Extracts from Euclid's *Elements of Geometry* Book III (T.L. Heath's Edition)

Transcribed by D. R. Wilkins

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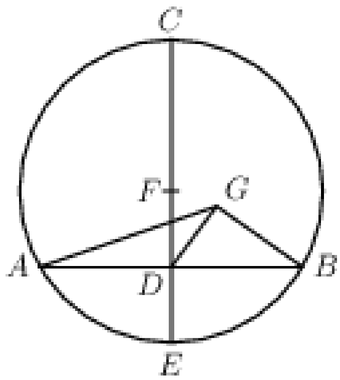
Proposition 1

To find the centre of a given circle.

Let ABC be the given circle;
thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D;
from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F;
I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre,
and let GA, GD, GB be joined.
Then, since AD is equal to DB, and DG is common,
the two sides AD, DG are equal to the two sides BD, DG respectively;
and the base GA is equal to the base GB, for they are radii;
therefore the angle ADG is equal to the angle DGB. [I. 8]
But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the the equal angles is right; [I Def. 10]
therefore the angle GDB is right.
But the angle FDB is also right;
Therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.
Therefore G is not the centre of the circle ABC.
Similarly we can prove that neither is any other point except F.
Therefore the point F is the centre of the circle ABC.

Porism. From this, it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

Q.E.F.

PROPOSITION 2

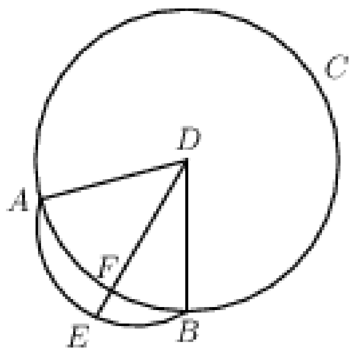
If on the circumference of a given circle two points be taken at random, the straight line joining the points wil fall within the circle.

Let ABC be a circle, and let two points A and B be taken at random on its circumference;

I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB;

let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.



Then since DA is equal to DB,

the angle DAE is also equal to the angle DBE. [I. 5] And, since one side AEB of the triangle DAE is produced,

the angle DEB is greater than the angle DAE. [I. 16]

But the angle DAE is equal to the angle DBE;

therefore the angle DEB is greater than the angle DBE. And the greater angle is subtended by the greater side; [I. 19]

therefore DB is greater than DE.

But DB is equal to DF;

therefore DF is greater than DE,

the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within. Therefore etc.

Q.E.D.

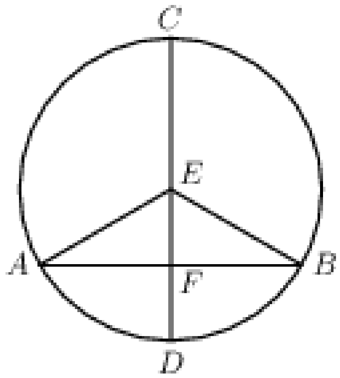
Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F;

I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common,

two sides are equal to two sides;

and the base EA is equal to the base EB;

therefore the angle AFE is equal to the angle BFE. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;

I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction,

since EA is equal to EB,

the angle EAF is also equal to the angle EBF. [I. 5]

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore AF is equal to FB.

Therefore etc.

Q.E.D.

Proposition 4

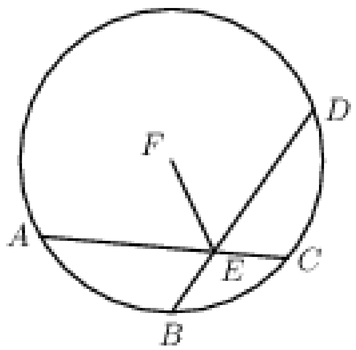
If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another in E;

I say that they do not bisect one another.

For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED;

let the centre of the circle $ABCD$ be taken [iii. 1], and let it be F; let FE be joined.



Then, since a straight line FE through the centre bisects a straight line AC not through the centre;

it also cuts it at right angles; [iii. 3]

therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD, it also cuts it at right angles; [iii. 3]

therefore the angle FEB is right.

But the angle FBA was also proved right;

therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc.

Q.E.D.

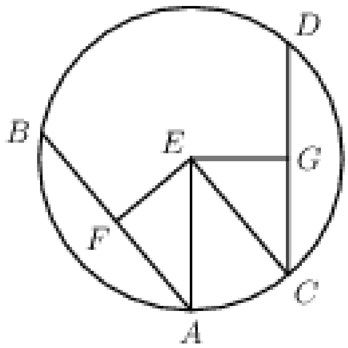
Proposition 14

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let $ABDC$ be a circle, and let AB, CD be equal straight lines in it;

I say that AB, CD are equally distant from the centre.

For let the centre of the circle $ABDC$ be taken [III. 1], and let it be E; from E let EF, EG be drawn perpendicular to AB, CD, and let AE, EC be joined.



Then, since a straight line EF through the centre cuts a straight line AB not through the centre at right angles, it also bisects it. [III. 3]

Therefore AF is equal to FB;

therefore AB is the double of AF.

For the same reason

CD is also the double of CG;

and AB is equal to CD;

therefore AF is also equal to CG.

And, since AE is equal to EC,

the square on AE is also equal to the square on EC.

But the squares on AF, EF are equal to the square on AE, for the angle at F is right;

and the squares on EG, GC are equal to the square on EC, for the angle at G is right; [I. 47]

therefore the squares on AF, FE are equal to the squares on CG, GE,

of which the square on AF is equal to the square on CG, for AF is equal to CG;

therefore the square on FE which remains is equal to the square on EG,

therefore EF is equal to EG.

But in a circle straight lines are said to be equally distant from the centre; that is, let EF be equal to EG.

I say that AB is also equal to CD.

For, with the same construction, we can prove, similarly, that AB is double of AF, and CD of CG.

And, since AE is equal to CE,

the square on AE is equal to the square on CE.

But the squares on EF, FA are equal to the square on AE, and the squares on EG, GC equal to the square on CE. [I. 47]

Therefore the squares on EF, FA are equal to the squares on EG, GC,

of which the square on EF is equal to the square on EG, for EF is equal to EG;

therefore the square on AF which remains is equal to the square on CG;

therefore AF is equal to CG.

And AB is double of AF, and CD double of CG;

therefore AB is equal to CD.

Therefore etc.

Q.E.D.

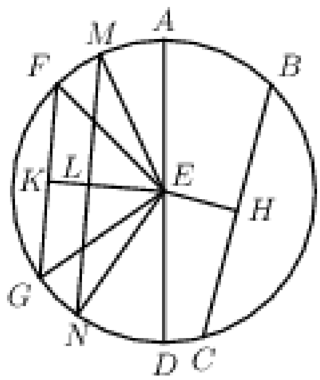
Proposition 15

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let $ABCD$ be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD, and FG more remote;

I say that AD is the greatest and BC greater than FG.

For from the centre E let EH, EK be drawn perpendicular to BC, FG.



Then, since BC is nearer to the centre and FG more remote, EK is greater than EH. [III. Def. 5]

Let EL be made equal to EH, through L let LM be drawn at right angles to EK and crried through to N, and let ME, EN, FE, EG be joined.

Then, since EH is equal to EL,

BC is also equal to MN. [III. 14]

Again, since AE is equal to EM, and ED to EN,

AD is equal to ME, EN.

But ME, EN are greater than MN, and MN is equal to BC;

therefore AD is greater than BC.

And since the two sides ME, EN are equal to the two sides FE, EG,

and the angle MEN greater than the angle FEG,

therefore the base MN is greater than the base FG. [I. 24]

But MN was proved equal to BC.

Therefore the diameter AD is the greatest and BC greater than FG.

Therefore etc.

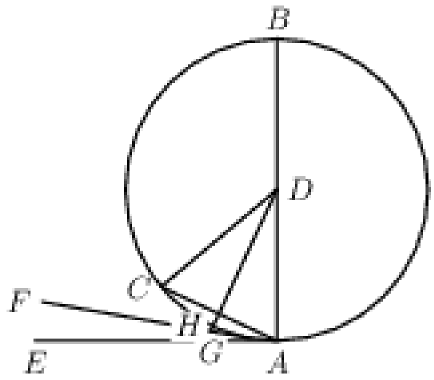
Q.E.D.

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.



Since DA is equal to DC,

the angle DAC is also equal to the angle ACD. [I. 5]

But the angle DAC is right;

therefore the angle ACD is also right:

thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible. [I. 17]

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

Let it fall as AE;

I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right,

and the angle DAG is less than a right angle,

AD is greater than DG. [I. 19]

But DA is equal to DH;

therefore DH is greater than DG, the less than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle,

and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilinear angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contined by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed;

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.—

PORISM. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Q.E.D.

Proposition 17

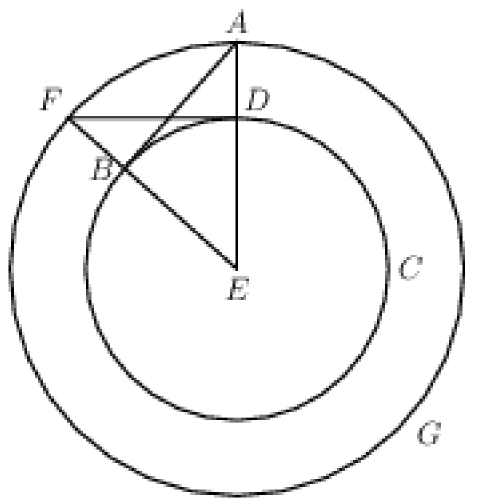
From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; thus it is required to draw from the point A a straight line touching the circle BCD.

For let the centre E of the circle be taken. [bkiii. 1]

let AE be joined, and with centre E and distance EA let the circle AFG be described;

from D let DF be drawn at right angles to EA, and let AF, AB be joined; I say that AB has been drawn from the point A touching the circle BCD.



For, since E is the centre of the circles BCD, AFG,

EA is equal to EF, and ED to EB;

therefore the two sides AE, EB are equal to the two sides FE, ED:

and they contain a common angle, the angle at E;

therefore the base DF is equal to the base AB,

and the triangle DEF is equal to the triangle BEA,

and the remaining angles to the remaining angles; [I. 4]

therefore the angle EDF is equal to the angle EBA.

But the angle EDF is right;

therefore the angle EBA is also right.

Now EB is a radius;

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.]

therefore AB touches the circle BCD.

Therefore from the given point A the straight line AB has been drawn touching the circle BCD.

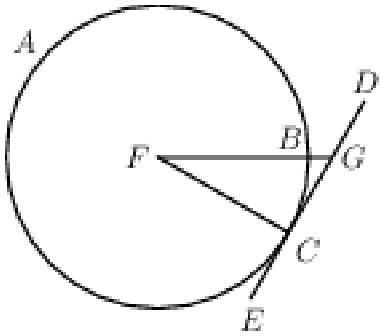
Proposition 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C;

I say that FC is perpendicular to DE.

For, if not, let FG be drawn from F perpendicular to DE.



Then, since the angle FGC is right,

the angle FCG is acute; [I. 17] and the greater angle is subtended by the greater side;

therefore FC is greater than FG.

But FC is equal to FB;

therefore FB is also greater than FG,

the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC;

therefore FC is perpendicular to DE. Therefore, etc.

Q.E.D.

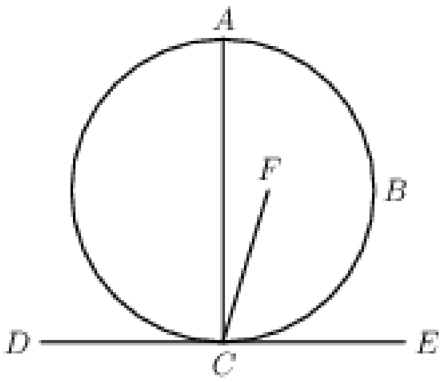
Proposition 19

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line DE touch the circle ABC at the point C, and from C let CA be drawn at right angles to DE;

I say that the centre of the circle is on AC.

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.



Since a straight line D touches the circle ABC,

and FC has been joined from the point of contact,

FC is perpendicular to DE; [III. 18]

therefore the angle FCE is right.

But the angle ACE is also right;

therefore the angle ACE is equal to the angle ACE,

the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except a point on AC. Therefore, etc.

Q.E.D.

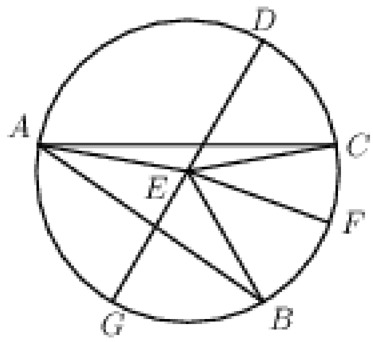
Proposition 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle at its centre, and the angle BAC an angle at the circumference, and let them have the same circumference BC as base;

I say that the angle BEC is double of the angle BAC.

For let AB be joined and drawn through to F.



Then, since EA is equal to EB,

the angle EAB is also equal to the angle EBA; [I. 5]

therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA; [I. 32] therefore the angle BEF is also double of the angle EAB.

For the same reason

the angle FEC is also double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.

Again let another straight line be inflected, and let there be another angle BDC; let DE be joined and produced to G.

Similarly then we can prove that the angle GEC is double of the angle EDC,

of which the angle GEB is double of the angle EDB;

therefore the angle BEC which remains is double of the angle BDC.

Therefore, etc.

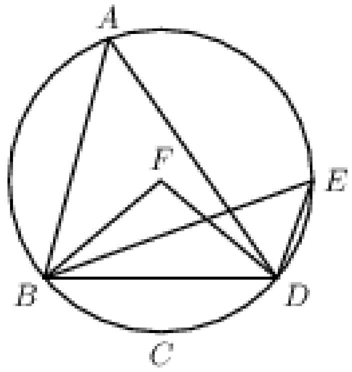
Q.E.D.

Proposition 21

In a circle the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD, BED be angles in the same segment $BAED$; I say that the angles BAD, BED are equal to one another.

For let the centre of circle $ABCD$ be taken, and let it be F; let BF, FD be joined.



Now, since the angle BFD is at the centre,
and the angle BAD at the circumference,
and they have the same circumference BCD as base,
therefore the angle BFD is double of the angle BAD [III. 20]

For the same reason
the angle BFD is also double of the angle BED;
therefore the angle BAD is equal to the angle BED.
Therefore, etc.

Q.E.D.

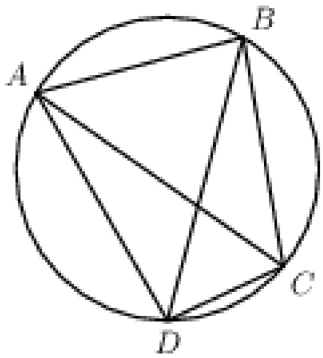
Proposition 22

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral in it;

I say that the opposite angles are equal to two right angles.

Let AC, BD be joined.



Then, since in any triangle the three angles are equal to two right angles, [I. 32]

the three angles CAB, ABC, BCA of the triangle ABC are equal to two right angles.

But the angle CAB is euqal to the angle BDC, for they are in the same segment $BADC$; [III. 21] and the angle ACB is equal to the angle ADB, for they are in the same segment $ADCB$;

therefore the whole angle ADC is equal to the angles BAC, ACB.

Let the angle ABC be added to each; therefore the angles ABC, BAC, ACB are equal to the angles ABC, ADC.

But the angles ABC, BAC, ACB are equal to two right angles;

therefore the angles ABC, ADC are also equal to two right angles.

Similarly we can prove that the angles BAD, DCB are also equal to two right angles.

Therefore, etc.

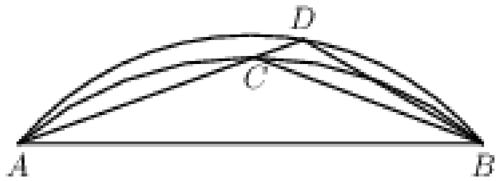
Q.E.D.

Proposition 23

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line AB let two similar and unequal segments of circles ACB, ADB be constructed on the same side;

let ACD be drawn through, and let CB, DB be joined.



Then, since the segment ACB is similar to the segment ADB,

and similar segments of circles are those which admit equal angles, [III. Def. 11]

the angle ACB is equal to the angle ADB, the exterior to the interior: which is impossible. [I. 16] Therefore, etc.

Q.E.D.

Proposition 24

Similar segments of circles on equal straight lines are equal to one another.

For let AEB, CFD be similar segments of circles on equal straight lines AB, CD;

I say that the segment AEB is equal to the segment CFD.

For, if the segement AEB be applied to CFD, and if the point A be placed on C and the straight line AB on CD,

the point B will also coincide with the point D, because AB is equal to CD;

and, AB coinciding with CD,

the segment AEB will also coincide with CFD.

For, if the straight line AB coincide with CD but the segment AEB do not coincide with CFD, it will either fall within it, or outside it;

or it will fall awry, as CGD, and a circle cuts a circle at more points than two: which is impossible. [III. 10]

Therefore, if the straight line AB be applied to CD, the segment AEB will nto fail to coincide with CFF also;

therefore it will coincide with it and will be equal to it.

Therefore, etc.

Q.E.D.

Proposition 25

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let ABC be the given segment of a circle;

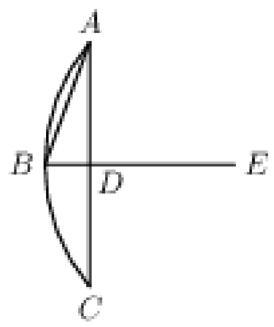
thus it is required to describe the complete circle belonging to the segment ABC, that is, of which it is a segment.

For let AC be bisected at D, let DB be drawn from the point$D$ at right angles to AC, and let AB be joined;

the angle ABD is then greater than, equal to, or less than the angle BAD.

First let it be greater;

and on the straight line BA, and at the point A on it, let the angle BAE be constructed equal to the angle ABD; let DB be drawn through to E, and let EC be joined.



Then, since the angle ABE is equal to the angle BAE, the straight line EB is also equal to EA. [I. 6]

And, since AD is equal to DC,

and DE is common,

the two sides AD, DE are equal to the two sides CD, DE respectively;

and the angle ADE is equal to the angle CD, for each is right;

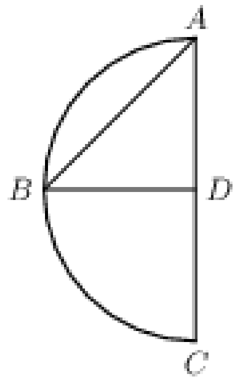
therefore the base AE is equal to the base CE; therefore the three straight lines AE, EB, EC are equal to one another.

Therefore the circle drawn with centre E and distance one of the straight line AE, EB, EC will also pass through the remaining points and will have been completed. [III. 9]

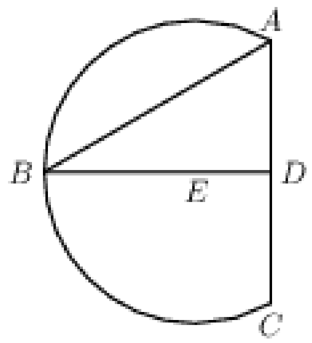
Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment ABC is less than a semicircle, because the centre E happens to be outside it.

Similarly, even if the angle ABD be equal to the angle BAD,
AD being equal ot each of hte two BD, DC,
the three straight lines DA, DB, C will be equal to one another,
D will be the centre of the completed circle,
and ABC will clearly be a semicircle.



But, if the angle ABD be less than the angle BAD, and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD, the centre will fall on DB within the segment ABC, and the segment ABC will clearly be greater than a semicircle.



Therefore, given a segment of a circle, the complete circle has been described.

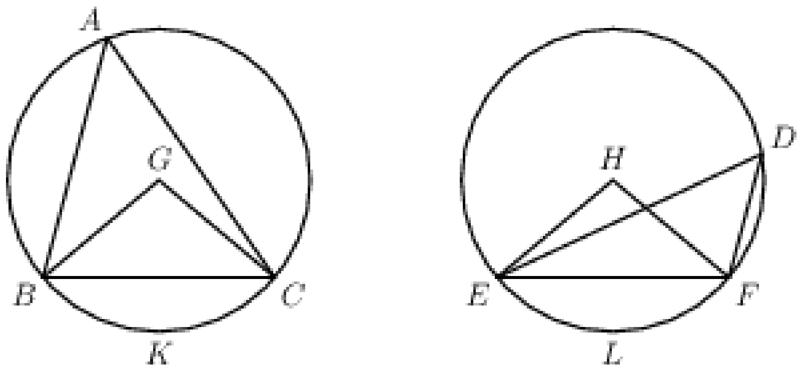
Q.E.F.

Proposition 26

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let ABC, DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC, EHF, and at the circumferences the angles BAC, EDF;

I say that the circumference BKC is equal to the circumference ELF.



For let BC, EF be joined.

Now, since the circles ABC, DEF are equal,

the radii are equal.

Thus the two straight lines BG, GC are equal to the two straight lines EH, HF;

and the angle at G is equal ot the angle at H;

therefore the base BC is equal to the base EF. [I. 4]

And, since the angle at A is equal to the angle at D,

the segment BAC is similar to the segment EDF; [III. Def. 11]

and there are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another; [III. 24]

therefore the segment BAC is equal to EDF.

But the whole circle ABC is also equal to the whole circle DEF;

therefore the circumference BKC which remains is equal to the circumference ELF.

Therefore etc.

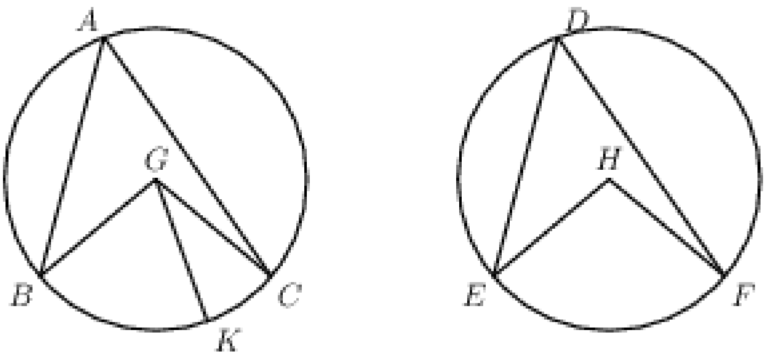
Q.E.D.

Proposition 27

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles ABC, DEF, on equal circumferences BC, EF, let the angles BGC, EHF stand at the centres G, H, and the angles BAC, EF at the circumferences;

I say that the angle BAC is equal to the angle EDF.



For, if the angle BGC is unequal to the angle EHF,
one of them is greater.

Let the angle BGC be greater: and on the straight line BG, and at the point G on it, let the angle BGK be constructed equal to the angle EHF.

Now equal angles stand on equal circumferences, when they are at the centres; [III. 26]
therefore the circumference BK is euqal to the circumference EF.

But EF is equal to BC;
Therefore BK is also equal to BC, the less to the greater: which is impossible.

Therefore the angle BGC is not unequal to the angle EHF;
therefore it is equal to it.

And the angle at A is half of the angle BGC,
and the angle at D half of the angle EHF; [III. 20]
therefore the angle at A is also equal to the angle at D.

Therefore etc.

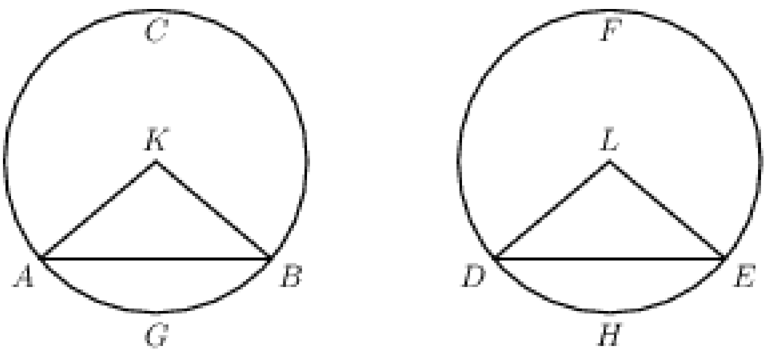
Q.E.D.

Proposition 28

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC, DEF be equal circles, and in the circles let AB, DE be equal straight lines cutting off ACB, DFE as greater circumferences and AGB, DHE as lesser;

I say that the greater circumference ACB is equal to the greater circumference DFE, and the less circumference AGB to DHE.



For let the centres K, L of the circles be taken, and let AK, KE, DL, LB be joined.

Now, since the circles are equal,

the radii are also equal;

therefore the two sides AK, KB are equal to the two sides DL, LE;

and the base AB is equal to the base DE;

therefore the angle AKB is equal ot the angle DLE. [I. 8]

But equal angles stand on equal circumferences, when they are at the centres; [III. 26] therefore the circumference AGB is equal to DHE.

And the whole circle ABC is also equal to the whole circle DEF; therefore the circumference ACB which remains is also equal to the circumference DFE which remains.

Therefore etc.

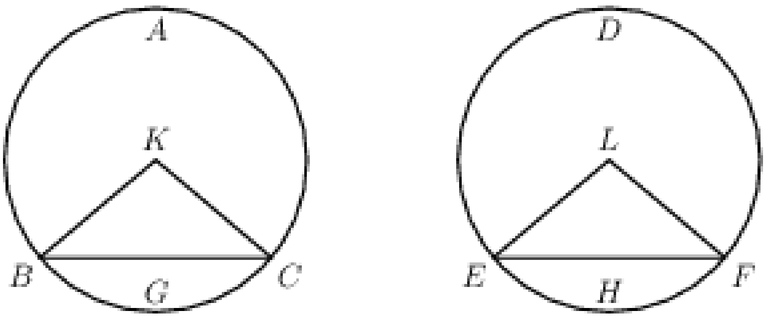
Q.E.D.

PROPOSITION 29

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and in them let equal circumferences BGC, EHF be cut off; and let the straight lines BC, EF be joined;

I say BC is equal to EF.



For let the centres of the circles be taken, and let them be K, L; let BK, KC, EL, LF be joined.

Now, since the circumference BGC is equal to the circumference EHF,

the angle BKC is also equal to the angle ELF.

III. 27

And, since the circles ABC, DEF are equal,

the radii are also equal;

therefore the two sides BK, KC are equal to the two sides EL, LF; and they contain equal angles;

therefore the base BC is equal to the base EF. [I. 4]

Therefore etc.

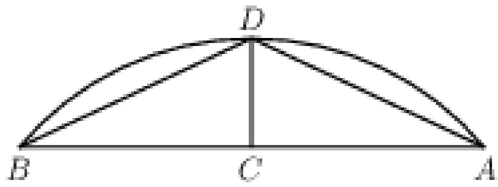
Q.E.D.

Proposition 30

To bisect a given circumference.

Let ADB be a given circumference;
thus it is required to bisect the circumference ADB.

Let AB be joined and bisected at C; from the point C let CD be drawn at right angles to the straight line AB, and let AD, DB be joined.



Then, since AC is equal to CB, and CD is common,
the two sides AC, CD are equal to the two sides BC, CD;
and the angle ACD is equal to the angle BCD, for each is right;
therefore the base AD is equal to the base DB. [I. 4]

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less; [III. 28]
and each of the circumferences AD, DB is less than a semicircle;
therefore the circumference AD is equal to the circumference DB.

Therefore the given circumference has been bisected at the point D.

Q.E.F.