Module MA2321: Analysis in Several Real Variables
Michaelmas Term 2018
Part I (Sections 1, 2, 3 and 4)

D. R. Wilkins

Contents

1 The Real Number System

1.1 A Concise Characterization of the Real Number System

The set $\mathbb{R}$ of *real numbers*, with its usual ordering and algebraic operations of addition and multiplication, is a Dedekind-complete ordered field.

We describe below what a *field* is, what an *ordered field* is, and what is meant by saying that an ordered field is *Dedekind-complete*.

1.2 Fields

Definition A *field* is a set $\mathbb{F}$ on which are defined operations of addition and multiplication, associating elements $x + y$ and xy of $\mathbb{F}$ to each pair x, y of elements of $\mathbb{F}$, for which the following axioms are satisfied:

(i) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *commutative*);

(ii) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *associative*);

(iii) there exists an element 0 of $\mathbb{F}$ with the property that $0 + x = x$ for all $x \in \mathbb{F}$ (i.e., there exists a *zero element* for the operation of addition on $\mathbb{F}$);

(iv) given any $x \in \mathbb{F}$, there exists an element $-x$ of $\mathbb{F}$ satisfying $x+(-x) = 0$ (i.e., *negatives* of elements of $\mathbb{F}$ always exist);

(v) $xy = yx$ for all $x, y \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *commutative*);

(vi) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *associative*);

(vii) there exists an element 1 of $\mathbb{F}$ with the property that $1x = x$ for all $x \in \mathbb{F}$ (i.e., there exists an *identity element* for the operation of multiplication on $\mathbb{F}$);

(viii) given any $x \in \mathbb{F}$ satisfying $x \neq 0$, there exists an element x^{-1} of $\mathbb{F}$ satisfying $xx^{-1} = 1$;

(ix) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$ (i.e., multiplication is *distributive* over addition).

The operations of subtraction and division are defined on a field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $x - y = x + (-y)$ for all elements x and y of $\mathbb{F}$, and moreover $x/y = xy^{-1}$ provided that $y \neq 0$.

1.3 Ordered Fields

Definition An *ordered field* consists of a field $\mathbb{F}$ together with an ordering $<$ on that field that satisfies the following axioms:—

(x) if x and y are elements of $\mathbb{F}$ then one and only one of the three statements $x < y$, $x = y$ and $y < x$ is true (i.e., the ordering satisfies the *Trichotomy Law*);

(xi) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ and $y < z$ then $x < z$ (i.e., the ordering is *transitive*);

(xii) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ then $x + z < y + z$;

(xiii) if x and y are elements of $\mathbb{F}$ which satisfy $0 < x$ and $0 < y$ then $0 < xy$.

We can write $x > y$ in cases where $y < x$. we can write $x \leq y$ in cases where either $x = y$ or $x < y$. We can write $x \geq y$ in cases where either $x = y$ or $y < x$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

1.4 Least Upper Bounds

Let S be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let S be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The following property is satisfied in some ordered fields but not in others.

> **Least Upper Bound Property**: given any non-empty subset S of $\mathbb{F}$ that is bounded above, there exists an element $\sup S$ of $\mathbb{F}$ that is the least upper bound for the set S.

Definition A *Dedekind-complete* ordered field $\mathbb{F}$ is an ordered field which has the Least Upper Bound Property.

1.5 Greatest Lower Bounds

Let S be a subset of an ordered field $\mathbb{F}$. A *lower bound* of S is an element l of $\mathbb{F}$ with the property that $l \leq x$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists. A *greatest lower bound* (or *infimum*) for a set S is a lower bound for that set that is greater than every other lower bound for that set. The greatest lower bound of the set S (if it exists) is denoted by $\inf S$.

Let $\mathbb{F}$ be a Dedekind-complete ordered field. Then, given any non-empty subset S of $\mathbb{F}$ that is bounded below, there exists a greatest lower bound (or *infimum*) $\inf S$ for the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

Remark It can be proved that any two Dedekind-complete ordered fields are isomorphic via an isomorphism that respects the ordering and the algebraic operations on the fields. The theory of *Dedekind cuts* provides a construction that yields a Dedekind-complete ordered field that can represent the system of real numbers. For an account of this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.6 Bounded Sets of Real Numbers

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field. Thus every non-empty subset S of $\mathbb{R}$ that is bounded above has a *least upper bound* (or *supremum*) $\sup S$, and every non-empty subset S of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or *infimum*) $\inf S$.

Let S be a non-empty subset of the real numbers that is bounded (both above and below). Then the closed interval $[\inf S, \sup S]$ is the smallest closed interval in the set $\mathbb{R}$ of real numbers that contains the set S. Indeed if $S \subset [a, b]$, where a and b are real numbers satisfying $a \leq b$, then $a \leq \inf S \leq \sup S \leq b$, and therefore

$$S \subset [\inf S, \sup S] \subset [a, b].$$

1.7 Absolute Values of Real Numbers

Let x be a real number. The *absolute value* $|x|$ of x is defined so that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$$

Lemma 1.1 *Let u and v be real numbers. Then $|u + v| \leq |u| + |v|$ and $|uv| = |u|\,|v|$.*

Proof Let u and v be real numbers. Then

$$-|u| \leq u \leq |u| \quad \text{and} \quad -|v| \leq v \leq |v|.$$

On adding inequalities, we find that

$$-(|u| + |v|) = -|u| - |v| \leq u + v \leq |u| + |v|,$$

and thus

$$u + v \leq |u| + |v| \quad \text{and} \quad -(u + v) \leq |u| + |v|.$$

Now the value of $|u + v|$ is equal to at least one of the numbers $u + v$ and $-(u + v)$. It follows that

$$|u + v| \leq |u| + |v|$$

for all real numbers u and v.

Next we note that $|u|\,|v|$ is the product of one or other of the numbers u and $-u$ with one or other of the numbers v and $-v$, and therefore its value is equal either to uv or to $-uv$. Because both $|u|\,|v|$ and $|uv|$ are non-negative, we conclude that $|uv| = |u|\,|v|$, as required. ∎

Lemma 1.2 *Let u and v be real numbers. Then* $\bigl||u| - |v|\bigr| \leq |u - v|$.

Proof It follows from Lemma 1.1 that

$$|u| = |v + (u - v)| \leq |v| + |u - v|.$$

Therefore $|u| - |v| \leq |u - v|$. Interchanging u and v, we find also that

$$|v| - |u| \leq |v - u| = |u - v|.$$

Now $\bigl||u| - |v|\bigr|$ is equal to one or other of the real numbers $|u| - |v|$ and $|v| - |u|$. It follows that $\bigl||u| - |v|\bigr| \leq |u - v|$, as required. ∎

1.8 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* $x_1, x_2, x_3, \ldots$ of real numbers associates to each positive integer j a corresponding real number x_j.

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing '$x_j \to p$ as $j \to +\infty$', or by writing '$\lim\limits_{j \to +\infty} x_j = p$'.

Let x and p be real numbers, and let ε be a strictly positive real number. Then $|x - p| < \varepsilon$ if and only if both $x - p < \varepsilon$ and $p - x < \varepsilon$. It follows that $|x - p| < \varepsilon$ if and only if $p - \varepsilon < x < p + \varepsilon$. The condition $|x - p| < \varepsilon$ essentially requires that the value of the real number x should agree with p to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p if and only if, given any positive real number ε, there exists some positive integer N such that $p - \varepsilon < x_j < p + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.3 *Every convergent sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number p. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $p-1 < x_j < p+1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $p-1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $p+1$. ∎

Proposition 1.4 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum and difference of these sequences are convergent, and*

$$\begin{aligned} \lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\ \lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j. \end{aligned}$$

Proof Throughout this proof let $p = \lim_{j \to +\infty} x_j$ and $q = \lim_{j \to +\infty} y_j$. It follows directly from the definition of limits that $\lim_{j \to +\infty} (-y_j) = -q$.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (p+q)| < \varepsilon$ whenever $j \geq N$. Now $x_j \to p$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - p| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - p| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - q| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (p+q)| &= |(x_j - p) + (y_j - q)| \leq |x_j - p| + |y_j - q| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to p + q$ as $j \to +\infty$.

On replacing y_j by $-y_j$ for all positive integers j, and using the result that $-y_j \to -q$ as $j \to +\infty$, we see that Thus $x_j - y_j \to p - q$ as $j \to +\infty$, as required. ∎

Lemma 1.5 *Let $x_1, x_2, x_3, \ldots$ be a convergent infinite sequence of real numbers, and let c be a real number. Then*

$$\lim_{j \to +\infty} (cx_j) = c \lim_{j \to +\infty} x_j.$$

Proof Let some strictly positive real number ε be given. Then a strictly positive real number ε_1 can be chosen so that $|c|\,\varepsilon_1 \leq \varepsilon$. There then exists some positive integer N such that $|x_j - p| < \varepsilon_1$ whenever $j \geq N$, where $p = \lim\limits_{j \to +\infty} x_j$. But then

$$|cx_j - cp| < |c|\,\varepsilon_1 \leq \varepsilon$$

whenever $j \geq N$. We conclude that $\lim\limits_{j \to +\infty} cx_j = cp$, as required. ∎

Proposition 1.6 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the product of these sequences is convergent, and*

$$\lim_{j \to +\infty} (x_j y_j) = \left(\lim_{j \to +\infty} x_j \right) \left(\lim_{j \to +\infty} y_j \right).$$

Proof Let $u_j = x_j - p$ and $v_j = y_j - q$ for all positive integers j where $p = \lim\limits_{j \to +\infty} x_j$ and $q = \lim\limits_{j \to +\infty} y_j$. Then

$$\begin{aligned}
\lim_{j \to +\infty} (u_j v_j) &= \lim_{j \to +\infty} (x_j y_j - x_j q - p y_j + pq) \\
&= \lim_{j \to +\infty} (x_j y_j) - q \lim_{j \to +\infty} x_j - p \lim_{j \to +\infty} y_j + pq \\
&= \lim_{j \to +\infty} (x_j y_j) - pq.
\end{aligned}$$

Let some strictly positive real number ε be given. It follows from the definition of limits that $\lim\limits_{j \to +\infty} u_j = 0$ and $\lim\limits_{j \to +\infty} v_j = 0$. Therefore there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. Thus $\lim\limits_{j \to +\infty} u_j v_j = 0$, and therefore $\lim\limits_{j \to +\infty} (x_j y_j) - pq = 0$. The result follows.

Proposition 1.7 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers, where $y_j \neq 0$ for all positive integers j and $\lim\limits_{j \to +\infty} y_j \neq 0$. Then the quotient of the sequences (x_j) and (y_j) is convergent, and*

$$\lim_{j \to +\infty} \frac{x_j}{y_j} = \frac{\lim\limits_{j \to +\infty} x_j}{\lim\limits_{j \to +\infty} y_j}.$$

Proof Let $p = \lim_{j\to+\infty} x_j$ and Let $q = \lim_{j\to+\infty} y_j$. Then

$$\frac{x_j}{y_j} - \frac{p}{q} = \frac{qx_j - py_j}{qy_j}$$

for all positive integers j. Now there exists some positive integer N_1 such that $|y_j - q| < \frac{1}{2}|q|$ whenever $j \geq N_1$. Then $|y_j| \geq \frac{1}{2}|q|$ whenever $j \geq N_1$, and therefore

$$\left|\frac{x_j}{y_j} - \frac{p}{q}\right| \leq \frac{2}{|q|^2}\,|qx_j - py_j|$$

whenever $j \geq N_1$.

Let some strictly positive real number ε be given. Applying Lemma 1.5 and Proposition 1.4, we find that

$$\lim_{j\to+\infty}(qx_j - py_j) = q\lim_{j\to+\infty} x_j - p\lim_{j\to+\infty} y_j = qp - pq = 0.$$

Therefore there exists some positive integer N satisfying $N \geq N_1$ with the property that

$$|qx_j - py_j| < \tfrac{1}{2}|q|^2\varepsilon$$

whenever $j \geq N$. But then

$$\left|\frac{x_j}{y_j} - \frac{p}{q}\right| < \varepsilon$$

whenever $j \geq N$. Thus

$$\lim_{j\to+\infty}\frac{x_j}{y_j} = \frac{p}{q},$$

as required. ∎

1.9 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.8 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.10 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

1.11 The Bolzano-Weierstrass Theorem

Theorem 1.9 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.8 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.8. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

2 Convergence in Euclidean Spaces

2.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 2.1 (Schwarz's Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Proposition 2.2 (Triangle Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.*

Proof Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Proposition 2.2) that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact that the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2 Convergence of Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 2.3 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

Proof Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n\left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

2.3 Limit Points of Subsets of Euclidean Spaces

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^n$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Lemma 2.4 *Let X be a subset of n-dimensional Euclidean space $\mathbf{R}^n$. A point $\mathbf{p}$ is a limit point of the set X if and only if, given any positive real number δ, the set*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set.

Proof Suppose that, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set. Then, for each positive real number δ, the set thus determined by δ must consist of more than just the single point $\mathbf{p}$, and therefore there exists $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus $\mathbf{p}$ is a limit point of the set X.

Now let $\mathbf{p}$ be an arbitrary point of $\mathbb{R}^n$. Suppose that there exists some positive real number δ_0 for which the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

is finite. If this set does not contain any points of X distinct from the point $\mathbf{p}$ then $\mathbf{p}$ is not a limit point of the set X. Otherwise let δ be the minimum value of $|\mathbf{x} - \mathbf{p}|$ as $\mathbf{x}$ ranges over all points of the finite set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

that are distinct from $\mathbf{p}$. Then $\delta > 0$, and $|\mathbf{x} - \mathbf{p}| \geq \delta$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Thus the point $\mathbf{p}$ is not a limit point of the set X. The result follows. ∎

Lemma 2.5 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$ and let $\mathbf{p} \in \mathbb{R}^n$. Then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.*

Proof Suppose that $\mathbf{p}$ is a limit point of X. Then, for each positive integer j, there exists a point $\mathbf{x}_j$ of X for which $0 < |\mathbf{x}_j - \mathbf{p}| < 1/j$. The points $\mathbf{x}_j$

satisfying this condition then constitute an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, that converge to the point $\mathbf{p}$.

Conversely suppose that $\mathbf{p}$ is some point of $\mathbb{R}^n$ that is the limit of some infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X that are all distinct from the point $\mathbf{p}$. Let some positive number δ be given. The definition of convergence ensures that there exists a positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. Moreover $|\mathbf{x}_j - \mathbf{p}| > 0$ for all positive integers j. Thus $0 < |\mathbf{x}_j - \mathbf{p}| < \delta$ when the positive integer j is sufficiently large. Thus the point $\mathbf{p}$ is a limit point of the set X, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{p}$ of X is said to be an *isolated point* of X if it is not a limit point of X.

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in X$. It follows immediately from the definition of isolated points that the point $\mathbf{p}$ is an isolated point of the set X if and only if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} = \{\mathbf{p}\}.$$

2.4 The Multidimensional Bolzano-Weierstrass Theorem

We introduce some terminology and notation for discussing convergence along subsequences of bounded sequences of points in Euclidean spaces. This will be useful in proving the multi-dimensional version of the Bolzano-Weierstrass Theorem.

Definition Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$, let J be an infinite subset of the set $\mathbb{N}$ of positive integers, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{p}$ is the *limit* of $\mathbf{x}_j$ as *j tends to infinity in the set J*, and write "$\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ in J" if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \in J$ and $j \geq N$.

The one-dimensional version of the Bolzano-Weierstrass Theorem (Theorem 1.9) is equivalent to the following statement:

> Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, there exists an infinite subset J of the set $\mathbb{N}$ of positive integers and a real number p such that $x_j \to p$ as $j \to +\infty$ in J.

Given an infinite subset J of $\mathbb{N}$, the elements of J can be labelled as $k_1, k_2, k_3, \ldots$, where $k_1 < k_2 < k_3 < \cdots$, so that k_1 is the smallest positive integer belonging of J, k_2 is the next smallest, etc. Therefore any standard result concerning convergence of sequences of points can be applied in the context of the convergence of subsequences of a given sequence of points.

The following result is therefore a direct consequence of the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9):

> Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given an infinite subset J of the set $\mathbb{N}$ of positive integers, there exists an infinite subset K of J and a real number p such that $x_j \to p$ as $j \to +\infty$ in K.

The above statement in fact corresponds to the following assertion:—

> Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given any subsequence
>
> $$x_{k_1}, x_{k_2}, x_{k_3}, \cdots$$
>
> of the given infinite sequence, there exists a convergent subsequence
>
> $$x_{k_{m_1}}, x_{k_{m_2}}, x_{k_{m_3}}, \ldots$$
>
> of the given subsequence. Moreover this convergent subsequence of the given subsequence is itself a convergent subsequence of the given infinite sequence, and it contains only members of the given subsequence of the given sequence.

The basic principle can be presented purely in words as follows:

> Given a bounded sequence of real numbers, and given a subsequence of that original given sequence, there exists a convergent subsequence of the given subsequence. Moreover this subsequence of the subsequence is a convergent subsequence of the original given sequence.

We employ this principle in the following proof of the Multidimensional Bolzano-Weierstrass Theorem.

Theorem 2.6 (Multidimensional Bolzano-Weierstrass Theorem) *Every bounded sequence of points in a Euclidean space has a convergent subsequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$, and, for each positive integer j, and for each integer i between 1 and n, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then

$$\mathbf{x}_j = \Big((\mathbf{x}_j)_1, (\mathbf{x}_j)_2, \ldots, (\mathbf{x}_j)_n\Big).$$

for all positive integers j. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9) that there exists an infinite subset J_1 of the set $\mathbb{N}$ of positive integers and a real number p_1 such that $(\mathbf{x}_j)_1 \to p_1$ as $j \to +\infty$ in J_1. Let k be an integer between 1 and $n-1$. Suppose that there exists an infinite subset J_k of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_k$ such that, for each integer i between 1 and k, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_k. It then follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_{k+1} of J_k and a real number p_{k+1}, such that $(\mathbf{x}_j)_{k+1} \to p_{k+1}$ as $j \to +\infty$ in J_{k+1}. Moreover the requirement that $J_{k+1} \subset J_k$ then ensures that, for each integer i between 1 and $k+1$, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_{k+1}. Repeated application of this result then ensures the existence of an infinite subset J_n of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_n$ such that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_n.

Let

$$J_n = \{k_1, k_2, k_3, \ldots\},$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\lim\limits_{j \to +\infty} (\mathbf{x}_{k_j})_i = p_i$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 2.3 that $\lim\limits_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$. The result follows. ∎

2.5 Cauchy Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if the following condition is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 2.7 *Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 2.8 (Cauchy's Criterion for Convergence) *An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 2.7. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

3 Open and Closed Sets in Euclidean Spaces

3.1 Open Sets in Euclidean Spaces

Definition Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a non-negative real number r, the *open ball* $B(\mathbf{p}, r)$ in $\mathbb{R}^n$ of *radius* r about $\mathbf{p}$ is defined to be the subset of $\mathbb{R}^n$ defined so that

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B(\mathbf{p}, r)$ is the set consisting of all points of $\mathbb{R}^n$ that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

The *open ball* $B(\mathbf{p}, r)$ of radius r about a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius r about $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = r\}.$$

Definition A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, \delta)$ is the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$, defined so that

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset V of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ about the origin. This open ball is an open set. Indeed let $\mathbf{x}$ be a point of V. Then $|\mathbf{x}| < 3$. Let $\delta = 3 - |\mathbf{x}|$. Then $\delta > 0$. Moreover if $\mathbf{y}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{y} - \mathbf{x}| < \delta$ then

$$|\mathbf{y}| = |\mathbf{x} + (\mathbf{y} - \mathbf{x})| \leq |\mathbf{x}| + |\mathbf{y} - \mathbf{x}| < |\mathbf{x}| + \delta = 3,$$

and therefore $\mathbf{y} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius about any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 3.1).

3.2 Open Sets in Subsets of Euclidean Spaces

Definition Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0,0,1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$) of radius δ about $(0,0,1)$ contains points (x,y,z) for which $x^2+y^2+z^2 \neq 1$. Thus the open ball of radius δ about the point $(0,0,1)$ is not a subset of N.

Lemma 3.1 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 3.2 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 3.3 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 3.4 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 3.1), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 3.3). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

3.3 Convergence of Sequences and Open Sets

Lemma 3.5 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 3.1. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

3.4 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 3.1 and Lemma 3.2 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\mathbf{x} \in X \setminus \bigcup_{S \in \mathcal{A}} S \iff \mathbf{x} \notin \bigcup_{S \in \mathcal{A}} S$$

$$
\begin{aligned}
&\iff \text{for all } S \in \mathcal{A}, \mathbf{x} \notin S \\
&\iff \text{for all } S \in \mathcal{A}, \mathbf{x} \in X \setminus S \\
&\iff \mathbf{x} \in \bigcap_{S \in \mathcal{A}} (X \setminus S),
\end{aligned}
$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$
\begin{aligned}
\mathbf{x} \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S \in \mathcal{A}} S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\
&\iff \mathbf{x} \in \bigcup_{S \in \mathcal{A}} (X \setminus S),
\end{aligned}
$$

and therefore

$$X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 3.3.

Proposition 3.6 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Lemma 3.7 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 3.5 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

3.5 Closed Sets and Limit Points

Lemma 3.8 *A subset F of n-dimensional Euclidean space $\mathbb{R}^n$ is closed in $\mathbb{R}^n$ if and only if it contains its limit points.*

Proof Let F be a closed set in $\mathbb{R}^n$ and let $\mathbf{p}$ be a limit point of F. It follows from Lemma 2.5 that there exists an infinite sequence of points of F that converges to the point $\mathbf{p}$. It then follows from Lemma 3.7 that $\mathbf{p} \in F$. Thus if the set F is closed then it contains its limit points.

Conversely let F be a subset of $\mathbb{R}^n$ that contains its limit points. Let $\mathbf{p} \in \mathbb{R}^n \setminus F$. Then $\mathbf{p}$ is not a limit point of F. It follows from the definition of limit points that there exists some positive real number δ for which

$$\{\mathbf{x} \in F : 0 < |\mathbf{x} - \mathbf{p}| < \delta\} = \emptyset.$$

It then follows from this that the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$ is contained in the complement of F. We conclude therefore that the complement of F in $\mathbb{R}^n$ is open in $\mathbb{R}^n$, and thus F is closed in $\mathbb{R}^n$, as required. ∎

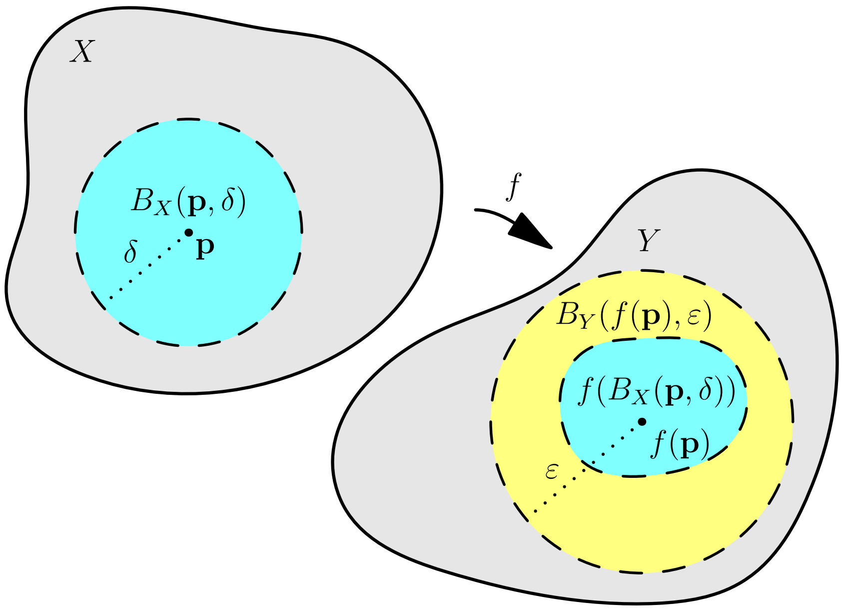
4 Limits and Continuity for Functions of Several Variables

4.1 Continuity of Functions of Several Real Variables

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

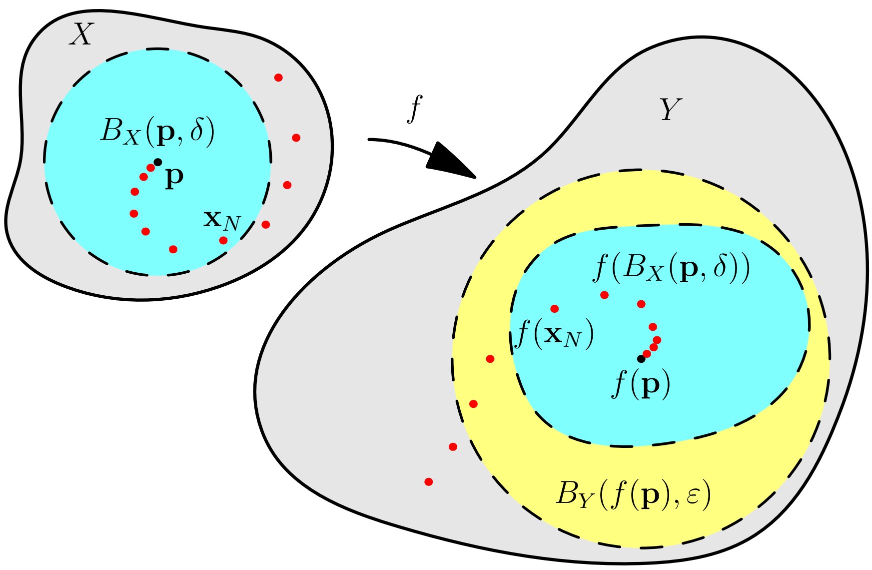


Lemma 4.1 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f: X \to Y$ and $g: Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.2 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| <$



δ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 4.3 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f: X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 4.1. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.4 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $m(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $m: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ be given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta \leq (1 + |u| + |v|)\delta \leq \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $m: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 4.5 *Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h: X \to \mathbb{R}^2$, $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 4.3, Lemma 4.4 and Lemma 4.1 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f: \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 4.5. It then follows from Proposition 4.3 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lemma 4.6 *Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|: X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

Proof Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

4.2 Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Proposition 4.7 *Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}^n$ has the property that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$. Let i be an integer between 1 and n, and let some positive real number ε be given. Then there exists some positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - q_i| \leq |f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ then $\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$ for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < |f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 4.8 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned}|f(\mathbf{x})+g(\mathbf{x})-(\mathbf{q}+\mathbf{r})| &\leq |f(\mathbf{x})-\mathbf{q}|+|g(\mathbf{x})-\mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.\end{aligned}$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))=\mathbf{q}+\mathbf{r},$$

as required. ∎

Definition Let $f\colon X\to\mathbb{R}^n$ be a function mapping some subset X of m-dimensional Euclidean space $\mathbb{R}^m$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. We say that $f(\mathbf{x})$ *remains bounded* as $\mathbf{x}$ tends to $\mathbf{p}$ in X if strictly positive constants C and δ can be determined so that $|f(\mathbf{x})|\leq C$ for all $\mathbf{x}\in X$ satisfying $0<|\mathbf{x}-\mathbf{p}|<\delta$.

Proposition 4.9 *Let $f\colon X\to\mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h\colon X\to\mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})=\mathbf{0}$. Suppose also that $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}}\Bigl(h(\mathbf{x})f(\mathbf{x})\Bigr)=\mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined so that $|h(\mathbf{x})|\leq C$ for all $\mathbf{x}\in X$ satisfying $0<|\mathbf{x}-\mathbf{p}|<\delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0<\varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})|<\varepsilon_0$ whenever $0<|\mathbf{x}-\mathbf{p}|<\delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta>0$, and if $0<|\mathbf{x}-\mathbf{p}|<\delta$ then $|h(\mathbf{x})|\leq C$ and $|f(\mathbf{x})|<\varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})|<C\varepsilon_0<\varepsilon.$$

The result follows. ∎

Proposition 4.10 *Let $f\colon X\to\mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h\colon X\to\mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})=0$. Suppose also that $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}}(h(\mathbf{x})f(\mathbf{x}))=\mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|h(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| \leq C$ and $|h(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.11 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) \cdot g(\mathbf{x}) \Big) = 0.$$

Proof Let some strictly positive real number ε be given. Now $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|g(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| < \varepsilon_0$ and $|g(\mathbf{x})| \leq C$. It then follows from Schwarz's Inequality (Proposition 2.1) that

$$|f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{x})|\,|g(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.12 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of $\mathbb{R}^n$ and let s be a real number. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = s.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})f(\mathbf{x}) = s\mathbf{q}.$$

Proof The functions f and h satisfy the equation

$$h(\mathbf{x})f(\mathbf{x}) = h(\mathbf{x})\Big(f(\mathbf{x}) - \mathbf{q}\Big) + (h(\mathbf{x}) - s)\mathbf{q} + s\mathbf{q},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) - \mathbf{q}\Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Big(h(\mathbf{x}) - s\Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that $|h(\mathbf{x}) - s| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|h(\mathbf{x})| < |s| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) = \mathbf{0}$$

(see Proposition 4.10). Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x}) - s)\, \mathbf{q} = \mathbf{0}.$$

It follows that

$$\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) \\
&= \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) + \lim_{\mathbf{x}\to\mathbf{p}} \Big(\Big(h(\mathbf{x}) - s\Big)\mathbf{q}\Big) + s\mathbf{q} \\
&= \mathbf{0} + s\mathbf{q},
\end{aligned}$$

as required. ∎

Lemma 4.13 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f\colon X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g\colon Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}),$$

as required. ∎

Proposition 4.14 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) = \mathbf{q} \cdot \mathbf{r}.$$

Proof The functions f and g satisfy the equation

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = (f(\mathbf{x}) - \mathbf{q}) \cdot g(\mathbf{x}) + \mathbf{q} \cdot (g(\mathbf{x}) - \mathbf{r}) + \mathbf{q} \cdot \mathbf{r},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(f(\mathbf{x}) - \mathbf{q}\Bigr) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Bigl(g(\mathbf{x}) - \mathbf{r}\Bigr) = \mathbf{0}.$$

Moreover there exists a strictly positive constant δ_0 such that $|g(\mathbf{x}) - \mathbf{r}| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|g(\mathbf{x})| < |\mathbf{r}| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\Bigl(f(\mathbf{x}) - \mathbf{q}\Bigr) \cdot g(\mathbf{x})\right) = 0$$

(see Proposition 4.11). Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\mathbf{q} \cdot \Bigl(g(\mathbf{x}) - \mathbf{r}\Bigr)\right) = 0.$$

It follows that

$$\begin{aligned}
& \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) \\
= \; & \lim_{\mathbf{x}\to\mathbf{p}} \left(\Bigl(f(\mathbf{x}) - \mathbf{q}\Bigr) \cdot g(\mathbf{x})\right) + \lim_{\mathbf{x}\to\mathbf{p}} \left(\mathbf{q} \cdot \Bigl(g(\mathbf{x}) - \mathbf{r}\Bigr)\right) + \mathbf{q} \cdot \mathbf{r} \\
= \; & \mathbf{q} \cdot \mathbf{r},
\end{aligned}$$

as required. ∎

Proposition 4.15 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X.*

Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ *and* $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ *both exist. Then so do* $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))$ *and* $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, *and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in X$ *and* $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ *then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

First Proof It follows from Proposition 4.8 (applied in the case when the target space is one-dimensional) that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

It follows from Proposition 4.12 (applied in the case when the target space is one-dimensional), or alternatively from Proposition 4.14, that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Let $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the reciprocal function defined so that $e(t) = 1/t$ for all non-zero real numbers t. Then the reciprocal function e is continuous. Applying the result of Lemma 4.13, we find that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{g(\mathbf{x})} = \lim_{\mathbf{x}\to\mathbf{p}} e(g(\mathbf{x})) = e\left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\right) = \frac{1}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})},$$

as required. ∎

Second Proof Let $q = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $r = \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$, and let $h\colon X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = (q, r)$$

(see Proposition 4.7).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u, v) = u + v$ and $m(u, v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 4.4). Also $f + g = s \circ h$ and $f \cdot g = m \circ f$. It follows from this that

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(h(\mathbf{x})) \\ &= s\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = s(q, r) = q + r, \end{aligned}$$

and

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(h(\mathbf{x})) \\ &= m\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = m(q, r) = qr \end{aligned}$$

(see Lemma 4.13).

Also

$$\lim_{\mathbf{x}\to\mathbf{p}} (-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = q - r.$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $e(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. ∎

Proposition 4.16 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ and $g: Y \to \mathbb{R}^k$ be functions satisfying $f(X) \subset Y$. Let $\mathbf{p}$ be a limit point of X in $\mathbb{R}^m$, let $\mathbf{q}$ be a limit point of Y in $\mathbb{R}^n$ let $\mathbf{r}$ be a point of $\mathbb{R}^k$. Suppose that the following three conditions are satisfied:*

(i) $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$;

(ii) $\lim_{\mathbf{y} \to \mathbf{q}} g(\mathbf{y}) = \mathbf{r}$;

(iii) there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$.

Then

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = \mathbf{r}.$$

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{y} \in Y$ satisfies $0 < |\mathbf{y} - \mathbf{q}| < \eta$. There then exists some positive real number δ_1 such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Also there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and $0 < |f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. But this then ensures that $|g(f(\mathbf{x})) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 4.17 *Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if* $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.

Proof The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x} - \mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f: X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

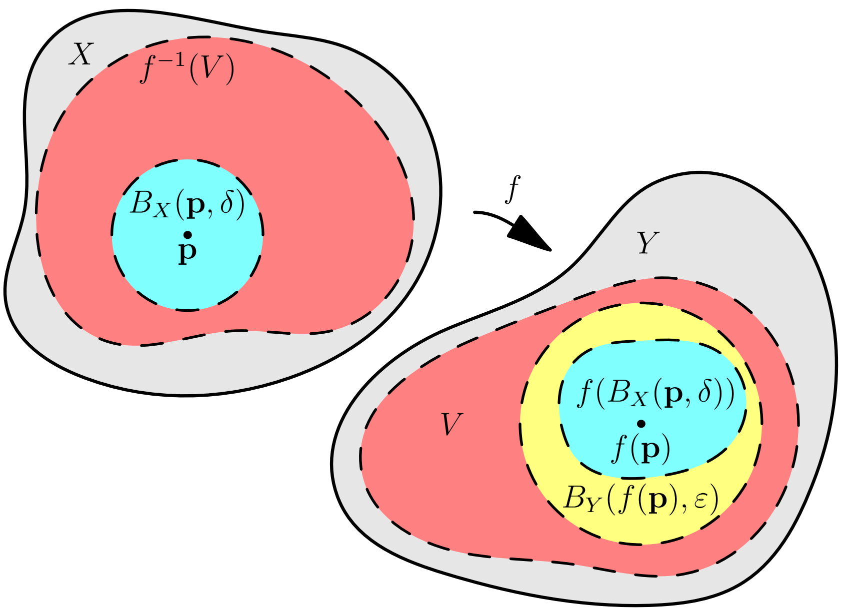
4.3 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

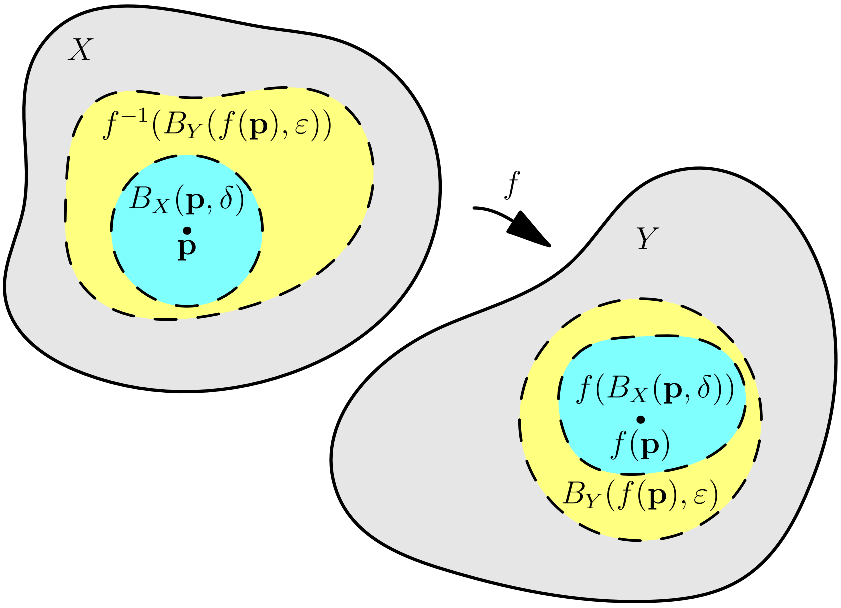
Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 4.18 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.



Conversely suppose that $f: X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is



an open set in Y, by Lemma 3.1, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f: X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

4.4 Limits and Neighbourhoods

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. A subset N of X is said to be a *neighbourhood* of $\mathbf{p}$ in X if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N.$$

Lemma 4.19 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X that is not an isolated point of X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in X such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$.

Proof This result follows directly from the definitions of limits and neighbourhoods. ∎

Remark Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a limit point of X that does not belong to X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in $X \cup \{\mathbf{p}\}$ such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$. Thus the result of Lemma 4.19 can be extended so as to apply to limits of functions taken at limit points of the domain that do not belong to the domain of the function.

4.5 The Multidimensional Extreme Value Theorem

Proposition 4.20 *Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a point $\mathbf{w}$ of X such that $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$.*

Proof Let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the function mapping each $\mathbf{x} \in X$ to $|f(\mathbf{x})|$ is continuous (see Lemma 4.6) and quotients of continuous functions are continuous where they are defined (see Lemma 4.5). It follows that the function $g\colon X \to \mathbb{R}$ is continuous.

Let

$$m = \inf\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$g(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{w}$ of $\mathbb{R}^n$.

Now the point $\mathbf{w}$ belongs to X because X is closed (see Lemma 3.7). Also

$$m \leq g(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $g(\mathbf{x}_{k_j}) \to m$ as $j \to +\infty$. It then follows from Lemma 4.2 that

$$g(\mathbf{w}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = m.$$

Then $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$, and therefore $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$, as required. ∎

Theorem 4.21 (The Multidimensional Extreme Value Theorem) *Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof It follows from Proposition 4.20 that the function f is bounded on X. It follows that there exists a real number C large enough to ensure that $f(\mathbf{x}) + C > 0$ for all $\mathbf{x} \in X$. It then follows from Proposition 4.20 that there exists some point $\mathbf{v}$ of X such that

$$f(\mathbf{x}) + C \leq f(\mathbf{v}) + C.$$

for all $\mathbf{x} \in X$. But then $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$. Applying this result with f replaced by $-f$, we deduce that there exists some $\mathbf{u} \in X$ such that $-f(\mathbf{x}) \leq -f(\mathbf{u})$ for all $\mathbf{x} \in X$. The result follows. ∎

4.6 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of $\mathbb{R}^m$. A function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 4.22 *Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f\colon X \to \mathbb{R}^n$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 2.6). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since

$$\lim_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0.$$

But then the sequences

$$f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$$

and

$$f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$$

would both converge to $f(\mathbf{p})$, since f is continuous (Lemma 4.2), and thus

$$\lim_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that

$$|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$$

for all j. We conclude therefore that there must exist some positive real number δ such that such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

4.7 Norms on Vector Spaces

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

The Euclidean norm $|.|$ is a norm on $\mathbb{R}^n$ defined so that

$$|(x_1, x_2, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

for all $(x_1, x_2, \ldots, x_n)$. There are other useful norms on $\mathbb{R}^n$. These include the norms $\|.\|_1$ and $\|.\|_{\mathrm{sup}}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\mathrm{sup}} = \mathrm{maximum}(|x_1|, |x_2|, \ldots, |x_n|)$$

for all $(x_1, x_2, \ldots, x_n)$.

Definition Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 4.23 *If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.*

Proof let $\|.\|_*$ and $\|.\|_{**}$ be norms on a real vector space X that are both equivalent to a norm $\|.\|$ on X. Then there exist constants c_*, c_{**}, C_* and C_{**}, where $0 < c_* \leq C_*$ and $0 < c_{**} \leq C_{**}$, such that

$$c_*\|x\| \leq \|x\|_* \leq C_*\|x\|$$

and

$$c_{**}\|x\| \leq \|x\|_{**} \leq C_{**}\|x\|$$

for all $x \in X$. But then

$$\frac{c_{**}}{C_*}\|x\|_* \leq \|x\|_{**} \leq \frac{C_{**}}{c_*}\|x\|_*.$$

for all $x \in X$, and thus the norms $\|.\|_*$ and $\|.\|_{**}$ are equivalent to one another. The result follows. ∎

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 4.24 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive real number C with the property that $\|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots,$$
$$\mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ be a point of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

Using Schwarz's Inequality, we see that

$$\begin{aligned} \|\mathbf{x}\| &= \left\| \sum_{j=1}^n x_j \mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x}|, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and

$$|\mathbf{x}| = \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The result follows. ∎

Lemma 4.25 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive constant C such that*

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\| \leq C|\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

It follows that

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

and

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|,$$

and therefore

$$\Big|\|\mathbf{y}\| - \|\mathbf{x}\|\Big| \leq \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The result therefore follows from Lemma 4.24. ∎

Theorem 4.26 *Any two norms on $\mathbb{R}^n$ are equivalent.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now it follows from Lemma 4.25 that the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous. Also S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. It therefore follows from the Extreme Value Theorem (Theorem 4.21) that there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. ∎