Module MA2321—Analysis in Several Real Variables.
Michaelmas Term 2017.
Assignment II

1. *Let* $\cosh y = \frac{1}{2}(e^y + e^{-y})$ *and* $\sinh y = \frac{1}{2}(e^y - e^{-y})$ *for all real numbers* y, *and let* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be the mapping from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *defined such that*

$$\varphi(x, y) = (\cos x\, \cosh y, -\sin x \sinh y)$$

for all real numbers x *and* y. *Let* $(p, q) \in \mathbb{R}^2$. *Determine the derivative* $(D\varphi)_{p.q}$ *of the mapping* φ *at* (p, q), *and determine the values of real numbers* r *and* θ *with the property that*

$$(D\varphi)_{p,q} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix},$$

expressing r *and* $\cos\theta$ *and* $\sin\theta$ *in terms of* p *and* q.

Let $\varphi(x, y) = (u, v)$. Then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\sin x\, \cosh y, \\
\frac{\partial u}{\partial y} &= \cos x\, \sinh y, \\
\frac{\partial v}{\partial x} &= -\cos x\, \sinh y, \\
\frac{\partial v}{\partial y} &= -\sin x\, \cosh y.
\end{aligned}$$

It follows that

$$(D\varphi)_{p,q} = \begin{pmatrix} -\sin p\, \cosh q & \cos p\, \sinh q \\ -\cos p\, \sinh q & -\sin p\, \cosh q \end{pmatrix}.$$

It follows that

$$\begin{aligned}
r^2 &= \sin^2 p\, \cosh^2 q + \cos^2 p\, \sinh^2 q \\
&= \sin^2 p\,(1 + \sinh^2 q) + \cos^2 p\, \sinh^2 q) \\
&= \sin^2 p + \sinh^2 q.
\end{aligned}$$

Thus $r = \sqrt{\sin^2 p + \sinh^2 q}$. (Alternative solution: $r = \sqrt{\cosh^2 q - \cos^2 p}$.)

It follows that

$$\cos\theta = -\frac{\sin p\, \cosh q}{\sqrt{\sin^2 p + \sinh^2 q}}, \quad \sin\theta = -\frac{\cos p\, \sinh q}{\sqrt{\sin^2 p + \sinh^2 q}}.$$

(Alternative solution:

$$\cos\theta = -\frac{\tan p}{\sqrt{\tan^2 p + \tanh^2 q}}, \qquad \sin\theta = -\frac{\tanh q}{\sqrt{\tan^2 p + \tanh^2 q}}.)$$

2. *For each positive real number k, let f_k be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $f_k(0,0) = 0$ and*

$$f_k(x,y) = \frac{x^4 + y^4}{(x^2 + y^2)^k}$$

for all points (x,y) of $\mathbb{R}^2$ distinct from $(0,0)$. Determine the values of the positive real number k for which the corresponding function f_k is continuous at $(0,0)$. Determine also the values of the positive real number k [N.B. "integer k in question as distributed"] for which the corresponding function f_k is differentiable at $(0,0)$.

Let $(x,y) \in \mathbb{R}^2$, where $(x,y) \neq (0,0)$. Then

$$f_k(tx,ty) = t^{4-2k} f(x,y),$$

$$\frac{f_k(tx,ty)}{\sqrt{x^2+y^2}} = t^{4-2k-1} f(x,y)$$

for all $t > 0$. It follows that if $(x,y) \neq (0,0)$ then

$$f_k(tx,ty) \to +\infty \text{ as } t \to 0^+$$

unless $4 - 2k \geq 0$. Thus the function f is not continuous at $(0,0)$ when $k > 2$. Also

$$\frac{f_k(tx,ty)}{\sqrt{x^2+y^2}} \to +\infty \text{ as } t \to 0^+$$

unless $4 - 2k - 1 \geq 0$. Thus the function f_k is not differentiable at $(0,0)$ when $k > \frac{3}{2}$.

Now $x^4 + y^4 \leq (x^2+y^2)^2$, and therefore

$$f_k(x,y) \leq (x^2+y^2)^{2-k}.$$

Suppose that $k < 2$. Given some positive real number ε, let $\delta_1 = \varepsilon^{\frac{1}{4-2k}}$. Then $\delta_1 > 0$, and if $\sqrt{x^2+y^2} < \delta_1$ then

$$0 \leq f_k(x,y) < \delta_1^{4-2k} = \varepsilon.$$

It follows that f_k is continuous at $(0,0)$ when $k < 2$.

Next suppose that $k < \frac{3}{2}$. Given some positive real number ε, let $\delta_2 = \varepsilon^{\frac{1}{4-2k}}$. Then $\delta_2 > 0$, and if $\sqrt{x^2+y^2} < \delta_2$ then

$$0 \leq \frac{f_k(x,y)}{\sqrt{x^2+y^2}} < \delta_2^{3-2k} = \varepsilon.$$

It follows that f_k is differentiable at $(0,0)$ when $k < \frac{3}{2}$.

Suppose that $k = 2$. Then

$$f_2(t,t) = \frac{2t^4}{4t^k} = \frac{1}{2}$$

when $t > 0$, but $f_2(0,0) = 0$. It follows that f_2 is not continuous at $(0,0)$.

Note that

$$\left.\frac{\partial f_{\frac{3}{2}}(x,y)}{\partial x}\right|_{(0,0)} = 0 \quad \text{and} \quad \left.\frac{\partial f_{\frac{3}{2}}(x,y)}{\partial y}\right|_{(0,0)} = 0.$$

Thus if the function $f_{\frac{3}{2}}$ were differentiable at $(0,0)$ then it would follow that

$$\lim_{(x,y)\to(0,0)} \frac{f_{\frac{3}{2}}(x,y)}{\sqrt{x^2+y^2}} = 0.$$

But this is not the case. Indeed

$$\frac{f_{\frac{3}{2}}(x,y)}{\sqrt{x^2+y^2}} = f_2(x,y)$$

for all $(x,y) \in \mathbb{R}^2$, and we have already shown that the limit of $f_2(x,y)$ as $(x,y) \to (0,0)$ does not exist. It follows therefore that the function $f_{\frac{3}{2}}$ is not differentiable at $(0,0)$.

Notes:

- The question as distributed inadvertently restricted k to integer values for discussing differentiability. Thus for the question, as distributed and assessed, it was sufficient to establish that f_k was differentiable for $k = 1$, but not differentiable for integers greater than or equal to two.

- An argument that, for example, shows that $\lim_{t\to 0^+} f_k(tx, ty)$ does not exist for $k > 2$ and is non-zero, for some (x, y), is enough to show that $\lim_{(x,y)\to(0,0)} f(x, y)$ either doesn't exist or is not equal to $(0, 0)$ in cases when $k \geq 2$. Thus establishes that, the condition that $0 < k < 2$ is a *necessary* condition for continuity at the origin. However this type of argument, by itself, does not prove that $0 < k < 2$ is a *sufficient* condition for continuity at the origin.
- Look at the expression for f_k in polar coordinates, we find that

  $$f_k(r\,\cos\theta, r\,\sin\theta) = r^{4-2k}(\cos^4\theta + \sin^4\theta).$$

  Now $\cos^4\theta+\sin^4\theta \leq 1$ for all real numbers θ. (For the purposes of the proof, it is sufficient to note that $\cos^4\theta+\sin^4\theta \leq 2$.) We then find that $f_k(r\,\cos\theta, r\,\sin\theta) \leq r^{4-2k}$. Alternatively, it follows from the inequality $x^4 + y^4 \leq (x^2 + y^2)^2$ that $f_k(x, y) \leq (x^2 + y^2)^{2-k}$. Now, by basic $\varepsilon - -\delta$, or something similar, $(x^2 + y^2)^{2-k} \to 0$ as $(x, y) \to (0, 0)$. So, by the Squeeze Theorem, $f_k(x, y) \to 0$ as $(x, y) \to (0, 0)$. Alternatively one can use the general proposition that if $f(x, y) = g(x, y)h(x, y)$, where $g(x, y) \to 0$ and $h(x, y)$ remains bounded as $(x, y) \to (0, 0)$, then $f(x, y) \to 0$ as $(x, y) \to (0, 0)$. There are various alternative ways of presenting the argument. But some such approach is needed to prove the existence of the two-dimensional limit required to show that the condition $0 < k < 2$ is a *sufficient* condition for the function f_k to be continuous at zero.
- In relation to differentiability for $0 < k < \frac{3}{2}$, one could compute first order partial derivatives away from zero, and show that they tend to the limit zero as $(x, y) \to (0, 0)$. But this involves unnessarary computations. In fact, looking at the values of the function f_k along the coordinate axes, it is clear that the first order partial derivatives of f_k with respect to $(0, 0)$ are zero at the origin. These partial derivatives would determine the derivative of the function f_k, were that function differentiable, and thus, if differentiable, the function f_k would have zero derivative at the origin. Thus, applying directly the definition of differentiability, we see that the function f_k is differentiable at the origin if and only if $f_{k-\frac{1}{2}}(x, y) \to 0$ as $(x, y) \to (0, 0)$. Moreover this is the case if and only if $f_{k-\frac{1}{2}}$ is continuous at the origin. Hence the sufficiency of the condition $0 < k < \frac{3}{2}$ to ensure differentiability in fact follows directly from the continuity result.