Propositions from Euclid's *Elements of Geometry*
Book IV (T.L. Heath's Edition)

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Definitions

1. A rectilineal figure is said to be **inscribed in a rectilineal figure** when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2. Similarly a figure is said to be **circumscribed about a figure** when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3. A rectilineal figure is said to be **inscribed in a circle** when each angle of the inscribed figure lies on the circumference of the circle. A rectilineal figure is said to be **circumscribed about a circle**, when each side of the circumscribed figure touches the circumference of the circle. Similarly a circle is said to be **inscribed in a figure** when the circumference of the circle touches each side of the figure in which it is circumscribed. A circle is said to be **circumscribed abut a figure** when the circumference of the circle passes through each angle of the figure about which it is circumscribed. A straight line is said to be **fitted into a circle** when its extremities are on the circumference of the circle.

Proposition 1

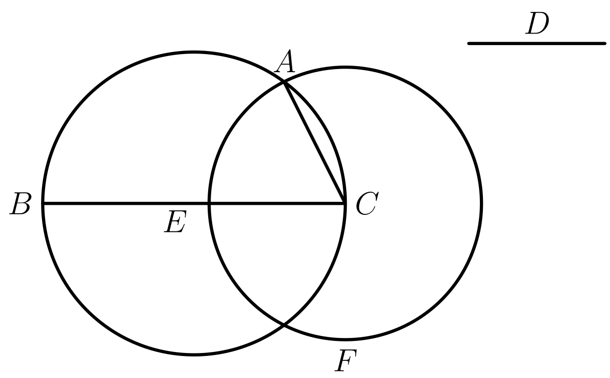
Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given line not greater than the diameter of the circle; thus it is required to fit into the circle ABC a straight line equal to the straight line D.

Let a diameter BC of the circle be drawn.

Then, if BC is equal to D, that which was enjoined will have been done; for BC has been fitted into the circle ABC equal to the straight line D.

But, if BC is greater than D, let CE be made equal to D, and with centre C and distance CE let the circle EF be described; let CA be joined.



Then, since the point C is the centre of the circle EAF, CA is equal to CE. But CE is equal to D; therefore D is also equal to CA.

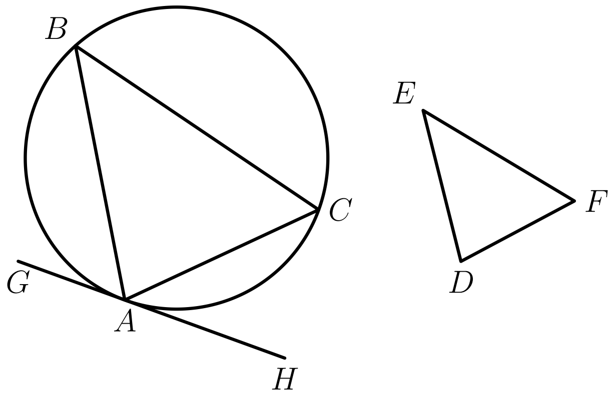
Therefore into the given circle ABC there has been fitted CA equal to the given straight line D.

Proposition 2

In a given circle to inscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to inscribe in the circle ABC a triangle equiangular with the triangle DEF.

Let GH be drawn touching the circle ABC at A [III. 16, Por.]; on the straight line AH, and at the point A on it, let the angle HAC be constructed equal to the angle DEF, and on the straight line AG, and at the point A on it, let the angle GAB be constructed equal to the angle DFE [I. 23]; let BC be joined.



Then, since a straight line AH touches the circle ABC, and from the point of contact at A the straight line AC is drawn across in the circle, therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle [III. 32]. But the angle HAC is equal to the angle DEF; therefore the angle ABC is also equal to the angle DEF. For the same reason the angle ACB is also equal to the angle DFE; therefore the remaining angle BAC is also equal to the remaining angle EDF.

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle.

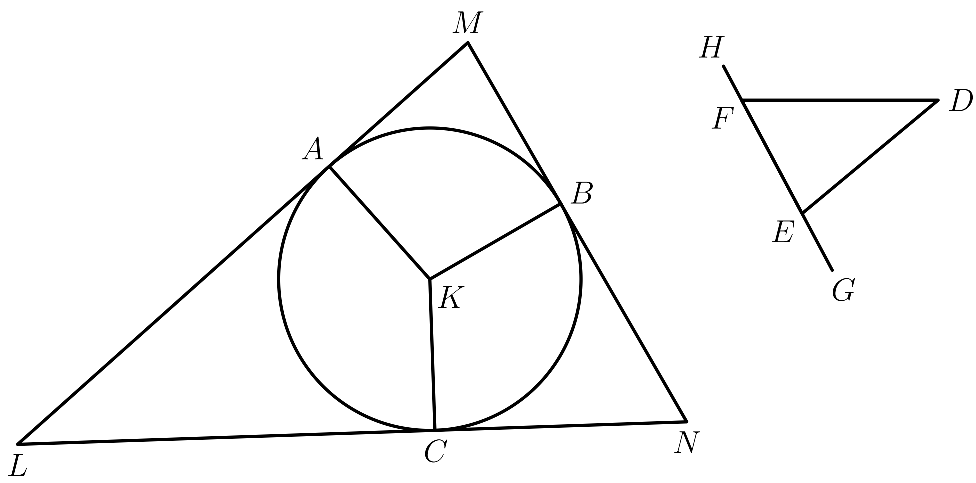
Q.E.F.

Proposition 3

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to circumscribe about the circle ABC a triangle equiangular with the triangle DEF.

Let EF be produced in both directions to the points G, H, let the centre K of the circle ABC be taken [III. 1], and let the straight line KB be drawn across at random; on the straight line KB, and at the point K on it, let the angle BKA be constructed equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C let LAM, MBN, NCL be drawn touching the circle ABC III. 16, Por..



Now, since LM, MN, NL touch the circle ABC at the points A, B, C, and KA, KB, KC have been joined from the centre K to the points A, B, C, therefore the angles at the points A, B, C are right [III. 18]. And, since the four angles of the quadrilateral $AMBK$ are equal to four right angles, inasmuch as $AMBK$ is in fact divisible into two triangles, and the angles KAM, KBM are right; therefore the remaining angles AKB, AMB are equal to two right angles. But the angles DEG, DEF are also equal to two right angles. [I. 13]; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which the angle AKB is equal to the angle DEG; therefore the angle AMB which remains is equal to the angle DEF which remains.

Similarly it can be proved that the angle LNB is also equal to the angle DFE; therefore the remaining angle MLN is equal to the angle EDF.

Therefore the triangle LMN is equiangular with the triangle DEF; and it has been circumscribed about the circle ABC. Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

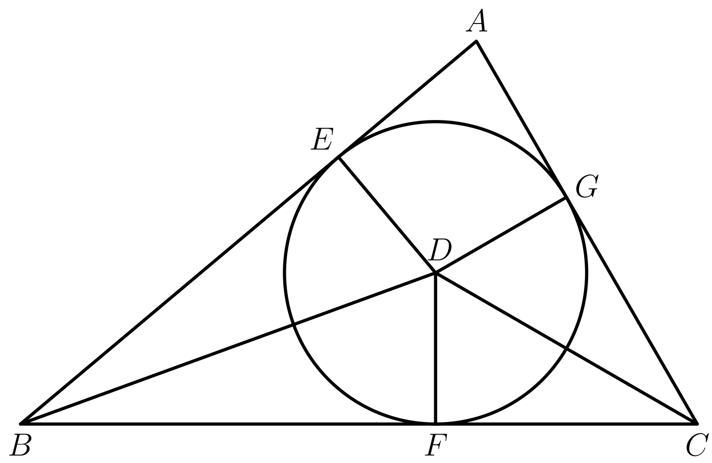
Q.E.F.

Proposition 4

In a given triangle to inscribe a circle.

Let ABC be the given triangle; thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB be bisected by the straight lines BD, CD [I. 9], and let these meet one another at the point D; from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA.



Now, since the angle ABD is equal to the angle CBD, and the right angle BED is also equal to the right angle BFD, EBD, FBD are two triangles having the two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles; therefore they will also have the remaining sides equal to the remaining sides; therefore DE is equal to DF.

For the same reason DG is also equal to DF. Therefore the three straight lines DE, DF, DG are equal to one another; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right. For if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle: which was proved absurd [III. 16]; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA; therefore it will touch them, and will be the circle inscribed in the triangle ABC [IV. Def. 5]. Let it be inscribed, as FGE. Therefore, in the given triangle ABC the circle EFG has been inscribed.

Q.E.F.

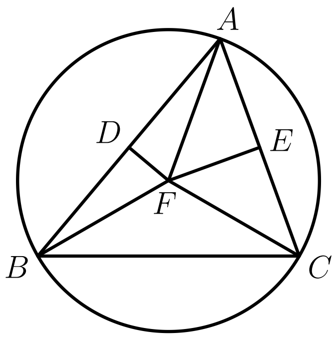
Proposition 5

About a given triangle to circumscribe a circle.

Let ABC be the given triangle; thus it is required to circumscribe a circle about the given triangle ABC.

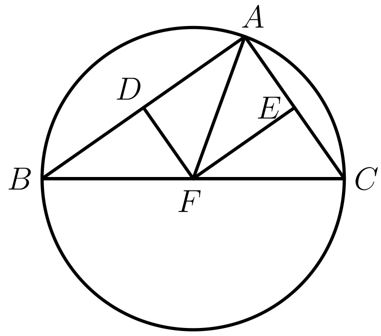
Let the straight lines AB, AC be bisected at the points D, E [I. 10], and from the points D, E let DE, DF be drawn at right angles to AB, AC; they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.



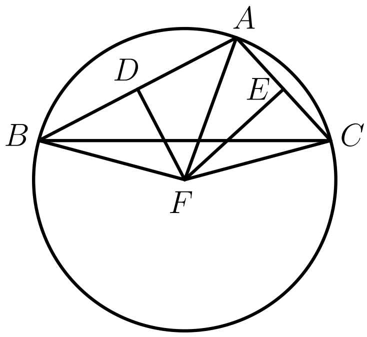
Then, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base FB [I. 4]. Similarly we can prove that CF is also equal to AF; so that FB is also equal to FC; therefore the three straight lines FA, FB, FC are equal to one another, Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC. Let it be circumscribed, as ABC.

Next, let DE, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.



Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.



Then again, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base BF [I. 4]. Similarly we can prove that CF is also equal to AF; so that BF is also equal to FC; therefore the circle described with centre F and distance on of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

Therefore about the given triangle a circle has been circumscribed.

Q.E.F.

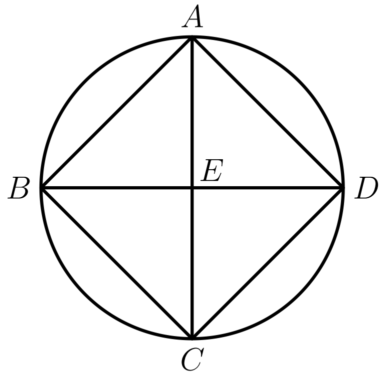
And it is manifest that, when the centre of the circle falls within the triange, the angle BAC, being in a segment greater than the semicircle, is less than a right angle; when the centre falls on the straight line BC, the angle BAC, being in a semicircle, is right; and when the centre of the circle falls outseide the triangle, the angle BAC, being in a segment less than a semicircle, is greater than a right angle [III. 31].

PROPOSITION 6

In a given circle to inscribe a square.

Let $ABCD$ be the given circle; thus it is required to inscribe a square in the circle $ABCD$.

Let two diameters AC, BD of the circle $ABCD$ be drawn at right angles to one another, and let AB, BC, CD, DA be joined.



Then, since BE is equal to ED, for E is the centre, and EA is common and at right angles, therefore the base AB is equal to the base AD. For the same reason each of the straight lines BC, CD is also equal to each of the straight lines AB, AD; therefore the quadrilateral $ABCD$ is equilateral.

I say next that it is also right-angled.

For, since the straight line BD is a diameter of the circle $ABCD$, therefore BAD is a semicircle; therefore the angle BAD is right III. 31. For the same reason each of the angles ABC, BCD, CDA is also right; therefore the quadrilateral $ABCD$ is right-angled. But it was also proved equilateral; therefore it is a square [I. Def. 22] and it has been inscribed in the circle $ABCD$. Therefore in the given circle the square $ABCD$ has been inscribed.

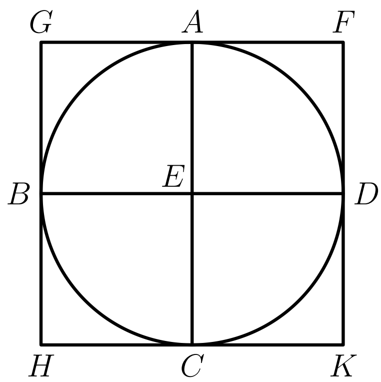
Q.E.F.

Proposition 7

About a given circle to circumscribe a square.

Let $ABCD$ be the given circle; thus it is required to circumscribe a square about the circle $ABCD$.

Let two diameters AC, BD of the circle $ABCD$ be drawn at right angles to one another, and through the points A, B, C, D let FG, GH, HK, KF be drawn touching the circle $ABCD$ [III. 16, Por.].



Then, since FG touches the circle ABC, and EA has been joined from the centre E to the point of contact at A, therefore the angles at A are right [III. 18]. For the same reason the angles at the points B, C, D are also right.

Now, since the angle AEB is right, and the angle EBG is also right [I. 28], therefore GH is parallel to AC. For the same reason AC is also parallel to FK, so that GH is also parallel to FK [I. 30]. Similarly, we can prove that each of the straight lines GF, HK is parallel to BED. Therefore GK, GC, AK, FB, BK are parallelograms; therefore GF is equal to HK and GH to FK [I. 34]. And, since AC is equal to BD, and AC is also equal to each of the straight lines GH, GK, whele BD is equal to each of the straight lines GF, HK [I. 34], therefore the quadrilateral $FGHK$ is equilateral.

I say next that it is also right-angled. For, since $GBEA$ is a parallelogram, and the angle AEB is right, therefore the angle AGB is also right [I. 34]. Similarly we can prove that the angles at H, K, F are also right. Therefore $FGHK$ is right-angled. But it was also proved equilateral; therefore it is a square; and it has been circumscribed about the circle $ABCD$. Therefore about the given circle a square has been circumscribed.

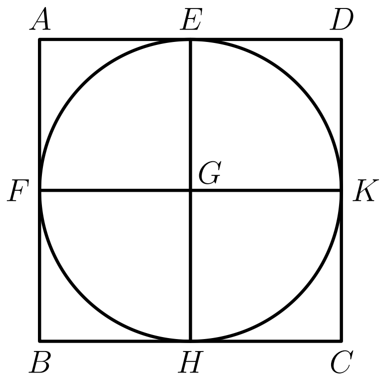
Q.E.F.

Proposition 8

In a given square to inscribe a circle.

Let $ABCD$ be the given square; thus it is required to inscribe a circle in the given square $ABCD$.

Let the straight lines AD, AB be bisected at the points E, F respectively [I. 10], through E let EH be drawn parallel to either AB or CD, and through F let FK be drawn parallel to either AD or BC [I. 31]; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are evidently equal.



Now, since AD is equal to AB, and AE is half of AD, and AF half of AB, therefore AE is equal to AF, so that the opposite sides are also equal; therefore FG is equal to GE. Similarly we can prove that each of the straight lines GH, GK is equal to each of the straight lines FG, GE; therefore the four straight lines GE, GF, GH, GK are equal to one another. Therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will pass also through the remaining points. And it will touch the straight lines AB,BC, CD, DA, because the angles at E, F, H, K are right. For, if the circle cuts AB, BC, CD, DA, the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle: which was proved absurd [III. 16]; therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will not cut the straight lines AB, BC, CD, DA. Therefore it will touch them, and will have been inscribed in the square $ABCD$. Therefore in the given square a circle has been inscribed.

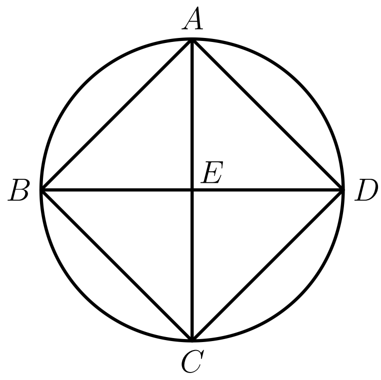
Q.E.F.

Proposition 9

About a given square to circumscribe a circle.

Let $ABCD$ be the given square; thus it is required to circumscribe a circle about the square $ABCD$.

For let AC, BD be joined, and let them cut one another at E. Then, since DA is equal to AB, and AC is common, therefore the two sides DA, AC are equal to the two sides BA, AC; and the base DC is equal to the base BC; therefore the angle DAC is equal to the angle BAC [I. 8]. Therefore the angle DAB has been bisected by AC. Similarly, we can prove that each of the angles ABC, BCD, CDA is bisected by the straight lines AC, DB.



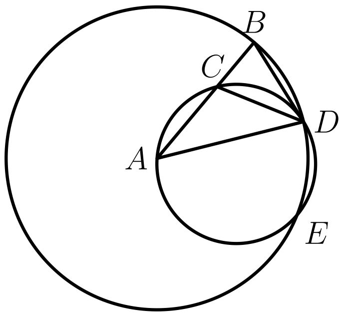
Now, since the angle DAB is equal to the angle ABC, and the angle EAB is half the angle DAB, and the angle EBA half the angle ABC, therefore the angle EAB is also equal to the angle EBA; so that the side EA is also equal to EB [I. 6]. Similarly we can prove that each of the straight lines EA, EB is equal to each of the straight lines EC, ED. Therefore the four straight lines EA, EB, EC, ED are equal to one another. Therefore the circle described with centre E and distance one of the straight lines EA, EB, EC, ED will pass also through the remaining points; and it will have been circumscribed about the square $ABCD$. Let it be circumscribed, as $ABCD$. Therefore about the given square a circle has been circumscribed.

Q.E.F.

Proposition 10

To construct an isosceles triangle havng each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA [II. 11]; with centre A and distance AB let the circle BDE be described, and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE [IV. 1]. Let AD, DC be joined, and let the circle ACD be circumscribed about the triangle ACD [IV. 5].



Then, since the rectangle AB, BC is equal to the square on AC, and AC is equal to BD, Therefore the rectangle AB, BC is equal to the square on BD.

And, since a point B has been taken outside the circle ACD, and from B the two straight lines BA, BD have fallen on the circle ACD, and one of them cuts it, while the other falls on it, and the rectangle AB, BC is equal to the square on BD, therefore BD touches the circle ACD [III. 37]. Since, then, BD touches it, and DC is drawn across from the contact at D, therefore the angle BDC is equal to the angle DAC in the alternate segment of the circle [III. 32]. Since, then, the angle BDC is equal to the angle DAC, let the angle CDA be added to each; therefore the whole angle BDA is equal to the two angles CDA, DAC. But the exterior angle BCD is equal to the angles CDA, DAC; therefore the angle BDA is also equal to the angle BCD. But the angle BDA is equal to the angle CBD, since the side AD is also equal to AB [I. 5]; so that the angle DBA is also equal to the angle BCD. Therefore the three angles BDA, DBA, BCD are equal to one another. And, since the angle DBC is equal to the angle BCD, the side BD is also equal to the side DC [I. 6]. But BD is by hypothesis equal to CA; therefore CA is also equal to CD, so that the angle CDA is also equal to the angle DAC

[I. 5]; therefore the angles CDA, DAC are double of the angle DAC. But the angle BCD is equal to the angles CDA, DAC; therefore the angle BCD is also double of the angle CAD. But the angle BCD is equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is also double of the angle DAB. Therefore the isosceles triangle ABD has been constructed having each of the angles at the base DB double of the remaining one.

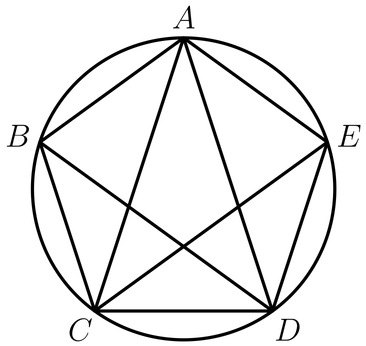
Q.E.F.

Proposition 11

In a given circle to inscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle; thus it is required to inscribe in the circle $ABCDE$ an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G, H double of the angle at F [IV. 10]; let there be inscribed in the circle $ABCDE$ the triangle ACD equiangular with the triangle FGH, so that the angle CAD is equal to the angle at F, and the angles at G, H respectively equal to the angles ACD, CDA [IV. 2]; therefore each of the angles ACD, CDA is also double of the angle ACD. Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB [I. 9], and let AB, BC, DE, EA be joined.



Then, since each of the angles ACD, CDA is double of the angle CAD, and they have been bisected by the straight lines CE, DB, therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. But equal angles stand on equal circumferences [III. 26]; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another. But equal circumferences are subtended by equal straight lines [III. 29]; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; therefore the pentagon $ABCDE$ is equilateral.

I say next that it is also equiangular.

For since the circumference AB is equal to the circumference DE, let BCD be adde to each; therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$. And the angle AED stands on the circumference $ABCD$, and the angle BAE on the circumference $EDCB$; therefore the angle BAE is also equal to the angle AED [III. 27]. For the same reason each of the angles ABC, BCD, CDE is also equal to each of the angles BAE, AED; therefore the pentagon $ABCDE$ is equiangular. But it was also proved equilateral; therefore in a given circle an equilateral and equiangular pentagon has been inscribed.

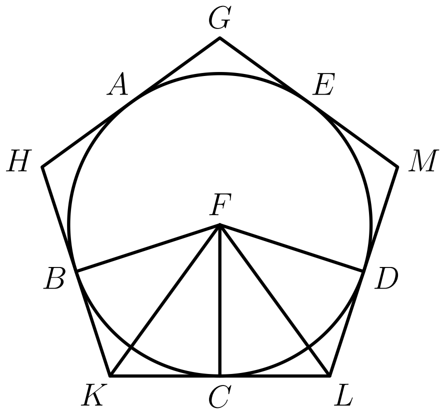
Q.E.F.

Proposition 12

About a given circle to circumscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle; thus it is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$.

Let A, B, C, D, E be conceived to be the angular points of the inscribed pentagon, so that the circumferences AB, BC, CD, DE, AE are equal [IV. 11] through A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle [III. 16, Por.]; let the centre F of the circle $ABCDE$ be taken [III. 1], and let FB, FK, FC, FL, FD be joined.



Then, since the straight line KL touches the circle $ABCDE$ at C, and FC has been joined from the centre F to the point of contact at C, therefore FC is perpendicular to KL; therefore each of the angles at C at right. For the same reason the angles at the points B, D are also right. And, since the angle FCK is right, therefore the square on FK is equal to the squares on FC, CK [I. 47]. For the same reason the square on FK is also equal to the squares on FB, BK; so that the squares on FC, CK are equal to the squares on FB, BK, of which the square on FC is equal to the square on FB; therefore the square on CK which remains is equal to the square on BK. Therefore BK is equal to CK. And, since FB is equal to FC, and FK common, the two sides BF, FK are equal to the two sides CF, FK; and the base BK equal to the base CK; therefore the angle BFK is equal to teh angle KFC [I. 8], and the angle BKF to the angle KFC. Therefore the angle BFC is double of the angle KFC, and the angle BKC of the angle FKC. For the same reason the angle CFD is also double of the angle CFL; and the angle DLC of the angle FLC.

Now, since the circumference BC is equal to CD, the angle BFC is also equal to the angle CFD [III. 27]. And the angle BFC is double of the angle

KFC, and the angle DFC of the angle LFC; therefore the angle KFC is also equal to the angle LFC. But the angle FCK is also equal to the angle FCL; therefore FKC, FLC are two triangles having two angle equal to two angles and one side equal to one side, namely FC which is common to them; therefore they will also have the remaining sides equal to the remaining sides, and the remaining angle to the remaining angle [I. 26]; therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC, And, since KC is equal to CL, therefore KL is double of KC. For the same reason it can be proved that HK is also double of BK. and BK is equal to KC; therefore HK is also equal to KL. Similarly each of the straight lines HG, GM, ML can be proved equal to each of the straight lines HK, KL; therefore the pentagon $GHKLM$ is equilateral.

I say next that it is also equiangular. For, since the angle FKC is equal to the angle FLC, and the angle HKL was proved double of the angle FKC, and the angle KLM double of the angle FLC, therefore the angle HKL is also equal to the angle KLM. Similarly each of the angles KHG, HGM, GML can also be proved equal to each of the angle HKL, KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another. Therefore the pentagon $GHKLM$ is equiangular. And it was also proved equilateral; and it has been circumscribed about the circle $ABCDE$.

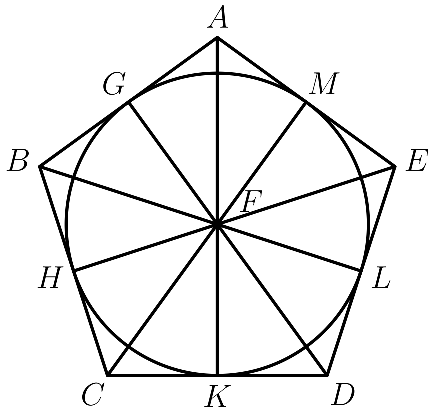
Q.E.F.

Proposition 13

In a given pentagon, which is equilateral and equiangular, to inscribe a circle.

Let $ABCDE$ be the given equilateral and equiangular pentagon; thus it is required to inscribe a circle in the pentagon $ABCDE$.

For let the angles BCD, CDE be bisected by the straight lines CF, DF respectively; and from the point F, at which the straight lines CF, DF meet one another, let the straight lines FB, FA, FE joined.



Then since BC is equal to CD, and CF common, the two sides BC, CF are equal to the two sides DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal to the base DF, and the triangle BCF is equal to the triangle DCF, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend [I. 4]. Therefore the angle CBF is equal to the angle CDF. And since the angle CDE is double of the angle CDF, and the angle CD is equal to the angle ABC, while the angle CDF is equal to the angle CBF; therefore the angle CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle FBC; therefore the angle ABC has been bisected by the straight line BF. Similarly it can be proved that the angles BAE, AED have also been bisected by the straight lines FA, FE respectively.

Now let FG, GH, FK, FL, FM be drawn from the point F perpendicular to the straight lines AB, BC, CD, DE, EA. Then, since the angle HCF is equal to the angle KCF, and the right angle FHC is also equal to the angle FKC, FHC, FKC are two triangles having two angles equal to two angles and one side equal to one side, namely FC which is common to them and subtends one of the equal angles; therefore they will also have the remaining sides equal to the remaining sides [I. 26]; therefore the perpendicular FH is equal to the perpendicular FK. Similarly it can be proved that each of the straight lines FL, FM, FG is also equal to each of the straight lines FH, FK; therefore the five straight lines FG, FH, FK, FL, FM are equal to one another. Therefore the circle described with centre F and distance one of the straight lines FG, FH, FK, FL, FM will pass also through the remaining points; and it will touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right. For if it does not touch them, but cuts them, it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle: which was proved absurd. Therefore the circle described with centre F and distance one of the straight lines FG, FH, FK, FL, FM will not cut the straight lines AB, BC, CD, DE, EA; therefore it will touch them. Let it be described, as $GHKLM$. Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed.

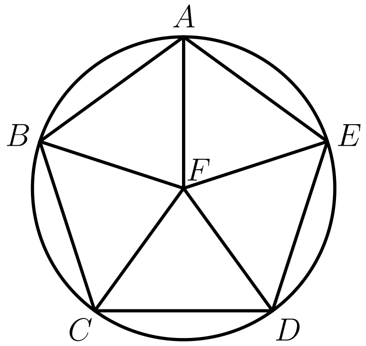
Q.E.F.

Proposition 14

About a given pentagon, which is equilateral and equiangular, to circumscribe a circle.

Let $ABCDE$ be the given pentagon, which is equilateral and equiangular; thus it is required to circumscribe a circle about the pentagon $ABCDE$.

Let the angles BCD, CDE be bisected by the straight lines CF, DF respectively, and from the point F, at which the straight lines meet, let the straight lines FB, FA, FE be joined to the points B, A, E.



Then in manner similar to the preceding it can be proved that the angles CBA, BAE, AED have also been bisected by the straight lines FB, FA, FE respectively.

Now, since the angle BCD is equal to the angle CDE, and the angle FCD is half of the angle BCD, and the angle CDF half of the angle CDE, therefore the angle FCD is also equal to the angle CDF, so that the side FC is also equal to the side FD [I. 6]. Similarly it can be provved that each of the straight lines FB, FA, FE is also equal to each of the straight lines FC, FD; therefore the five straight lines FA, FB, FC, FD, FE are equal to one another. Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC, FD, FE will pass also through the remaining points, and will have been circumscribed. Let it be circumscribed, and let it be $ABCDE$. Therefore about the given pentagon, which is equilateral and equiangular, a circle has been circumscribed.

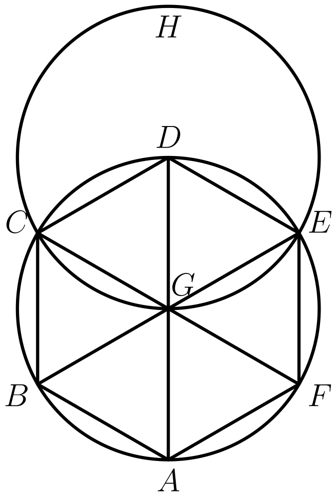
Q.E.F.

Proposition 15

In a given circle to inscribe an equilateral and equiangular hexagon.

Let $ABCDEF$ be the given circle; thus it is required to inscribe an equilateral and equiangular hexagon in the circle $ABCDEF$.

Let the diameter AD of the circle $ABCDEF$ be drawn; let the centre G of the circle be taken, and with centre D and distance DG let the circle $EGCH$ be described; let EG, CG be joined and carried through to the points B, F, and let AB, BC, CD, DE, EF, FA be joined. I say that the hexagon $ABCDEF$ is equilateral and equiangular.



For, since the point G is the centre of the circle $ABCDEF$, GE is equal to GD. Again, since the point D is the centre of the circle CGH, DE is equal to DG, But GE was proved equal to GD; therefore GE is also equal to ED; therefore the triangle EGD is equilateral; and therefore its three angles EGD, GD, DEG are equal to one another, inasmuch as, in isosceles triangles, the angles at the base are equal to one another [I. 5]. And the three angles of the triangle are equal to two right angles [I. 32]; therefore the angle EGD is one-third of two right angles. Similarly the angle DGC can also be proved to be one third of two right angles. And, since the straight line CG standing on EB makes the adjacent angles EGC, CGB equal to two right angles, therefore the remaining angle CGB is also one third of two right angles. Therefore the angles EGD, DGC, CGB are equal to one another; so that the angles vertical to them, the angles BGA, AGF, FGE are equal. Therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another. But equal angles stand on equal circumferences [III. 26]; therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another. And equal circumferences are subtended by equal straight lines [III. 29]; therefore the six straight lines are equal to one another; therefore the hexagon $ABCDEF$ is equilateral.

I say next that it is also equiangular.

For, since the circumference FA is equal to the circumference ED, let the circumference $ABCD$ be added to each; therefore the whole $FABCD$ is equal to the whole $EDCBA$; and the angle FED standes on the circumference $FABCD$, and the angle AFE on the circumference $EDCBA$; therefore the angle AFE is equal to the angle DEF [III. 27]. Similarly it can be proved that the remaining angles of the hexagon $ABCDEF$ are also severally equal to each of the angles AFE, FED; therefore the hexagon $ABCDEF$ is equiangular. But it was also proved equilateral; and it has been inscribed in the circle $ABCDEF$. Therefore in the given circle an equilateral and equiangular hexagon has been inscribed.

Q.E.F.

PORISM. From this it is manifest that the side of the hexagon is equal to the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

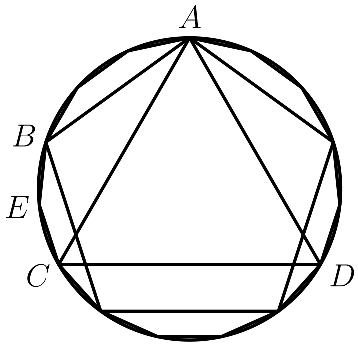
Q.E.F.

Proposition 16

In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Let $ABCD$ be the given circle; thus it is required to inscribe in the circle $ABCD$ a fifteen angled figure which shall be both equilateral and equiangular.

In the circle $ABCD$ let there be inscribed a side AC of the equilateral triangle inscribed in it, and a side AB of an equilateral pentagon; therefore, of the equal segments of which there are fifteen in the circle $ABCD$, there will be five in the circumference ABC which is one-third of the circle, and there will be three in the circumference AB which is one-fifth of the circle; therefore the remainder BC there will be two of the equal segments.



Let BC be bisected at E; therefore each of the circumferences BE, EC is a fifteenth of the circle $ABCD$. If therefore we join BE, EC and fit into the circle $ABCD$ straight lines equal to them and in contiguity, a fifteen-angled figure which is both equilateral and equiangular will have been inscribed in it.

Q.E.F.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in a given fifteen-angled figure and circumscribe one about it.

Q.E.F.