes m.a and m.b respectively. The proof in the other case where AE is greater than AB is similar.