Course MA2321: Michaelmas Term 2018.

Solutions to Assignment 1.

To be handed in by Thursday 1st November, 2018.

1. *Let X be the subset of $\mathbb{R}^3$ defined such that*

$$X = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 < 4\}.$$

Determine whether or not the set X is open in $\mathbb{R}^3$. Determine also whether or not the set X is closed in $\mathbb{R}^3$. [Justify your answers with appropriate logical reasoning or examples.]

The set X is not open in $\mathbb{R}^3$. The point $(1, 0, 0)$ belongs to the set X, but any open ball of radius δ about this point contains points $(u, 0, 0)$ of $\mathbb{R}^3$ with $0 \leq< u < 1$ and $u > 1 - \delta$, and such points do not belong to the set X. Thus no open ball of positive radius centred on the point $(1, 0, 0)$ is contained within the set X.

The set X is not closed in $\mathbb{R}^3$. The point $(2, 0, 0)$ belongs to the complement of the set X, but any open ball of radius δ about this point contains points $(u, 0, 0)$ of $\mathbb{R}^3$ with $1 \leq u < 2$ and $u > 2 - \delta$, and such points do belong to the set X. Thus no open ball of positive radius centred on the point $(2, 0, 0)$ is contained within the complement of the set X. Thus the complement $\mathbb{R}^3 \setminus X$ of X is not open, and therefore the set X itself is not closed.

2. *Let Y be the subset of $\mathbb{R}^2$ defined such that*

$$Y = \{(x, y) \in \mathbb{R}^2 : \textit{there exists } n \in \mathbb{Z} \textit{ such that } (x - n)^2 + y^2 < 1\}.$$

(In other words, a point (x, y) of $\mathbb{R}^2$ belongs to Y if and only if some integer n, depending on the values of x and y, can be found for which $(x - n)^2 + y^2 < 1$.) Determine whether or not the set Y is open in $\mathbb{R}^2$. Determine also whether or not the set Y is closed in $\mathbb{R}^2$. [Justify your answers with appropriate logical reasoning or examples.]

The set Y is open in $\mathbb{R}^2$. It is the union of open balls centred on the points $(n, 0)$ as n ranges over the set $\mathbb{Z}$ of integers. Any open ball is an open set, and any union of open sets is an open set. Therefore the set Y is an open set in $\mathbb{R}^2$.

The set Y is not closed in $\mathbb{R}^2$. The point $(0, 1)$ belongs to the complement of the set Y, but any open ball of radius δ about this point contains points $(0, u)$ of $\mathbb{R}^2$ with $0 \leq u < 1$ and $u > 1 - \delta$, and such points do belong to the set Y. Thus no open ball of positive radius centred on the point $(0, 1)$ is contained within the complement of the set Y. Thus the complement $\mathbb{R}^2 \setminus Y$ of Y is not open, and therefore the set Y itself is not closed.

3. *Let Z be the subset of $\mathbb{R}^2$ defined such that*

$$Z = \{(x, y) \in \mathbb{R}^2 : \textit{there exists } n \in \mathbb{Z} \textit{ such that } (x - n)^2 + y^2 \leq 1\}.$$

Determine whether or not the set Z is open in $\mathbb{R}^2$. Determine also whether or not the set Z is closed in $\mathbb{R}^2$. [Justify your answers with appropriate logical reasoning or examples.]

The set Z is not open in $\mathbb{R}^2$. The point $(0, 1)$ belongs to the set Z, but any open ball of radius δ about this point contains points $(0, u)$ of $\mathbb{R}^2$ with $0 \leq u < 1$ and $u > 1 - \delta$, and such points do not belong to the set Z. Thus no open ball of positive radius centred on the point $(0, 1)$ is contained within the set Z.

The set Z is closed in $\mathbb{R}^2$. Let (u, v) be a point of the complement of Z. An integer M can be found large enough to ensure that $-M + 2 < u < M - 2$. If n is an integer satisfying $|n| \geq M$ then the closed ball of radius 1 about the point $(n, 0)$ does not intersect the open ball of radius 1 about the point (u, v). Let F be the union of the closed balls of radius 1 about the points $(n, 0)$ for which n is an integer satisfying $|n| < M$. The set F is then a finite union of closed balls. Moreover closed balls are closed sets, and any finite union of closed sets is closed. It follows that F is a closed set. Therefore some real number δ satisfying $0\delta < 1$ can be found so that the open ball of radius δ about (u, v) does not intersect the set F. Then the open ball of radius δ about (u, v) does not intersect the set Z. This shows that the complement of the set Z is open, and therefore the set Z itself is closed in $\mathbb{R}^2$.

Alternative Solution (with credit to members of the class).

Proof that Z is not open as above.

A set Z is closed in $\mathbb{R}^2$ if and only if it contains its limit points, and this is the case if and only if the limit of every convergent sequence of points in Z belongs to Z. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in Z, and let $\mathbf{p}$ be the limit of this sequence. We show that $\mathbf{p} \in Z$. Now the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is bounded, because all convergent sequences are bounded. Therefore there exists a positive integer M

such that $\mathbf{x}_j \in \bigcup_{n=-M}^{M} \overline{B}((n,0),1)$, where $\overline{B}((n,0),1)$ denotes the closed ball of radius 1 about the point $(n,0)$. Now closed balls are closed sets, and therefore $\bigcup_{n=-M}^{M} \overline{B}((n,0),1)$, being a finite union of closed sets must itself be a closed set. It follows from this that $\mathbf{p} \in \bigcup_{n=-M}^{M} \overline{B}((n,0),1)$, and therefore $\mathbf{p} \in Z$. We conclude therefore that the set Z is indeed closed in $\mathbb{R}^2$.

4. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function of two real-variables defined such that*

$$f(x,y) = \begin{cases} \dfrac{4x(x^2+y^2)}{4x^2+(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) *Determine the values of the function f at all points lying on the circle of radius 1 centred at the point $(1,0)$.*

The circle consists of those points (x,y) of the plane for which $(x-1)^2+y^2=1$. But $(x-1)^2+y^2=1$ if and only if $x^2+y^2=2x$, in which case $4x(x^2+y^2) = 4x^2+(x^2+y^2)^2$. It follows that the function f has the value 1 at all points of this circle with the exception of the point $(0,0)$, where the function has the value 0.

(b) *Prove that $\lim_{t\to 0} f(tu,tv) = 0$ for all points (u,v) of $\mathbb{R}^2$.*

First suppose that $u=0$. Then $f(0,tv)=0$ for all real numbers t, and therefore $\lim_{t\to 0} f(0,tv) = 0$.

Next suppose that $u \neq 0$. Then

$$f(tu,tv) = \frac{4t^3u(u^2+v^2)}{4t^2u^2+t^4(u^2+v^2)^2} = \frac{4tu(u^2+v^2)}{4u^2+t^2(u^2+v^2)^2}$$

whenever $t \neq 0$. Now

$$\lim_{t\to 0}(4tu(u^2+v^2)) = 0 \quad \text{and} \quad \lim_{t\to 0} 4u^2+t^2(u^2+v^2)^2 = 4u^2.$$

Moreover $4u^2 \neq 0$. It follows that

$$\lim_{t\to 0} f(tu,tv) = \frac{\lim_{t\to 0}(4tu(u^2+v^2))}{\lim_{t\to 0}(4u^2+t^2(u^2+v^2)^2)} = \frac{0}{4u^2} = 0.$$

Thus the identity $\lim_{t\to 0} f(tu,tv) = 0$ holds both when $u=0$ and also when $u \neq 0$, and thus holds in all required cases.

(c) *Determine whether or not the function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *is continuous at* $(0,0)$. *[Justify your answer with appropriate logical reasoning or examples.]*

The function f is not continuous at zero. Indeed $f(0,0) = 0$. But, given any positive real number δ, there exist points of the circle $x^2 + y^2 = 2x$ distinct from $(0,0)$ that lie within a distance δ of $(0,0)$, and the function f has the value 1 at such points. Therefore there cannot exist a positive real number δ with the property that $|f(x,y) - f(0,0)| < \frac{1}{2}$ at all points (x,y) satisfying $\sqrt{x^2 + y^2} < \delta$. Thus the definition of continuity is not satisfied at $(0,0)$.