Module MA2321: Analysis in Several Real Variables
Michaelmas Term 2017
Part II (Sections 5 and 6)

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Contents

5 The Riemann Integral in One Dimension

5.1 Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Definition A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$.

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$.

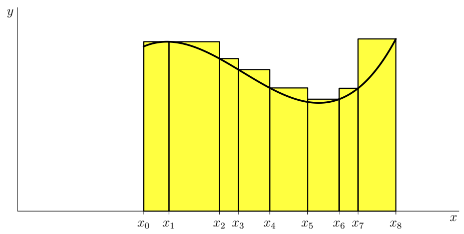
Clearly $L(P, f) \le U(P, f)$. Moreover $\sum\limits_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a),$$

for any real numbers m and M satisfying $m \le f(x) \le M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned}
\mathcal{U}\int_a^b f(x)\,dx &= \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\
\mathcal{L}\int_a^b f(x)\,dx &= \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}.
\end{aligned}$$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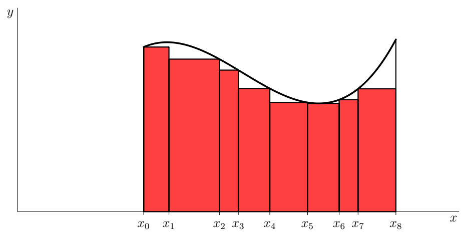
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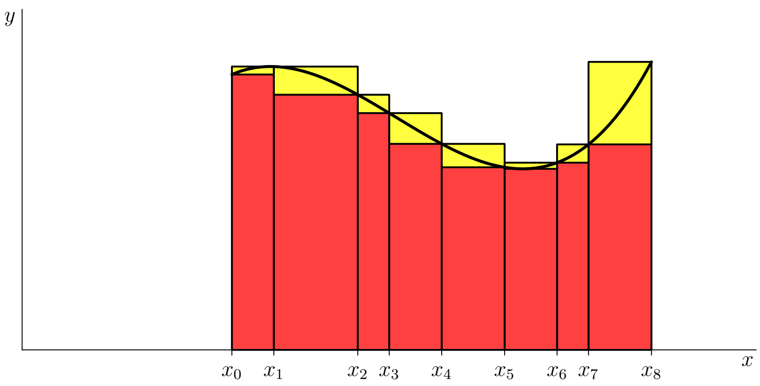
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The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P,f) \leq L(P,g)$ and $U(P,f) \leq U(P,g)$ for all partitions P of $[a,b]$.

Definition Let P and R be partitions of $[a,b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 5.1 *Let R be a refinement of some partition P of $[a,b]$. Then*

$$L(R,f) \geq L(P,f) \quad \textit{and} \quad U(R,f) \leq U(P,f)$$

for any bounded function $f\colon [a,b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$

given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R,f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R,f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P,f) \leq L(R,f) \leq U(R,f) \leq U(P,f)$, as required. ∎

Given any two partitions P and Q of $[a,b]$ there exists a partition R of $[a,b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 5.2 *Let f be a bounded real-valued function on the interval $[a,b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a,b]$, and let R be a common refinement of P and Q. It follows from Lemma 5.1 that $L(P,f) \leq L(R,f) \leq U(R,f) \leq U(Q,f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P,f) \leq U(Q,f)$ as P ranges over all possible partitions of the interval $[a,b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q,f)$ for all partitions Q of $[a,b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a,b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0,1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0,1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$
\begin{aligned}
L(P_n, f) &= \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{ci}{n} + d - \frac{c}{n} \right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{ci}{n} + d \right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}
$$

It follows that

$$\lim_{n \to +\infty} L(P_n, f) = \frac{c}{2} + d$$

and

$$\lim_{n \to +\infty} U(P_n, f) = \frac{c}{2} + d$$

Now $L(P_n, f) \leq \mathcal{L} \int_a^b f(x)\, dx \leq \mathcal{U} \int_a^b f(x)\, dx \leq U(P_n, f)$ for all positive integers n. It follows that $\mathcal{L} \int_a^b f(x)\, dx = \frac{1}{2}c + d = \mathcal{U} \int_a^b f(x)\, dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\, dx = \frac{1}{2}c + d$.

Example Let $f\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L} \int_0^1 f(x)\, dx = 0$ and $\mathcal{U} \int_0^1 f(x)\, dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

5.2 Basic Properties of the Riemann Integral

Lemma 5.3 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$
\begin{aligned}
\mathcal{U} \int_a^b (-f(x))\, dx &= -\mathcal{L} \int_a^b f(x)\, dx, \\
\mathcal{L} \int_a^b (-f(x))\, dx &= -\mathcal{U} \int_a^b f(x)\, dx.
\end{aligned}
$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned} m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \end{aligned}$$

Then the lower and upper sums of f for the partition P are given by the formulae

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Now

$$\begin{aligned} &\sup\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &\qquad = -\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = -m_i, \\ &\inf\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &\qquad = -\sup\{f(x) : x_{i-1} \leq x \leq x_i\} = -M_i \end{aligned}$$

It follows that

$$\begin{aligned} U(P, -f) &= \sum_{i=1}^{n} (-m_i)(x_i - x_{i-1}) = -L(P, f), \\ L(P, -f) &= \sum_{i=1}^{n} (-M_i)(x_i - x_{i-1}) = -U(P, f). \end{aligned}$$

We have now shown that

$$U(P, -f) = -L(P, f) \quad \text{and} \quad L(P, -f) = -U(P, f)$$

for all partitions P of the interval $[a, b]$. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\, dx &= \inf\{U(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{L}\int_a^b f(x)\, dx \end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P,-f) : P \text{ is a partition of } [a,b]\}\\
&= \sup\{-U(P,f) : P \text{ is a partition of } [a,b]\}\\
&= -\inf\{U(P,f) : P \text{ is a partition of } [a,b]\}\\
&= -\mathcal{U}\int_a^b f(x)\,dx.
\end{aligned}$$

This completes the proof. ∎

Lemma 5.4 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, and let P be a partition of the interval $[a,b]$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P,f) + L(P,g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P,f) + U(P,g).$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned}
L(P,f) &= \sum_{i=1}^n m_i(f)(x_i - x_{i-1}),\\
L(P,g) &= \sum_{i=1}^n m_i(g)(x_i - x_{i-1}),\\
L(P,f+g) &= \sum_{i=1}^n m_i(f+g)(x_i - x_{i-1}),
\end{aligned}$$

where

$$\begin{aligned}
m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\},\\
m_i(g) &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\},\\
m_i(f+g) &= \inf\{f(x)+g(x) : x_{i-1} \leq x \leq x_i\}
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \geq m_i(f) \quad \text{and} \quad g(x) \geq m_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \geq m_i(f) + m_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $m_i(f) + m_i(g)$ is a lower bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The greatest lower bound for this set is $m_i(f + g)$. Therefore

$$m_i(f + g) \geq m_i(f) + m_i(g).$$

It follows that

$$\begin{aligned}
L(P, f + g) &= \sum_{i=1}^{n} m_i(f + g)(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{n} (m_i(f) + m_i(g))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) \\
&= L(P, f) + L(P, g).
\end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\
U(P, f + g) &= \sum_{i=1}^{n} M_i(f + g)(x_i - x_{i-1}),
\end{aligned}$$

where

$$\begin{aligned}
M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(g) &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(f + g) &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \leq M_i(f) \quad \text{and} \quad g(x) \leq M_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \leq M_i(f) + M_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $M_i(f) + M_i(g)$ is an upper bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The least upper bound for this set is $M_i(f + g)$. Therefore

$$M_i(f + g) \leq M_i(f) + M_i(g).$$

It follows that

$$\begin{aligned}
U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (M_i(f) + M_i(g))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}) \\
&= U(P, f) + U(P, g).
\end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 5.5 *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f + g$ and $f - g$ are Riemann-integrable on $[a, b]$, and moreover*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of $[a,b]$ for which

$$L(P,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$L(Q,g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R,f) \geq L(P,f) \quad \text{and} \quad L(R,g) \geq L(P,g).$$

Applying Lemma 5.4, and the definition of the lower Riemann integral, we see that

$$\begin{aligned}
&\mathcal{L}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \geq \ L(R,f+g) \geq L(R,f) + L(R,g) \\
&\qquad \geq \ L(P,f) + L(Q,g) \\
&\qquad > \ \left(\int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon\right) + \left(\int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon\right) \\
&\qquad > \ \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon
\end{aligned}$$

We have now shown that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx, \qquad \int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

have values that have no dependence whatsoever on the value of ε. It follows that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We can deduce a corresponding inequality involving the upper integral of $f+g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 5.3). We find that

$$\begin{aligned}\mathcal{L}\int_a^b (-f(x)-g(x))\,dx &\geq \int_a^b (-f(x))\,dx + \int_a^b (-g(x))\,dx \\ &= -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{U}\int_a^b (f(x)+g(x))\,dx &= -\mathcal{L}\int_a^b (-f(x)-g(x))\,dx \\ &\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}&\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\ &\qquad\leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\qquad\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &\qquad\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}\mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

Thus the function $f+g$ is Riemann-integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Then, replacing g by $-g$, we find that

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

as required. ∎

Proposition 5.6 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof First suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a,b]$, and therefore there exist partitions Q and R of $[a,b]$ for which

$$L(Q,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q,f) \leq L(P,f) \leq U(P,f) \leq U(R,f).$$

(see Lemma 5.1). It follows that

$$U(P,f) - L(P,f) \leq U(R,f) - L(Q,f) < \varepsilon.$$

Now suppose that $f\colon [a,b] \to \mathbb{R}$ is a bounded function on $[a,b]$ with the property that, given any positive real number ε, there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P,f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P,f) - L(P,f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a, b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Proposition 5.7 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{u_0, u_1, \ldots, u_m\}$, where $u_0, u_1, \ldots, u_m$ are real numbers satisfying

$$a = u_0 < u_1 < u_2 < \cdots u_{m-1} < u_m = b,$$

and $R = \{v_0, v_1, \ldots, v_n\}$, where $v_0, v_1, \ldots, v_n$ are real numbers satisfying

$$b = v_0 < v_1 < v_2 < \cdots v_{n-1} < v_n = c.$$

Then

$$P = \{a, u_1, u_2, \ldots, u_{m-1}, b, v_1, v_2, \ldots, v_{n-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\displaystyle\int_a^c f(x)\,dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx \geq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma 5.3). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$

It follows that

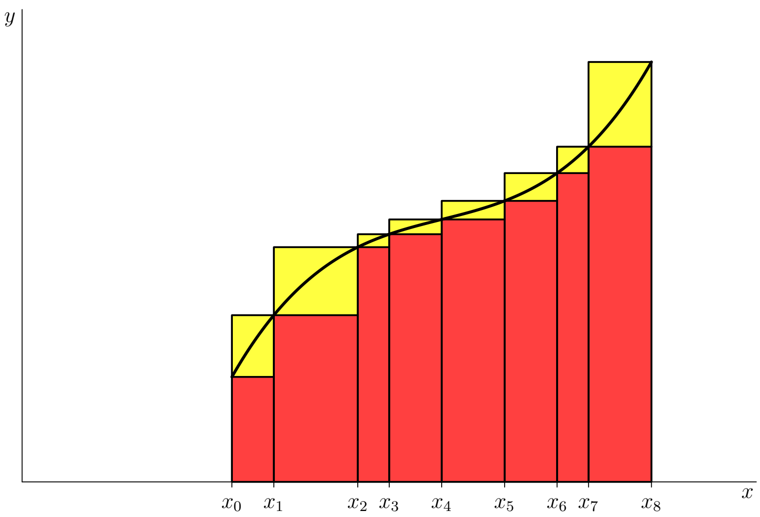
$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The result follows. ∎

5.3 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a, b] \to \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a, b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a, b] \to \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

Proposition 5.8 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.*



Proof Let $f\colon [a, b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a, b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$, and therefore the function f is bounded on $[a, b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a, b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

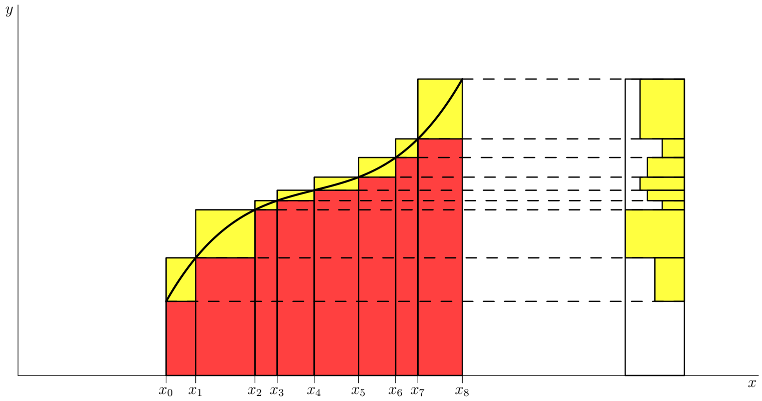
and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$. The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P,f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that



$$\begin{aligned} & U(P,f) - L(P,f) \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a,b]$.

Now let $f\colon [a,b] \to \mathbb{R}$ be a non-increasing function on $[a,b]$. Then $-f$ is a non-decreasing function on $[a,b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a,b]$. It follows that the function f itself must be Riemann-integrable on $[a,b]$, as required. ∎

Corollary 5.9 *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a,b]$.

Proof The result follows immediately on applying the results of Proposition 5.7 and Proposition 5.8. ∎

Remark The result and proof-strategy of Proposition 5.8 are to be found in their essentials in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

5.4 Integrability of Continuous functions

Theorem 5.10 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a,b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a,b]$. Then f is bounded above and below on the interval $[a,b]$, and moreover $f\colon [a,b] \to \mathbb{R}$ is uniformly continuous on $[a,b]$. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a,b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a,b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b-a) \\ &\qquad \leq \quad L(P,f) \leq U(P,f) \\ &\qquad \leq \quad \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b-a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b-a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

5.5 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 5.10). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 5.11 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$. Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a, b]$. Also

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{f(x)}{h}\int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x + h$. It follows that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt$$

whenever $x + h \in [a, b]$. But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x + h$, and therefore

$$-\varepsilon_0|h| \leq \int_x^{x+h} (f(t) - f(x))\,dt \leq \varepsilon_0|h|.$$

It follows that

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \varepsilon_0 < \varepsilon$$

whenever $x + h \in [a, b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎

6 The Multidimensional Riemann Integral

6.1 Partitions of Closed Cells

Definition We define a *closed n-cell* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \dots, n\},$$

where $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ are real numbers satisfying $u_i < v_i$ for $i = 1, 2, \dots, n$.

Definition Let C be a closed n-cell in $\mathbb{R}^n$. Then there are uniquely-determined real numbers $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ satisfying $u_i < v_i$ for $i = 1, 2, \dots, n$ for which

$$C = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \dots, n\}.$$

We define the *interior* of the n-cell C to be the open set $\operatorname{int}(C)$ defined such that

$$\operatorname{int}(C) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : u_i < x_i < v_i \text{ for } i = 1, 2, \dots, n\}.$$

Definition Let C be a closed n-cell in $\mathbb{R}^n$. Then there are uniquely-determined real numbers $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ satisfying $u_i < v_i$ for $i = 1, 2, \dots, n$ for which

$$C = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \dots, n\}.$$

We define the *content* of the n-cell C to be the positive real number $\mu(C)$ defined by the formula

$$\mu(C) = \prod_{i=1}^{n} (v_i - u_i),$$

where $\prod_{i=1}^{n} (v_i - u_i)$ denotes the product of the quantities $v_i - u_i$ for $i = 1, 2, \dots, n$.

We now develop some notation and terminology for use in discussing partitions of closed n-cells in $\mathbb{R}^n$.

Given sets $X_1, X_2, \dots, X_n$, the *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of those sets is the set consisting of all ordered n-tuples $(x_1, x_2, \dots, x_n)$ with the property that $x_i \in X_i$ for $i = 1, 2, \dots, n$.

Thus for example let $[a, b]$ and $[c, d]$ be closed intervals, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. The Cartesian product of these two closed intervals is a closed rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$, where

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b_1 \text{ and } c \leq y \leq d\}.$$

This closed rectangle is a closed 2-cell in $\mathbb{R}^2$, and moreover any closed 2-cell in $\mathbb{R}^2$ is the Cartesian product of 2 closed intervals in $\mathbb{R}^2$.

More generally, any n-cell in $\mathbb{R}^n$ is the Cartesian product of n closed intervals of positive length. The content of the n-cell is then the product of the lengths of those closed intervals.

Indeed let C be a closed n-cell in $\mathbb{R}^n$. This closed cell is determined by real numbers u_i and v_i for $i = 1, 2, \ldots, n$, where $u_i < v_i$ for all i and

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, n\}.$$

The n-cell C is thus the Cartesian product

$$[u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n]$$

of the closed intervals $[u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n]$.

Let P_i be a partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ induce a partition P of the closed n-cell C, where

$$C = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n],$$

partitions this n-cell as a collection of closed subcells that meet one another only along parts of their boundaries. Specifically let

$$P_i = \{w_{i,0}, w_{i,1}, \ldots, w_{i,k_i}\}$$

for $i = 1, 2, \ldots, n$, where

$$u_i = w_{i,0} < w_{i,1} < \cdots < w_{i,k_i} = v_i.$$

The partition P_i then decomposes the closed interval $[u_i, v_i]$ as a collection of subintervals $[w_{i,j_i-1}, w_{i,j_i}]$ where the index j_i ranges over the integers from 1 to k_i.

Let

$$\Omega(P) = \{(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n : 1 \leq j_i \leq k_i \text{ for } i = 1, 2, \ldots, n\}.$$

Given $\alpha \in \Omega(P)$, there exist integers $j_1, j_2, \ldots, j_n$ for which $1 \leq j_i \leq k_i$ for $i = 1, 2, \ldots, n$ and $\alpha = (j_1, j_2, \ldots, j_n)$. Let $C_{P,\alpha}$, or $C_{(j_1,j_2,\ldots,j_n)}$, denote the closed n-cell in $\mathbb{R}^n$ defined so that

$$\begin{aligned} C_{P,\alpha} &= C_{(j_1,j_2,\ldots,j_n)} \\ &= [w_{1,j_1-1}, w_{1,j_1}] \times [w_{2,j_2-1}, w_{2,j_2}] \times \cdots \times [w_{n,j_n-1}, w_{n,j_n}] \\ &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : w_{i,j_i-1} \leq x_i \leq w_{i,j_i} \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Then the closed n-cell C is the union of the closed subcells $C_{P,\alpha}$ as α ranges over the set $\Omega(P)$. Moreover, if two of these subcells intersect one another, then they intersect only along parts of their boundaries, and thus the interiors of these subcells are disjoint.

Proposition 6.1 *Let C be a closed n-cell in $\mathbb{R}^n$, let*

$$[u_1, v_1] \times [u_2, v_2], \ldots, [u_n, v_n]$$

be the closed intervals of positive length whose Cartesian product is the n-cell C, and let P_i be a partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ induce a partition P of the closed n-cell C as the union of closed subcells $C_{P,\alpha}$, where the index α ranges over a finite set $\Omega(P)$. Each element α of this indexing set $\Omega(P)$ is an n-tuple of integers $(j_1, j_2, \ldots, j_n)$, where j_i numbers the corresponding subinterval in the partition P_i of the interval $[u_i, v_i]$, and the corresponding subcell $C_{P,\alpha}$ of C is the Cartesian product of those subintervals. Moreover the subcells $C_{P,\alpha}$ for $\alpha \in \Omega(P)$ meet, if at all, only along parts of their boundaries, and thus the interiors of these subcells are disjoint.

Let C be a closed n-cell in $\mathbb{R}^n$. This n-cell is a product of n closed intervals

$$[u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n].$$

Let P_i be a partition of the interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ determine a partition P of the closed n-cell with indexing set $\Omega(P)$ in the manner described in Proposition 6.1. The elements of this indexing set $\Omega(P)$ are n-tuples of integers. These n-tuples label the closed subcells of C determined by the partition P. We refer to these elements of $\Omega(P)$ as *multi-indices*.

Let α be a multi-index in the indexing set $\Omega(P)$ for the partition P of the closed n-cell induced by partitions of the closed intervals $[u_i, v_i]$ whose Cartesian product is the n-cell C. Let k_i denote the number of subintervals in the partition of the ith interval $[u_i, v_i]$ occurring as a factor in the Cartesian product. Then $\alpha = (j_1, j_2, \ldots, j_n)$, where j_i is an integer between 1 and k_i for $i = 1, 2, \ldots, n$. The closed subcell $C_{P,\alpha}$ that corresponds to the multi-index α is then determined as follows:

$$C_{P,\alpha} = [w_{1,j_1-1}, w_{1,j_1}] \times [w_{2,j_2-1}, w_{2,j_2}] \times \cdots \times [w_{n,j_n-1}, w_{n,j_n}],$$

where $[w_{i,j_i-1}, w_{i,j_i}]$ is the j_ith subinterval occuring in the partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. The content $\mu(C_{P,\alpha})$ of the closed n-cell $C_{P,\alpha}$ is then given by the formula

$$\mu(C_{P,\alpha}) = \prod_{i=1}^{n} (w_{i,j_i} - w_{i,j_i-1}).$$

Proposition 6.2 *Let C be a closed n-cell in $\mathbb{R}^n$ with content $\mu(C)$, and let P be a partition of C induced by partitions of the closed intervals whose Cartesian product is the closed n-cell C. Let $\Omega(P)$ be the indexing set for the partition P, and for all multi-indices $\alpha \in \Omega(P)$, let $C_{P,\alpha}$ be the corresponding closed subcell in the partition of the closed n-cell C, and let $\mu(C_{P,\alpha})$ denote the content of $C_{P,\alpha}$. Then*

$$\mu(C) = \sum_{\alpha \in \Omega(P)} \mu(C_{P,\alpha}).$$

Proof Let

$$C = [u_1, v_1] \times [u_2, v_2], \ldots, [u_n, v_n],$$

where, for each i between 1 and n, u_i and v_i are real numbers satisfying $u_i < v_i$. Then

$$\mu(C) = \prod_{i=1}^{n} (v_i - u_i).$$

Let the partition P of C be induced by partitions P_i of $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Moreover let

$$P_i = \{w_{i,0}, w_{i,1}, \ldots, w_{i,k_i}\},$$

where $w_{i,0}, w_{i,1}, w_{i,2}, \ldots, w_{i,k_i}$ are real numbers for $j = 1, 2, \ldots, k_i$ and

$$u_i = w_{i,0} < w_{i,1} < \cdots < w_{i,k_i} = v_i.$$

The content $\mu(C_{(j_1,j_2,\ldots,j_n)})$ of the closed subcell $C_{(j_1,j_2,\ldots,j_n)}$ in the partition of C corresponding to the multi-index $(j_1, j_2, \ldots, j_n)$ is then given by the formula

$$\mu(C_{(j_1,j_2,\ldots,j_n)}) = \prod_{i=1}^{n} (w_{i,j_i} - w_{i,j_i-1}).$$

It follows that

$$\begin{aligned}\sum_{j_n=1}^{k_n} \mu(C_{(j_1,j_2,\ldots,j_n)}) &= \left(\prod_{i=1}^{n-1} (w_{i,j_i} - w_{i,j_i-1})\right) \times \left(\sum_{j_n=1}^{k_n} (w_{i,n_i} - w_{i,j_n-1})\right) \\ &= \left(\prod_{i=1}^{n-1} (w_{i,j_i} - w_{i,j_i-1})\right) \times (v_n - u_n).\end{aligned}$$

The proposition therefore follows from a straightforward application of the Principle of Mathematical Induction, using induction on the dimension n of the n-cell, and making use of the above identity in establishing the inductive step. ∎

Definition Let C be an n-cell in $\mathbb{R}^n$ and let P and R be partitions of C, where P is induced by partitions $P_1, P_2, \ldots, P_n$ of the closed intervals whose Cartesian product is the n-cell C and the partition R is induced by partitions $R_1, R_2, \ldots, R_n$ of those same closed intervals. We say that the partition R is a *refinement* of the partition P if $P_i \subset R_i$ for $i = 1, 2, \ldots, n$.

The following result follows directly from the definition of refinements of partitions of closed n-cells in $\mathbb{R}^n$.

Lemma 6.3 *Let C be an n-cell in $\mathbb{R}^n$ and let P and R be partitions of C. Then, for each multi-index β belonging to the indexing set $\Omega(R)$ for the partition R of C, there exists a unique multi-index α belonging to the indexing set $\Omega(P)$ for the partition P of C for which the subcells $C_{R,\beta}$ and $C_{P,\alpha}$ of C for the partitions P and R determined by the multi-indices β and α respectively satisfy the inclusion $C_{R,\beta} \subset C_{P,\alpha}$.*

Lemma 6.4 *Let C be a closed n-cell in $\mathbb{R}^n$, and let P and Q be partitions of C. Then there exists a partition R of C that is a common refinement of the partitions P and Q.*

Proof Let

$$C = [u_1, v_1] \times [u_2, v_2], \ldots, [u_n, v_n],$$

where, for each i between 1 and n, u_i and v_i are real numbers satisfying $u_i < v_i$. Then there are partitions P_i and Q_i of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$ so that the partitions $P_1, P_2, \ldots, P_n$ of the respective closed intervals induce the partition P of C and the partitions $Q_1, Q_2, \ldots, Q_n$ of those same closed intervals induce the partition Q of C. Let $R_i = P_i \cup Q_i$ for $i = 1, 2, \ldots, n$. Then R_i is a partition of the interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$ that is a common refinement of the partitions P_i and Q_i of the interval $[u_i, v_i]$. Let R be the partition of the closed n-cell C induced by the partitions $R_1, R_2, \ldots, R_n$ of the respective closed intervals. Then the partition R of C is the required common refinement of the partitions P and Q of C. ∎

6.2 Multidimensional Darboux Sums

Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$. A partition P of the n-cell C represents C as the union of a collection of closed n-cells $C_{P,\alpha}$ contained in C indexed by a finite set $\Omega(P)$. Distinct n-cells in this collection intersect, if at all, only along parts of their boundaries, and therefore the interiors of the subcells of C determined by the partition P are disjoint. Thus each point of C belongs to the interior of at most one cell in the collection of closed subcells into which the n-cell C is partitioned. Also the content $\mu(C)$ of the n-cell C is the sum of the contents of the subcells determined by the partition, and thus

$$\mu(C) = \sum_{\alpha \in \Omega(P)} \mu(C_{P,\alpha})$$

(see Proposition 6.2).

Definition Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, let P be a partition of C, and let $\Omega(P)$ denote the indexing set for the partition P, and, for each $\alpha \in \Omega(P)$, let

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \quad \text{and} \quad M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\},$$

where $\mu(C_{P,\alpha})$ denotes the content of the closed subcell $C_{P,\alpha}$ of C indexed by α. Then the Darboux lower sum $L(P, f)$ and the Darboux upper sum $U(P, f)$ are defined by the formulae

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, \mu(C_{P,\alpha}).$$

Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$. Then the definition of the Darboux lower and upper sums ensures that $L(P, f) \leq U(P, f)$ for all partitions P of the n-cell C.

Let C be a closed n-cell in $\mathbb{R}^n$, and let P and R be partitions of C, where P is determined by partitions $P_1, P_2, \ldots, P_n$ of the closed intervals whose Cartesian product is the closed n-cell C and R is determined by partitions $R_1, R_2, \ldots, R_n$ of those same closed intervals. We recall that the partition R is a *refinement* of P if and only if $P_i \subset R_i$ for $i = 1, 2, \ldots, n$.

Lemma 6.5 *Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, and let P and R be partitions of C. Suppose that R is a refinement of P. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f).$$

Proof Let the cells of the partitions P and R be indexed by indexing sets $\Omega(P)$ and $\omega(R)$ respectively. Also, for each $\alpha \in \Omega(P)$, let $C_{P,\alpha}$ be the cell of the partition P determined by α, and, for each $\beta \in \Omega(R)$, let $C_{R,\beta}$ be the cell of the partition R determined by β. Then, given a subcell $C_{R,\beta}$ of C, indexed by some element β of the indexing set $\Omega(R)$ for the partition R, there exists a uniquely-determined subcell $C_{P,\alpha}$ of C, indexed by some element α of the indexing set $\Omega(P)$ for the partition P, for which $C_{R,\beta} \subset C_{P,\alpha}$. (see Lemma 6.3). It follows that there is a unique well-defined function $\lambda: \Omega(R) \to \Omega(P)$ characterized by the requirement that, for each multi-index β belonging to the indexing set $\Omega(R)$ for the partition R, the element $\lambda(\beta)$ of the indexing set $\Omega(P)$ for the partition P is the unique multi-index in $\Omega(P)$ for which $C_{R,\beta} \subset C_{P,\lambda(\beta)}$. Now

$$\begin{aligned} U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, \mu(C_{P,\alpha}), \\ L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha}), \\ U(R, f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, \mu(C_{R,\beta}), \\ L(R, f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, \mu(C_{R,\beta}), \end{aligned}$$

where

$$\begin{aligned} M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ M_{R,\beta} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{R,\beta}\}, \\ m_{R,\beta} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{R,\beta}\} \end{aligned}$$

for all $\alpha \in \Omega(P)$ and $\beta \in \Omega(R)$. Also

$$M_{R,\beta} \leq M_{P,\lambda(\beta)} \quad \text{and} \quad m_{R,\beta} \geq m_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, because $C_{R,\beta} \subset C_{P,\lambda(\beta)}$. Now the partition R of C determines a partition of each cell $C_{P,\alpha}$ of the partition P, decomposing the cell $C_{P,\alpha}$ as a union of the sets $C_{R,\beta}$ for which $\lambda(\beta) = \alpha$. It follows from Proposition 6.2 that

$$C_{P,\alpha} = \sum_{\beta \in \Omega(R;\alpha)} \mu(C_{R,\beta})$$

where

$$\Omega(R;\alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$. Therefore

$$\begin{aligned} L(R,f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, \mu(C_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} m_{R,\beta}\, \mu(C_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} \mu(C_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha}) \\ &= L(P,f). \end{aligned}$$

Similarly

$$\begin{aligned} U(R,f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, \mu(C_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} M_{R,\beta}\, \mu(C_{R,\beta}) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} \mu(C_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, \mu(C_{P,\alpha}) \\ &= U(P,f). \end{aligned}$$

This completes the proof. ∎

Lemma 6.6 *Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, and let P and Q be partitions of C. Then then the Darboux sums of the function f for the partitions P and Q satisfy $L(P,f) \leq U(Q,f)$.*

Proof There exists a partition R of C that is a common refinement of the partitions P and Q of C. (Lemma 6.4.) Moreover $L(R,f) \geq L(P,f)$ and

$U(R, f) \leq U(Q, f)$ (Lemma 6.5). It follows that

$$L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f),$$

as required. ▮

6.3 The Multidimensional Riemann-Darboux Integral

Definition Let C be an n-cell in $\mathbb{R}^n$, and let $f: C \to \mathbb{R}$ be a bounded real-valued function on C. The *lower Riemann integral* and the *upper Riemann integral*, denoted by

$$\mathcal{L} \int_C f(\mathbf{x})\, d\mu \quad \text{and} \quad \mathcal{U} \int_C f(\mathbf{x})\, d\mu$$

respectively, are defined such that

$$\begin{aligned} \mathcal{L} \int_C f(\mathbf{x})\, d\mu &= \sup\{L(P, f) : P \text{ is a partition of } C\}, \\ \mathcal{U} \int_C f(\mathbf{x})\, d\mu &= \inf\{U(P, f) : P \text{ is a partition of } C\}. \end{aligned}$$

Lemma 6.7 *Let f be a bounded real-valued function on an n-cell C in $\mathbb{R}^n$. Then*

$$\mathcal{L} \int_C f(\mathbf{x})\, dx \leq \mathcal{U} \int_C f(\mathbf{x})\, dx.$$

$$\mathcal{L} \int_C f(\mathbf{x})\, d\mu \leq \mathcal{U} \int_C f(\mathbf{x})\, d\mu.$$

Proof The inequality $L(P, f) \leq L(Q, f)$ holds for all partitions P and Q of the closed n-cell C (Lemma 6.6). It follows that, for a fixed partition Q, the upper sum $U(Q, f)$ is an upper bound on all the lower sums $L(P, f)$, and therefore

$$\mathcal{L} \int_C f(\mathbf{x})\, dx \leq U(Q, f).$$

The lower Riemann integral is then a lower bound on all the upper sums, and therefore

$$\mathcal{L} \int_C f(\mathbf{x})\, d\mu \leq \mathcal{U} \int_C f(\mathbf{x})\, d\mu.$$

as required. ▮

Definition A bounded function $f\colon C \to \mathbb{R}$ on a closed n-cell C in $\mathbb{R}^n$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on C if

$$\mathcal{U}\int_C f(\mathbf{x})\,d\mu = \mathcal{L}\int_C f(\mathbf{x})\,d\mu,$$

in which case the *Riemann integral* $\int_C f(\mathbf{x})\,d\mu$ (or *Darboux integral*) of f on X is defined to be the common value of $\mathcal{U}\int_C f(\mathbf{x})\,d\mu$ and $\mathcal{L}\int_C f(\mathbf{x})\,d\mu$.

Lemma 6.8 *Let $f\colon C \to \mathbb{R}$ be a bounded function on a closed n-cell C in $\mathbb{R}^n$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned} \mathcal{U}\int_C (-f(\mathbf{x}))\,d\mu &= -\mathcal{L}\int_C f(\mathbf{x})\,d\mu, \\ \mathcal{L}\int_C (-f(\mathbf{x}))\,d\mu &= -\mathcal{U}\int_C f(\mathbf{x})\,d\mu. \end{aligned}$$

Proof Let P be a partition of C, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $C_{P,\alpha}$. Then the lower and upper sums of f for the partition P satisfy the equations

$$L(P,f) = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\,\mu(C_{P,\alpha}), \quad U(P,f) = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\,\mu(C_{P,\alpha}),$$

where

$$\begin{aligned} m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}. \end{aligned}$$

Now

$$\begin{aligned} \sup\{-f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} &= -\inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} = -m_{P,\alpha}, \\ \inf\{-f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} &= -\sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} = -M_{P,\alpha} \end{aligned}$$

It follows that

$$\begin{aligned} U(P,-f) &= \sum_{\alpha\in\Omega(P)} (-m_{P,\alpha})\mu(C_{P,\alpha}) = -L(P,f), \\ L(P,-f) &= \sum_{\alpha\in\Omega(P)} (-M_{P,\alpha})\mu(C_{P,\alpha}) = -U(P,f). \end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval C. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned}
&\mathcal{U}\int_C (-f(\mathbf{x}))\,d\mu \\
&= \inf\{U(P,-f) : P \text{ is a partition of } C\} \\
&= \inf\{-L(P,f) : P \text{ is a partition of } C\} \\
&= -\sup\{L(P,f) : P \text{ is a partition of } C\} \\
&= -\mathcal{L}\int_C f(\mathbf{x})\,d\mu
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathcal{L}\int_C (-f(\mathbf{x}))\,d\mu \\
&= \sup\{L(P,-f) : P \text{ is a partition of } C\} \\
&= \sup\{-U(P,f) : P \text{ is a partition of } C\} \\
&= -\inf\{U(P,f) : P \text{ is a partition of } C\} \\
&= -\mathcal{U}\int_C f(\mathbf{x})\,d\mu.
\end{aligned}$$

This completes the proof. ∎

Lemma 6.9 *Let $f: C \to \mathbb{R}$ and $g: C \to \mathbb{R}$ be bounded functions on a closed n-cell C in $\mathbb{R}^n$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P,f) + L(P,g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P,f) + U(P,g).$$

Proof Let P be a partition of C, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $C_{P,\alpha}$. Then

$$L(P,f) = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}),$$

$$
\begin{aligned}
L(P, g) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(g) \mu(C_{P,\alpha}), \\
L(P, f + g) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f + g) \mu(C_{P,\alpha}), \\
U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) \mu(C_{P,\alpha}), \\
U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) \mu(C_{P,\alpha}), \\
U(P, f + g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f + g) \mu(C_{P,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(f + g) &= \inf\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \\
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(f + g) &= \sup\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}
\end{aligned}
$$

for $\alpha \in \Omega(P)$.

Now

$$
m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f) \quad \text{and} \quad m_{P,\alpha}(g) \leq g(\mathbf{x}) \leq M_{P,\alpha}(g).
$$

for all $\mathbf{x} \in C_{P,\alpha}$. Adding, we see that

$$
m_{P,\alpha}(f) + m_{P,\alpha}(g) \leq f(\mathbf{x}) + g(\mathbf{x}) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g)
$$

for all $\mathbf{x} \in C_{P,\alpha}$, and therefore $M_{P,\alpha}(f) + M_{P,\alpha}(g)$ is an upper bound for the set

$$
\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.
$$

and $m_{P,\alpha}(f) + m_{P,\alpha}(g)$ is a lower bound for the same set. The least upper bound and greatest lower bound for this set are $M_{P,\alpha}(f+g)$ and $m_{P,\alpha}(f+g)$ respectively. Therefore

$$
\begin{aligned}
m_{P,\alpha}(f) + m_{P,\alpha}(g) &\leq m_{P,\alpha}(f + g) \\
&\leq M_{P,\alpha}(f + g) \\
&\leq M_{P,\alpha}(f) + M_{P,\alpha}(g).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&U(P, f+g) \\
&\quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f+g)\mu(C_{P,\alpha}) \\
&\quad \leq \sum_{\alpha \in \Omega(P)} (M_{P,\alpha}(f) + M_{P,\alpha}(g))\mu(C_{P,\alpha}) \\
&\quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}) + \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g)\mu(C_{P,\alpha}) \\
&\quad = U(P, f) + U(P, g).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&L(P, f+g) \\
&\quad = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f+g)\mu(C_{P,\alpha}) \\
&\quad \geq \sum_{\alpha \in \Omega(P)} (m_{P,\alpha}(f) + m_{P,\alpha}(g))\mu(C_{P,\alpha}) \\
&\quad = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}) + \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(g)\mu(C_{P,\alpha}) \\
&\quad = L(P, f) + L(P, g).
\end{aligned}
$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \qquad \blacksquare$$

Proposition 6.10 *Let $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-cell C. Then the functions $f+g$ and $f-g$ are Riemann-integrable on C, and moreover*

$$
\begin{aligned}
&\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu \\
&\quad = \int_C f(\mathbf{x})\, d\mu + \int_C g(\mathbf{x})\, d\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_C (f(\mathbf{x}) - g(\mathbf{x}))\, d\mu \\
&\quad = \int_C f(\mathbf{x})\, d\mu - \int_C g(\mathbf{x})\, d\mu.
\end{aligned}
$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of C for which

$$L(P, f) > \int_C f(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_C g(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad L(R, g) \geq L(P, g).$$

Applying Lemma 6.9, and the definition of the lower Riemann integral, we see that

$$\begin{aligned} & \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu \\ \geq\ & L(R, f+g) \geq L(R, f) + L(R, g) \\ \geq\ & L(P, f) + L(Q, g) \\ >\ & \left(\int_C f(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon\right) \\ & \qquad + \left(\int_C g(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon\right) \\ >\ & \int_C f(\mathbf{x})\, d\mu + \int_C g(\mathbf{x})\, d\mu - \varepsilon \end{aligned}$$

We have now shown that

$$\begin{aligned} & \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu \\ >\ & \int_C f(\mathbf{x})\, d\mu + \int_C g(\mathbf{x})\, d\mu - \varepsilon \end{aligned}$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu, \quad \int_C f(\mathbf{x})\, d\mu$$

and

$$\int_C g(\mathbf{x})\, d\mu$$

have values that have no dependence whatsoever on the value of ε.

It follows that

$$\begin{aligned} &\mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ \geq\ & \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

We can deduce a corresponding inequality involving the upper integral of $f+g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 6.8). We find that

$$\begin{aligned} &\mathcal{L}\int_C (-f(\mathbf{x}) - g(\mathbf{x}))\,d\mu \\ \geq\ & \int_C (-f(\mathbf{x}))\,d\mu + \int_C (-g(\mathbf{x}))\,d\mu \\ =\ & -\int_C f(\mathbf{x})\,d\mu - \int_C g(\mathbf{x})\,d\mu \end{aligned}$$

and therefore

$$\begin{aligned} &\mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ =\ & -\mathcal{L}\int_C (-f(\mathbf{x}) - g(\mathbf{x}))\,d\mu \\ \leq\ & \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned} \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu &\leq \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &\leq \mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &\leq \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned} \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu &= \mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &= \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

Thus the function $f + g$ is Riemann-integrable on C, and

$$\begin{aligned}&\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ = &\int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu.\end{aligned}$$

Then, replacing g by $-g$, we find that

$$\begin{aligned}&\int_C (f(\mathbf{x}) - g(\mathbf{x}))\,d\mu \\ = &\int_C f(\mathbf{x})\,d\mu - \int_C g(\mathbf{x})\,d\mu.\end{aligned}$$

as required. ∎

Proposition 6.11 *Let $f: C \to \mathbb{R}$ be a bounded function on a closed n-cell C in $\mathbb{R}^n$. Then the function f is Riemann-integrable on C if and only if, given any positive real number ε, there exists a partition P of C with the property that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof First suppose that $f: C \to \mathbb{R}$ is Riemann-integrable on C. Let some positive real number ε be given. Then

$$\int_C f(\mathbf{x})\,d\mu$$

is equal to the common value of the lower and upper integrals of the function f on C, and therefore there exist partitions Q and R of C for which

$$L(Q, f) > \int_C f(\mathbf{x})\,d\mu - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_C f(\mathbf{x})\,d\mu + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 6.5). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f\colon C \to \mathbb{R}$ is a bounded function on C with the property that, given any positive real number ε, there exists a partition P of C for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of C for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$\begin{aligned} L(P, f) &\leq \mathcal{L}\int_C f(\mathbf{x})\,d\mu \\ &\leq \mathcal{U}\int_C f(\mathbf{x})\,d\mu \leq U(P, f), \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{U}\int_C f(\mathbf{x})\,d\mu - \mathcal{L}\int_C f(\mathbf{x})\,d\mu \\ < U(P, f) - L(P, f) < \varepsilon. \end{aligned}$$

Thus the difference between the values of the upper and lower integrals of f on C must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_C f(\mathbf{x})\,d\mu = \mathcal{L}\int_C f(\mathbf{x})\,d\mu.$$

This completes the proof. ∎

Lemma 6.12 *Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}. \end{aligned}$$

Then

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

for all $u, v \in X$.

Proof Let $u, v \in X$. Then either $f(v) \geq f(u)$ or $f(u) \geq f(v)$. In the case where $f(v) \geq f(u)$ the inequalities $m_X(f) \leq f(u) \leq f(v) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. In the case where $f(u) \geq f(v)$ the inequalities $m_X(f) \leq f(v) \leq f(u) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. The result follows. ∎

Lemma 6.13 *Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(|f|) &= \sup\{|f(x)| : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(|f|) &= \inf\{|f(x)| : x \in X\}. \end{aligned}$$

Then

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f).$$

Proof Let δ be a positive real number. Then there exist $u, v \in X$ such that

$$m_X(|f|) \leq |f(u)| < m_X(|f|) + \delta$$

and

$$M_X(|f|) - \delta < |f(v)| \leq M_X(|f|).$$

Then

$$|f(v)| - |f(u)| > M_X(|f|) - m_X(|f|) - 2\delta.$$

But

$$|f(v)| - |f(u)| \leq |f(v) - f(u)|,$$

(because $|f(v)| \leq |f(u)| + |f(v) - f(u)|$) and

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

(see Lemma 6.12). It follows that

$$\begin{aligned} M_X(|f|) - m_X(|f|) - 2\delta &< |f(v)| - |f(u)| \leq |f(v) - f(u)| \\ &\leq M_X(f) - m_X(f). \end{aligned}$$

But the values of $M_X(|f|) - m_X(|f|)$ and $M_X(f) - m_X(f)$ are independent of δ, where $\delta > 0$. It follows that

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f),$$

as required. ∎

Let X be a non-empty set, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X. We denote by $f \cdot g\colon X\mathbb{R}$ the product function defined such that We denote by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.

Lemma 6.14 *Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be bounded real-valued functions defined on a non-empty set X, let C be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in X$, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(g) &= \sup\{g(x) : x \in X\}, \\ M_X(f \cdot g) &= \sup\{f(x)g(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(g) &= \inf\{g(x) : x \in X\}, \\ m_X(f \cdot g) &= \inf\{f(x)g(x) : x \in X\}. \end{aligned}$$

Then

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Proof Let u and v be elements of the set X. Then

$$f(v)g(v) - f(u)g(u) = (f(v) - f(u))g(v) + f(u)(g(v) - g(u)),$$

and therefore

$$\begin{aligned} &|f(v)g(v) - f(u)g(u)| \\ &\qquad \leq \ |f(v) - f(u)|\,|g(v)| + |f(u)|\,|g(v) - g(u)|, \\ &\qquad \leq \ K\Big(|f(v) - f(u)| + |g(v) - g(u)|\Big). \end{aligned}$$

Now $|f(v) - f(u)| \leq M_X(f) - m_X(f)$ and $|g(v) - g(u)| \leq M_X(g) - m_X(g)$ and (see Lemma 6.12). Therefore

$$|f(v)g(v) - f(u)g(u)| \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Now, given any positive real number δ, elements u and v of X can be chosen so that

$$m_X(f \cdot g) \leq f(u)g(u) < m_X(f \cdot g) + \delta$$

and

$$M_X(f \cdot g) - \delta < f(v)g(v) \leq M_X(f \cdot g).$$

Then

$$f(v)g(v) - f(u)g(u) > M_X(f \cdot g) - m_X(f \cdot g) - 2\delta.$$

It follows that

$$M_X(f \cdot g) - m_X(f \cdot g) - 2\delta < K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big)$$

for all positive real numbers δ, and therefore

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big),$$

as required. ∎

Proposition 6.15 *Let $f: C \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-cell C in $\mathbb{R}^n$, and let $|f|: C \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in C$. Then the function $|f|$ is Riemann-integrable on C, and*

$$\left|\int_C f(\mathbf{x})\, d\mu\right| \leq \int_C |f(\mathbf{x})|\, d\mu.$$

Proof Let P be a partition of the n-cell C. We first show that the Darboux sums $U(P, f)$ and $L(P, f)$ of the function f on C and the Darboux sums $U(P, |f|)$ and $L(P, |f|)$ of the function $|f|$ on C satisfy the inequality

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f).$$

Let $\Omega(P)$ be the indexing set for the partition P of C, and let

$$\begin{aligned} M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ M_{P,\alpha}(|f|) &= \sup\{|f(\mathbf{x})| : \mathbf{x} \in C_{P,\alpha}\}, \\ m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ m_{P,\alpha}(|f|) &= \inf\{|f(\mathbf{x})| : \mathbf{x} \in C_{P,\alpha}\} \end{aligned}$$

for $\alpha \in \Omega(P)$. It follows from Lemma 6.13 that

$$M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|) \leq M_{P,\alpha}(f) - m_{P,\alpha}(f)$$

for $\alpha \in \Omega(P)$. Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}), \\ L(P, |f|) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(|f|)\mu(C_{P,\alpha}), \\ U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}), \\ U(P, |f|) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(|f|)\mu(C_{P,\alpha}). \end{aligned}$$

It follows that

$$\begin{aligned} U(P,|f|) - L(P,|f|) &= \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|))\mu(C_{P,\alpha}) \\ &\leq \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(f) - m_{P,\alpha}(f))\mu(C_{P,\alpha}) \\ &= U(P,f) - L(P,f). \end{aligned}$$

Let some positive real number ε be given. It follows from Proposition 6.11 that there exists a partition P of C such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Then

$$U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f) < \varepsilon.$$

Proposition 6.11 then ensures that the function $|f|$ is Riemann-integrable on C.

Now $-|f(\mathbf{x})| \leq f(\mathbf{x}) \leq |f(\mathbf{x})|$ for all $\mathbf{x} \in C$. It follows that

$$\begin{aligned} -\int_C |f(\mathbf{x})|\,d\mu &\leq \int_C f(\mathbf{x})\,d\mu \\ &\leq \int_C |f(\mathbf{x})|\,d\mu. \end{aligned}$$

It follows that

$$\left|\int_C f(\mathbf{x})\,d\mu\right| \leq \int_C |f(\mathbf{x})|\,d\mu,$$

as required. ∎

Proposition 6.16 *Let $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded n-cell C in $\mathbb{R}^n$. Then the function $f \cdot g$ is Riemann-integrable on C, where $(f \cdot g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in C$.*

Proof The functions f and g are bounded on C, and therefore there exists some positive real number K with the property that $|f(\mathbf{x})| \leq K$ and $|g(\mathbf{x})| \leq K$ for all $\mathbf{x} \in C$.

Let P be a partition of the n-cell C. We first show that the Darboux sums $U(P,f)$, $U(P,g)$, $U(P,f\cdot g)$, $L(P,f)$, $L(P,g)$ and $L(P,f\cdot g)$ of the functions f, g and $f\cdot g$ on C satisfy the inequality

$$\begin{aligned} &U(P,f\cdot g) - L(P,f\cdot g) \\ &\quad\leq K\Big(U(P,f) - L(P,f) + U(P,g) - L(P,g)\Big). \end{aligned}$$

Let $\Omega(P)$ be the indexing set of the partition P of the n-cell C, and for all $\alpha \in \Omega(P)$, let $\mu(C_{P,\alpha})$ denote the content of the closed subcell of C for the partition P that corresponds to the multi-index α, and let

$$\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(f \cdot g) &= \sup\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \\
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(f \cdot g) &= \inf\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.
\end{aligned}$$

Now it follows from Lemma 6.14 that

$$\begin{aligned}
&M_{P,\alpha}(f \cdot g) - m_{P,\alpha}(f \cdot g) \\
&\qquad \leq K\Big(M_{P,\alpha}(f) - m_{P,\alpha}(f) + M_{P,\alpha}(g) - m_{P,\alpha}(g)\Big).
\end{aligned}$$

for $\alpha \in \Omega(P)$. On multiplying both sides of this inequality by the content $\mu(C_{P,\alpha})$ of the subcell $C_{P,\alpha}$ of the partition indexed by α and summing over all integers between 1 and n, we find that

$$\begin{aligned}
&U(P, f \cdot g) - L(P, f \cdot g) \\
&\qquad \leq K\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big),
\end{aligned}$$

where

$$\begin{aligned}
U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}), \\
U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g)\mu(C_{P,\alpha}), \\
U(P, f \cdot g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f \cdot g)\mu(C_{P,\alpha}), \\
L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}), \\
L(P, g) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(g)\mu(C_{P,\alpha}), \\
L(P, f \cdot g) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f \cdot g)\mu(C_{P,\alpha}),
\end{aligned}$$

Let some positive real number ε be given. It follows from Proposition 6.11 that there exist partitions Q and R of the closed n-cell C for which

$$U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 6.5 that

$$U(P, f) - L(P, f) \leq U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(P, g) - L(P, g) \leq U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Combining the various inequalities obtained in the course of the proof, we find that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \; K\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big) \\ & \quad < \; \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed n-cell C with the property that

$$U(P, f \cdot g) - L(P, f \cdot g) < \varepsilon.$$

It follows from Proposition 6.11 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎

6.4 Integrability of Continuous Functions

Theorem 6.17 *Let C be a closed n-cell in $\mathbb{R}^n$. Then any continuous real-valued function on C is Riemann-integrable.*

Proof Let $f: C \to \mathbb{R}$ be a continuous real-valued function on C. Then f is bounded above and below on C, and moreover $f: C \to \mathbb{R}$ is uniformly continuous on C. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(\mathbf{u}) - f(\mathbf{w})| < \varepsilon$ whenever $\mathbf{u}, \mathbf{w} \in C$ satisfy $|\mathbf{u} - \mathbf{w}| < \delta$.

Choose a partition P of the n-cell C such that each cell in the partition has diameter less than δ. Let $\Omega(P)$ be an index set which indexes the cells of the partition P and, for each $\alpha \in \Omega(P)$ let $C_{P,\alpha}$ be the corresponding cell of the partition P of C. Also let $\mathbf{p}_\alpha$ be a point of $C_{P,\alpha}$ for all $\alpha \in \Omega(P)$. Then $|\mathbf{x} - \mathbf{p}_\alpha| < \delta$ for all $\mathbf{x} \in C_{P,\alpha}$. Thus if

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}$$

then

$$f(\mathbf{p}_\alpha) - \varepsilon \leq m_{P,\alpha} \leq M_{P,\alpha} \leq f(\mathbf{p}_\alpha) + \varepsilon$$

for all $\alpha \in \Omega(P)$. It follows that

$$\begin{aligned} & \sum_{i=1}^{n} f(\mathbf{p}_\alpha)\mu(C_{P,\alpha}) - \varepsilon\mu(C) \\ \leq \; & L(P, f) \leq U(P, f) \\ \leq \; & \sum_{i=1}^{n} f(\mathbf{p}_\alpha)\mu(C_{P,\alpha}) + \varepsilon\mu(C), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$\begin{aligned} 0 \;\leq\; & \mathcal{U}\int_C f(x)\, d\mu - \mathcal{L}\int_C f(x)\, d\mu \\ \leq\; & U(P, f) - L(P, f) \leq 2\varepsilon\mu(C). \end{aligned}$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_C f(x)\, d\mu = \mathcal{L}\int_C f(x)\, d\mu,$$

and thus the function f is Riemann-integrable on C. ∎

6.5 Repeated Integration

Let C be an n-cell in $\mathbb{R}^n$, given by

$$\begin{aligned} C &= \prod_{i=1}^{n} [a_i, b_i] \\ &= \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}, \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers which satisfy $a_i \leq b_i$ for each i. Given any continuous real-valued function f on C, let us denote by $\mathcal{I}_C(f)$ the repeated integral of f over the n-cell C whose value is

$$\int_{x_n=a_n}^{b_n} \Bigl(\cdots \int_{x_2=a_2}^{b_2} \Bigl(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\Bigr)\, dx_2 \ldots\Bigr)\, dx_n.$$

(Thus $\mathcal{I}_C(f)$ is obtained by integrating the function f first over the coordinate x_1, then over the coordinate x_2, and so on).

Note that if $m \leq f(\mathbf{x}) \leq M$ on C for some constants m and M then

$$m\,\mu(C) \leq \mathcal{I}_C(f) \leq M\,\mu(C).$$

We shall use this fact to show that if f is a continuous function on some n-cell C in $\mathbb{R}^n$ then

$$\mathcal{I}_C(f) = \int_C f(\mathbf{x})\, d\mu$$

(i.e., $\mathcal{I}_C(f)$ is equal to the Riemann integral of f over C).

Theorem 6.18 *Let f be a continuous real-valued function defined on some n-cell C in $\mathbb{R}^n$, where*

$$C = \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

Then the Riemann integral

$$\int_C f(\mathbf{x})\, d\mu$$

of f over C is equal to the repeated integral

$$\int_{x_n=a_n}^{b_n} \Bigl(\cdots \int_{x_2=a_2}^{b_2} \Bigl(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\Bigr)\, dx_2 \ldots\Bigr)\, dx_n.$$

Proof Given a partition P of the n-cell C, we denote by $L(P, f)$ and $U(P, f)$ the quantities so that

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, \mu(C_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, \mu(C_{P,\alpha})$$

where $\Omega(P)$ is an indexing set that indexes the cells of the partition P, and where, for all $\alpha \in \Omega(P)$, $\mu(C_{P,\alpha})$ is the content of the cell $C_{P,\alpha}$ of the partition P indexed by α,

$$m_{P,\alpha}(f) = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\},$$

and

$$M_{P,\alpha}(f) = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.$$

Now

$$m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f)$$

for all $\alpha \in \Omega(P)$ and $\mathbf{x} \in C_{P,\alpha}$, and therefore

$$m_{P,\alpha}(f)\, \mu(C_{P,\alpha}) \leq \mathcal{I}_{C,\alpha}(f) \leq M_{P,\alpha}(f)\, \mu(C_{P,\alpha})$$

for all $\alpha \in \Omega(P)$. Summing these inequalities as α ranges over the indexing set $\Omega(P)$, we find that

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, \mu(C_{P,\alpha}) \\ &\leq \sum_{\alpha \in \Omega(P)} \mathcal{I}_{C,\alpha}(f) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, \mu(C_{P,\alpha}) \\ &= U(P, f). \end{aligned}$$

But

$$\sum_{\alpha \in \Omega(P)} \mathcal{I}_{C,\alpha}(f) = \mathcal{I}_C(f).$$

It follows that

$$L(P, f) \leq \mathcal{I}_C(f) \leq U(P, f).$$

The Riemann integral of f is equal to the supremum of the quantities $L(P, f)$ as P ranges over all partitions of the n-cell C, hence

$$\int_C f(\mathbf{x})\, d\mu \leq \mathcal{I}_C(f).$$

Similarly the Riemann integral of f is equal to the infimum of the quantities $U(P, f)$ as P ranges over all partitions of the n-cell C, hence

$$\mathcal{I}_C(f) \leq \int_C f(\mathbf{x})\, d\mu.$$

Hence

$$\mathcal{I}_C(f) = \int_C f(\mathbf{x})\, d\mu,$$

as required. ∎

Note that the order in which the integrations are performed in the repeated integral plays no role in the above proof. We may therefore deduce the following important corollary.

Corollary 6.19 *Let f be a continuous real-valued function defined over some closed rectangle C in $\mathbb{R}^2$, where*

$$C = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\}.$$

Then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Proof It follows directly from Theorem 6.18 that the repeated integrals

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \text{ and } \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

are both equal to the Riemann integral of the function f over the rectangle C. Therefore these repeated integrals must be equal. ∎

Example Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that

$$f(x, y) = \begin{cases} \dfrac{4xy(x^2 - y^2)}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Set $u = x^2 + y^2$. Then

$$f(x, y) = \frac{2x(2x^2 - u)}{u^3} \frac{\partial u}{\partial y},$$

and therefore, when $x \neq 0$,

$$\begin{aligned} \int_{y=0}^1 f(x, y)\, dy &= \int_{u=x^2}^{x^2+1} \left(\frac{4x^3}{u^3} - \frac{2x}{u^2} \right) du \\ &= \left[-\frac{2x^3}{u^2} + \frac{2x}{u} \right]_{u=x^2}^{x^2+1} \\ &= -\frac{2x^3}{(x^2 + 1)^2} + \frac{2x}{x^2 + 1} \\ &= \frac{2x}{(x^2 + 1)^2} \end{aligned}$$

It follows that

$$\begin{aligned}\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx &= \int_{x=0}^{1}\frac{2x}{(x^2+1)^2}\,dx \\ &= \left[-\frac{1}{x^2+1}\right]_0^1 = \frac{1}{2}.\end{aligned}$$

Now $f(y,x) = -f(x,y)$ for all x and y. Interchanging x and y in the above evaluation, we find that

$$\begin{aligned}\int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy &= \int_{x=0}^{1}\left(\int_{y=0}^{1} f(y,x)\,dy\right)\,dx \\ &= -\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx \\ &= -\frac{1}{2}.\end{aligned}$$

Thus

$$\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx \neq \int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy.$$

when

$$f(x,y) = \frac{4xy(x^2-y^2)}{(x^2+y^2)^3}$$

for all $(x,y) \in \mathbb{R}^2$ distinct from $(0,0)$. Note that, in this case $f(2t,t) \to +\infty$ as $t \to 0^+$, and $f(t,2t) \to -\infty$ as $t \to 0^-$. Thus the function f is not continuous at $(0,0)$ and does not remain bounded as $(x,y) \to (0,0)$.