Course MA2321: Michaelmas Term 2018.

Assignment 2.

To be handed in by Monday 19th November, 2018.

1. *Throughout this question let $f\colon \mathbb{R} \to \mathbb{R}$ be the cosine function, where $f(x) = \cos x$, let b be real number satisfying $0 < b < \pi$, and, for each positive integer m, let P_m be the partition of $[0, b]$ given by $P_m = \{x_0, x_1, \ldots, x_m\}$ where $x_k = kb/m$ for $k = 0, 1, \ldots, m$.*

   (a) *Show that, for each positive integer m the upper sum $U(P_m, f)$ for the cosine function f satisfies $U(P_m, f) = F(b, b/m)$, where*

$$F(b,h) = \frac{h(1 - \cos b - \cos h + \cos(b - h))}{2(1 - \cos h)}.$$

   The restriction $0 < b < \pi$ ensures that the cosine function is decreasing on $[0, b]$. Therefore the maximum value of $\cos x$ on the interval $[x_{k-1}, x_k]$ is attained at x_{k-1} for $k = 1, 2, \ldots, m$. Moreover each subinterval of the partition is of length b/m. It follows that

$$U(P_m, f) = \frac{b}{m} \sum_{k=0}^{m-1} \cos\left(\frac{kb}{m}\right).$$

   Let $z = \cos(b/m) + i \sin(b/m)$, where $i = \sqrt{-1}$, and let $h = b/m$. Then $\cos(kb/m) + i \sin(kb/m) = z^k$ for $k = 0, 1, \ldots, m$, and therefore

$$\begin{aligned}
U(P_m, f) &= \frac{b}{m} \operatorname{Re}\left[\sum_{k=0}^{m-1} z^k\right] = \frac{b}{m} \operatorname{Re}\left[\frac{1 - z^m}{1 - z}\right] \\
&= h\operatorname{Re}\left[\frac{1 - \cos b - i\,\sin b}{1 - \cos h - i\,\sin h}\right] \\
&= h\operatorname{Re}\left[\frac{(1 - \cos b - i\,\sin b)(1 - \cos h + i\,\sin h)}{(1 - \cos h)^2 + \sin^2 h}\right] \\
&= h\,\frac{(1 - \cos b)(1 - \cos h) + \sin b\,\sin h}{1 - 2\cos h + \cos^2 h + \sin^2 h} \\
&= h\,\frac{1 - \cos b - \cos h + \cos b\,\cos h + \sin b\,\sin h}{2(1 - \cos h)} \\
&= h\,\frac{1 - \cos b - \cos h + \cos(b - h)}{2(1 - \cos h)} \\
&= F(b, h) = F(b, b/m).
\end{aligned}$$

[In answering part (a), you may find it helpful to make use of the following results:

$$\cos m\theta + i\,\sin m\theta = (\cos\theta + i\,\sin\theta)^m \quad \textit{for all } m \in \mathbb{Z} \textit{ where } i = \sqrt{-1};$$

$$\sum_{k=0}^{m-1} z^k = \frac{1-z^m}{1-z} \quad \textit{for all } z \in \mathbb{C} \textit{ satisfying } z \neq 1;$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} \quad \textit{for all } a,b,c,d \in \mathbb{R} \textit{ with } (c,d) \neq (0,0).]$$

(b) *Show that* $\lim_{h\to 0} F(b,h) = \sin b$.

$$\begin{aligned}
\lim_{h\to 0} F(b,h) &= \lim_{h\to 0} \frac{h^2}{2(1-\cos h)} \times \lim_{h\to 0} \frac{1-\cos b - \cos h + \cos(b-h)}{h} \\
&= \lim_{h\to 0} \frac{1-\cos h}{h} - + \lim_{h\to 0} \frac{\cos(b-h)-\cos b}{h} \\
&= -\left.\frac{d(\cos\theta)}{d\theta}\right|_{\theta=b} = \sin b.
\end{aligned}$$

[The following may be useful: $\lim_{h\to 0} \frac{2(1-\cos h)}{h^2} = 1$; $\frac{d(\cos x)}{dx} = -\sin x$.*]*

(c) *Explain why* $U(P_m,f) - L(P_m,f) \leq 2b/m$.

$$U(P_m,f) = \frac{b}{m}\sum_{k=0}^{m-1}\cos\left(\frac{kb}{m}\right), \quad L(P_m,f) = \frac{b}{m}\sum_{k=1}^{m}\cos\left(\frac{kb}{m}\right).$$

Therefore

$$U(P_m,f) - L(P_m,f) = \frac{b}{m}(1-\cos b) \leq \frac{2b}{m}.$$

(d) *Without making use of the Fundamental Theorem of Calculus and its consequences, prove, on the basis of the results stated above, that the cosine function* f *is Riemann-integrable on the interval* $[0,b]$*, and that* $\int_0^b \cos x\,dx = \sin b$.

$$\lim_{m\to+\infty} U(P_m, f) = \lim_{m\to+\infty} F(b, b/m) = \lim_{h\to 0} F(b, h) = \sin b.$$

Also

$$\lim_{m\to+\infty} (U(P_m, f) - L(P_m, f)) = 0.$$

It follows that

$$\lim_{m\to+\infty} L(P_m, f) = \sin b.$$

Now

$$L(P_m, f) \leq \mathcal{L}\int_0^b cosx\,dx \leq \mathcal{U}\int_0^b cosx\,dx \leq U(P_m, f)$$

for all integers m, and therefore

$$0 \leq \mathcal{U}\int_0^b cosx\,dx - \mathcal{L}\int_0^b cosx\,dx \leq U(P_m, f) - L(P_m, f)$$

For all integers m. The term on the right hand side tends to zero as $m \to +\infty$. It follows that

$$\mathcal{U}\int_0^b cosx\,dx = \mathcal{L}\int_0^b cosx\,dx = \sin b.$$

Thus the cosine function is Riemann-integrable on $[0, b]$ with integral equal to $\sin b$, as required.

2. *Let f be a function that is 5 times differentiable on an open interval containing the closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that*

$$f(a) = f'(a) = f''(a) = f(b) = f'(b) = f''(b) = 0.$$

Prove that there exists some real number s satisfying $a < s < b$ for which $f^{(5)}(s) = 0$.

Applying Rolle's Theorem to f on $[a, b]$, there exists some real number c satisfying $a < c < b$ for which $f'(c) = 0$. Then applying Rolle's Theorem to f' on $[a, c]$ and $[c, b]$, there exist real numbers d and e satisfying $a < d < e < b$ for which $f''(d) = 0$ and $f''(e) = 0$. Then applying

Rolle's Theorem to f'' on $[a, d]$, $[d, e]$ and $[e, b]$, there exist real numbers p, q, r satisfying $a < p < d$, $d < q < e$, $e < r < b$ for which

$$f'''(p) = f'''(q) = f'''(r) = 0.$$

There then exist real numbers u and v satisfying $p < u < q < v < r$ for which $f^{(4)}(u) = f^{(4)}(v) = 0$. There then exists a real number s satisfying $u < s < v$ for which $f^{(5)}(s) = 0$. Moreover $a < s < b$, as required.