Course MA2321: Michaelmas Term 2017.

Assignment 1.

To be handed in by Thursday 23rd November, 2017.

1. *Let a, b and c be fixed real numbers, and let*

$$X = \{(x, y, z) \in \mathbb{R}^3 : x < a \text{ and } y < b \text{ and } z < c\}.$$

(a) *Let $(u, v, w) \in X$. What is the largest value of δ for which the open ball of radius δ about the point (u, v, w) is contained in the set X? [Express δ in terms of a, b, c, u, v and w.]*

$$\delta = \min(a - u, b - v, c - w).$$

(b) *Is the set X closed in $\mathbb{R}^3$? [Justify your answer.]*

The set X is not closed. Indeed the point (a, b, c) does not intersect the set, but, for any positive real number δ, the open disk of radius δ about (a, b, c) contains points (x, y, z) for which $x < a$, $y < b$ and $z < c$. Such points (x, y, z) belong to the set X. Therefore the complement of X in $\mathbb{R}^3$ is not open in $\mathbb{R}^3$, and thus X itself is not closed in $\mathbb{R}^3$.

2. *Let $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ be continuous real-valued functions on $\mathbb{R}$. Prove that*

$$\{(x, y) \in \mathbb{R}^2 : g(x) < y < h(x)\}$$

is an open set in $\mathbb{R}^2$. [Hint: consider ways in which you could apply Proposition 4.18 of the lecture notes for module MA2321 in Michaelmas Term 2017.]

Let $\varphi: \mathbb{R}^2 \to \mathbb{R}$ and $\psi: \mathbb{R}^2 \to \mathbb{R}$ be defined so that $\varphi(x, y) = y - g(x)$ and $\psi(x, y) = h(x) - y$ for all $(x, y) \in \mathbb{R}^2$. Also let $P = \{t \in \mathbb{R} : t > 0\}$. Then P is open in $\mathbb{R}$. Now the set X in question satisfies $X = X_1 \cap X_2$, where

$$\begin{aligned} X_1 &= \{(x, y) \in \mathbb{R}^3 : g(x) < y\} = \{(x, y) \in \mathbb{R}^3 : \varphi(x, y) > 0\} = \varphi^{-1}(P), \\ X_2 &= \{(x, y) \in \mathbb{R}^3 : y < h(x)\} = \{(x, y) \in \mathbb{R}^3 : \psi(x, y) > 0\} = \psi^{-1}(P), \end{aligned}$$

It follows from Proposition 4.18 that the sets X_1 and X_2 are open in $\mathbb{R}^2$. Thus the given set X is the intersection of two open sets, and is thus itself open in $\mathbb{R}^2$.

3. *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be the real-valued function on* $\mathbb{R}^2$*, defined so that*

$$f(x,y) = \begin{cases} \dfrac{2xy^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Standard theorems of analysis in one and several variables ensure that the function f *is continuous at all points* (x,y) *for which* $(x,y) \neq (0,0)$*. (See in particular Proposition 4.5 in the lecture notes for module MA2321 Notes in Michaelmas Term 2017.) Also the partial derivatives* $\dfrac{\partial f(x,y)}{\partial x}$ *and* $\dfrac{\partial f(x,y)}{\partial y}$ *are defined for all* $(x,y) \in \mathbb{R}^2$*. Indeed if* $(x,y) \neq (0,0)$ *then*

$$\frac{\partial f(x,y)}{\partial x} = \frac{2y^2(y^2-3x^2)}{(x^2+y^2)^3}, \qquad \frac{\partial f(x,y)}{\partial y} = \frac{4xy(x^2-y^2)}{(x^2+y^2)^3},$$

and if $(x,y) = 0$ *then*

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(0,0)} = 0,$$

because $f(x,0) = 0$ *for all real numbers* x *and* $f(0,y) = 0$ *for all real numbers* y*.*

(a) *Determine, for each real number* x*, the value of* $\lim\limits_{y\to 0} f(x,y)$*.*

For fixed x satisfying $x \neq 0$ the function $f(x,y)$ is a continuous function of y, and therefore $\lim\limits_{y\to 0} f(x,y) = f(x,0) = 0$. If $x \neq 0$ then $f(x,y) = 0$ for all real numbers y, and therefore $\lim\limits_{y\to 0} f(x,y) = f(x,0) = 0$. Thus $\lim\limits_{y\to 0} f(x,y) = f(x,0) = 0$ for all real numbers x.

(b) *Let* $g\colon \mathbb{R} \to \mathbb{R}$ *be the function on* $\mathbb{R}$ *determined such that* $g(y) = \int\limits_{x=0}^{1} f(x,y)\,dx$ *for all real numbers* y*. Determine the value of* $g(y)$ *for all real numbers* y*, and determine the value of* $\lim\limits_{y\to 0} g(y)$*.*

Let $y \neq 0$, and let $u = x^2 + y^2$. Applying the rule for Integration by Parts, we find that

$$g(y) \;=\; \int_{x=0}^{1} \frac{y^2}{(x^2+y^2)^2}\,\frac{du}{dx}\,dx = \int_{u=y^2}^{y^2+1} \frac{y^2}{u^2}\,du$$

$$
\begin{aligned}
&= \left[-\frac{y^2}{u}\right]_{u=y^2}^{y^2+1} \\
&= 1 - \frac{y^2}{y^2+1} = \frac{1}{y^2+1}.
\end{aligned}
$$

It follows that

$$
\lim_{y\to 0} g(y) = 1 - \lim_{y\to 0} \frac{y^2}{y^2+1} = 1,
$$

because the function sending y to $\dfrac{y^2}{y^2+1}$ is continuous at $y=0$. However $f(x,0)=0$ for all real numbers x and therefore

$$
g(0) = \int_{x=0}^{1} f(x,0)\,dx = 0.
$$

(c) *Is it the case that*

$$
\lim_{y\to 0} \int_{x=0}^{1} f(x,y)\,dx = \int_{x=0}^{1} \left(\lim_{y\to 0} f(x,y)\right)\,dx?
$$

Parts (a) and (b) demonstrate that this is not the case.

(d) *For each real number b, determine the maximum value of $f(x,y)$ on the line $y=b$, and determine the value of x for which $f(x,b)$ attains its maximum value.*

Suppose that $b \neq 0$. At the point where the function attains is maximum value on the line $y=b$, the function will be positive and the partial derivative of the function with respect to x will be zero. This happens when $b^2 - 3x^2 = 0$. The maximum is on the line $y=b$ is thus attained at $\left(\dfrac{1}{\sqrt{3}}\,b, b\right)$, and has value $\frac{3}{8}\sqrt{3}\,b^{-1}$.

(e) *Is the function f continuous at $(0,0)$? [Fully justify your answer.]*

The function f is not continuous at $(0,0)$. Compositions of continuous functions are continuous. If it were the case that the function f were continuous at $(0,0)$ then the function sending all real numbers t to $f(t,\sqrt{3}t)$ would be continuous at $t=0$. But this function has value $\frac{3}{8}t^{-1}$ when $t \neq 0$, and therefore cannot be continuous at zero. This function therefore cannot be a composition of continuous functions, and therefore the function f cannot be continuous at zero. [N.B., there are multiple ways to justify the result of (e).]