Module MA2321: Analysis in Several Real Variables
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Section 6: The Multidimensional Riemann-Darboux Integral

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Contents

6 The Multidimensional Riemann Integral

6.1 Rectangles and Partitions

Let X_i be a subset of $\mathbb{R}$ for $i = 1, 2, \ldots, n$, where n is some positive integer. The Cartesian product

$$X_1 \times X_2 \times \cdots \times X_n$$

of the sets $X_1, X_2, \ldots, X_n$ is the subset of $\mathbb{R}^n$ defined such that

$$\begin{aligned} & X_1 \times X_2 \times \cdots \times X_n \\ & \quad = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \in X_i \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

We use the notation

$$\prod_{i=1}^{n} X_i$$

to denote the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$.

Definition We define a *closed n-dimensional rectangle* in n-dimensional Euclidean space $\mathbb{R}^n$ to be Cartesian product of closed intervals in the real line.

A closed n-dimensional rectangle can thus be represented as a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\},$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers such that $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

An n-dimensional rectangle may be referred to as an *n-rectangle.*

The *interior* of the closed *n-rectangle*

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}$$

is the open set

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i < x_i < b_i \text{ for } i = 1, 2, \ldots, n\},$$

for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

In other words, the interior of the closed n-rectangle $\prod_{i=1}^{n} [a_i, b_i]$ is the open set $\prod_{i=1}^{n} (a_i, b_i)$.

Definition Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let K be the n-rectangle defined so that

$$S = \{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}.$$

The *volume* (or *content*) $v(K)$ of K is defined so that

$$v(S) = \prod_{i=1}^{n} (b_i - a_i) = (a_1 - b_1)(a_1 - b_2) \cdots (a_n - b_n).$$

Let a and b be real numbers, where $a \leq b$. A partition of the closed interval $[a, b]$ is represented as a finite set P which includes the endpoints a and b of the interval and whose elements belong to the interval. The elements of such a partition P can be listed as $x_0, x_1, x_2, \ldots, x_m$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b.$$

Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let P_i be a partition of the closed interval $[a_i, b_i]$ for each i. We can then write

$$P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\},$$

where

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i$$

for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m(i)$. Let K be the closed n-rectangle defined so that $K = \prod_{i=1}^{n} [a_i, b_i]$. Then the partitions $P_1, P_2, \ldots, P_n$ of the closed intervals

$$[a_1, b_1],\ [a_2, b_2], \ldots\ [a_n, b_n]$$

determine a partition $P_1 \times P_2 \times \cdots \times P_n$ of the n-rectangle K as a union of smaller closed n-rectangles $K_{j_1, j_2, \ldots, j_n}$, where j_i is an integer between 1 and $m(i)$ for $i = 1, 2, \ldots, n$, and where, for given integers $j_1, j_2, \ldots, j_n$ satisfying $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$, the closed n-rectangle $K_{j_1, j_2, \ldots, j_n}$ is defined so that

$$K_{j_1, j_2, \ldots, j_n} = \prod_{i=1}^{n} [x_{i, j_i - 1}, x_{i, j_i}].$$

Definition Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let K be the closed n-rectangle in $\mathbb{R}^n$ defined such that $K = \prod_{i=1}^{n} [a_i, b_i]$.

A *partition* of K is the decomposition of K as a union of closed n-rectangles $K_{j_1,j_2,\ldots,j_n}$ that is determined by partitions $P_1, P_2, \ldots, P_n$ of

$$[a_1, b_1], [a_2, b_2], \ldots [a_n, b_n]$$

respectively, where, for each integer i between 1 and n, the partition P_i is representable in the form

$$P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\}$$

for real numbers $x_{i,0}, x_{i,1}, x_{i_2}, \ldots, x_{i,m(i)}$ that satisfy

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i,$$

and where $K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$ for all integers $j_1, j_2, \ldots, j_n$ that satisfy $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$.

Proposition 6.1 *Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let $K = \prod_{i=1}^{n} [a_i, b_i]$. Let the partition P_i of $[a_i, b_i]$ be represented in the form $P_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,m(i)}\}$ for $i = 1, 2, \ldots, n$, where*

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i.$$

Then the volume $v(K)$ of the n-rectangle K satisfies

$$v(K) = \sum_{j_1=1}^{m(1)} \sum_{j_2=2}^{m(2)} \cdots \sum_{j_n=1}^{m(n)} v(K_{j_1,j_2,\ldots,j_n}),$$

where $K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$ for all n-tuples $(j_1, j_2, \ldots, j_n)$ of integers satisfying $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$.

Proof We must prove that

$$\prod_{i=1}^{n} (b_i - a_i) = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_n=1}^{m(n)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

First we note that

$$b_n - a_n = \sum_{i_n=1}^{m(n)} (x_{n,j_n} - x_{n,j_n-1}).$$

It follows directly that the result holds in the case when $n = 1$.

Suppose that $n > 1$ and that the result is known to hold for all partitions of $(n-1)$-dimensional rectangles in $\mathbb{R}^{n-1}$. Applying the result to the rectangle $\prod_{i=1}^{n-1} [a_i, b_i]$ in $\mathbb{R}^{n-1}$, we find that

$$\begin{aligned} &\prod_{i=1}^{n-1} (b_i - a_i) \\ &\quad = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_{n-1}=1}^{m(n-1)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n-1,j_{n-1}} - x_{n-1,j_{n-1}-1}). \end{aligned}$$

It follows that

$$\prod_{i=1}^{n} (b_i - a_i) = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_n=1}^{m(n)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

Thus if the result holds all partitions of $(n-1)$-dimensional rectangles in $\mathbb{R}^{n-1}$ then it also holds for all partitions of n-dimensional rectangles in $\mathbb{R}^n$. The result follows. ∎

We now introduce "multi-index" notation in order to reduce the complexity of the notation involved in analysing the properties of n-dimensional rectangles, partitions of such rectangles, and of real-valued functions defined on such rectangles.

Let K be an closed n-dimensional closed rectangle in $\mathbb{R}^n$, let

$$[a_1, b_1], \ [a_2, b_2], \ldots, \ [a_n, b_n]$$

be closed intervals such that $K = \prod_{i=1}^{n} [a_i, b_i]$, where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let P be a partition of K. Then there exists a partition P_i of $[a_i, b_i]$ for $i = 1, 2, \ldots, n$ such that

$$P = P_1 \times P_2 \times \cdots \times P_k.$$

We let $P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\}$ for $i = 1, 2, \ldots, n$, where $x_{i,0}, x_{i,1}, x_{i_2}, \ldots, x_{i,m(i)}$ are real numbers that satisfy

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i.$$

Now the n-rectangle K is the union of smaller n-rectangles $K_{j_1,j_2,\ldots,j_n}$ determined by the partition P, where

$$K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$$

for $i = 1, 2, \ldots, n$ and $j_i = 1, 2, \ldots, m(i)$. We refer to these n-rectangles $K_{j_1,j_2,\ldots,j_n}$ as the *cells* determined by the partition P of the rectangle K. Each cell $K_{j_1,j_2,\ldots,j_n}$ is identified by a "multi-index" $(j_1, j_2, \ldots, j_n)$. Such "multi-indices" are typically denoted by Greek letters $\alpha, \beta, \gamma, \ldots$.

Accordingly we let

$$\Omega(P) = \{(j_1, j_2, \ldots, j_n) : i = 1, 2, \ldots, n \text{ and } j_i = 1, 2, \ldots, m(i)\}.$$

Then $\Omega(P)$ is the set consisting of the multi-indices that identify cells of the partition P. Given a multi-index α, where $\alpha = (j_1, j_2, \ldots, j_n)$ for some $(j_1, j_2, \ldots, j_n) \in \Omega(P)$, we can denote by $K_{P,\alpha}$ the cell $K_{j_1,j_2,\ldots,j_n}$ of the partition corresponding to the multi-index α. The result Proposition 6.1 can then be expressed by the identity

$$v(K) = \sum_{\alpha \in \Omega(P)} v(K_{P,\alpha})$$

where $v(K)$ denotes the volume of the rectangle K and $v(K_{P,\alpha})$ denotes the volume of the cell $K_{P,\alpha}$ for all $\alpha \in \Omega(P)$.

Definition Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. We say that the partition R is a *refinement* of P if every cell of the partition R is contained within a cell of the partition P.

Lemma 6.2 *Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. Let the partition P represent K as a union of cells $K_{P,\alpha}$, where the index α ranges over an indexing set $\Omega(P)$, and where the interiors of the cells are disjoint. Similarly let the partition R represent K as a union of cells $K_{R,\beta}$, where the index β ranges over an indexing set $\Omega(R)$, and where the interiors of the cells are disjoint. Suppose that the partition R is a refinement of the partition P. Then there is a well-defined function $\lambda \colon \Omega(R) \to \Omega(P)$ characterized by the requirement that, for every $\beta \in \Omega(R)$, the cell $K_{P,\lambda(\beta)}$ of the partition P is the unique cell of that partition for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$.*

Proof The definition of the cells of the partitions P and R ensures that the interiors of these cells are non-empty. Moreover if a cell $K_{R,\beta}$ of the refinement R is contained in a cell $K_{P,\alpha}$ of the partition P then the interior of $K_{R,\beta}$ is contained in the interior of $K_{P,\alpha}$. But the interiors of the cells of the partition P are disjoint, and therefore the interior of $K_{R,\beta}$ cannot intersect the interiors of two or more cells of the partition P. Therefore $K_{R,\beta}$ can be contained in at most one cell of the partition P. But the definition of refinements ensures that $K_{R,\beta}$ is contained in the interior of at least one cell of the partition P. The result follows. ∎

Lemma 6.3 *Let K be an n-dimensional rectangle in $\mathbb{R}^n$, and let P and Q be partitions of K. Then there exists a partition R of K that is a common refinement of the partitions P and Q.*

Proof Let $K = \prod_{i=1}^{n} [a_i, b_i]$, where a_i and b_i are real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. It follows from the definition of partitions that there exist partitions P_i and Q_i of the closed bounded interval $[a_i, b_i]$ for $i = 1, 2, \ldots, n$ such that

$$P = P_1 \times P_2 \times \cdots \times P_n$$

and

$$Q = Q_1 \times Q_2 \times \cdots \times Q_n.$$

For each i, P_i and Q_i are finite sets containing the endpoints a_i and b_i of the interval whose other elements all belong to the interval. Let $R_i = P_i \cup Q_i$ for $i = 1, 2, \ldots, n$, and let

$$R = R_1 \times R_2 \times \cdots \times R_n.$$

Then R is a partition of K that is a common refinement of the partitions P and Q of K. The result follows. ∎

6.2 Multidimensional Darboux Sums

Let $f\colon K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$. A partition P of the n-rectangle K represents K as the union of a collection

$$\{K_{P,\alpha} : \alpha \in \Omega(P)\}$$

of n-rectangles contained in K. The interior of each of these n-rectangles is a non-empty open set in $\mathbb{R}^n$, and distinct n-rectangles in this collection intersect, if at all, only along their boundaries. Thus each point of K belongs to the interior of at most one rectangle in the collection

$$\{K_{P,\alpha} : \alpha \in \Omega(P)\}.$$

Also the volume $v(K)$ of the n-dimensional rectangle K is the sum of the volumes of the cells of the partition, and thus

$$v(K) = \sum_{\alpha \in \Omega(P)} K_{P,\alpha}.$$

Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. Then the partition R is a refinement of P if every cell of the partition R is contained within a cell of the partition P. We have shown that if the partition R of K is a refinement of a partition P of K, and if the cells of the partitions P and R of K are indexed by indexing sets $\Omega(P)$ and $\Omega(R)$ respectively, then there is a well-defined function $\lambda: \Omega(R) \to \Omega(P)$ characterized by the property that, for each $\beta \in \Omega(P)$, the cell $K_{P,\lambda(\beta)}$ is the unique cell of the partition P for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$ (see Lemma 6.2). We have also shown that, given any two partitions P and Q of K, there exists a partition R of K that is a common refinement of P and Q. (see Lemma 6.3.)

Remark The previous discussion contains more details regarding how the partition of K is implemented, and how the cells of the partition are constructed, and how they can be indexed. The results just described will be essential in the following discussion. But the details of how the cells of the partition are indexed is immaterial to the following discussion, and we could at this point choose an ordering of the cells of a given partition, and use this ordering to represent the indexing set $\Omega(P)$ associated with a partition P of an n-dimensional rectangle K as a set of consecutive integers indexing the cells of the partition P in accordance with the chosen ordering of those cells. The cells determined by the partition P of K could then be denoted as

$$K_{P,1}, \;\; K_{P,2}, \;\; \cdots K_{P,r},$$

where r is the number of cells resulting from the partition P of K.

Definition Let $f: K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, let P be a partition of K, and let the cells of this partition be indexed by the set $\Omega(P)$. For each element α of the indexing set $\Omega(P)$, let $K_{P,\alpha}$ denotes the cell of the partition indexed by α, let $v(K_P, \alpha)$ denote the volume of $K_{P,\alpha}$, and let

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

Then the Darboux lower sum $L(P, f)$ and the Darboux upper sum $U(P, f)$ are defined by the formulae

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}),$$

Let $f: K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$. Then the definition of the Darboux lower and upper sums ensures that $L(P, f) \leq U(P, f)$ for all partitions P of the n-rectangle K.

Lemma 6.4 *Let $f: K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, and let P and R be partitions of K. Suppose that R is a refinement of P. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f).$$

Proof Let the cells of the partitions P and R be indexed by indexing sets $\Omega(P)$ and $\omega(R)$ respectively. Also, for each $\alpha \in \Omega(P)$, let $K_{P,\alpha}$ be the cell of the partition P determined by α, and, for each $\beta \in \Omega(R)$, let $K_{R,\beta}$ be the cell of the partition R determined by β. Then there is a well-defined function $\lambda: \Omega(R) \to \Omega(P)$ characterized by the requirement that, for every $\beta \in \Omega(R)$, the cell $K_{P,\lambda(\beta)}$ of the partition P is the unique cell of that partition for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$ (see Lemma 6.2). Now

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha}), \\ U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}), \end{aligned}$$

where

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

for all $\alpha \in \Omega(P)$. Similarly

$$\begin{aligned} L(R, f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, v(K_{R,\beta}), \\ U(R, f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, v(K_{R,\beta}), \end{aligned}$$

where

$$m_{R,\beta} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

and

$$M_{R,\beta} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

for all $\beta \in \Omega(R)$.

Now

$$m_{R,\beta} \geq m_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, because $K_{R,\beta} \subset K_{P,\lambda(\beta)}$. Also the partition R of K determines a partition of each cell $K_{P,\alpha}$ of that partition P, decomposing the cell $K_{P,\alpha}$ as a union of the sets $K_{R,\beta}$ for which $\lambda(\beta) = \alpha$. It follows that

$$K_{P,\alpha} = \sum_{\beta\in\Omega(R;\alpha)} v(K_{R,\beta})$$

where

$$\Omega(R;\alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$ (see Proposition 6.1). Therefore

$$\begin{aligned} L(R,f) &= \sum_{\beta\in\Omega(R)} m_{R,\beta}\, v(K_{R,\beta}) \\ &= \sum_{\alpha\in\Omega(P)} \sum_{\beta\in\Omega(R;\alpha)} m_{R,\beta}\, v(K_{R,\beta}) \\ &\geq \sum_{\alpha\in\Omega(P)} m_{P,\alpha} \sum_{\beta\in\Omega(R;\alpha)} v(K_{R,\beta}) \\ &\geq \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha}) \\ &= L(P,f). \end{aligned}$$

An analogous argument applies to upper sums. Now

$$M_{R,\beta} \geq M_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, where

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

for all $\alpha \in \Omega(P)$ and

$$M_{R,\beta} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

for all $\beta \in \Omega(R)$, because $K_{R,\beta} \subset K_{P,\lambda(\beta)}$. Also

$$K_{P,\alpha} = \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta})$$

where

$$\Omega(R;\alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$, as before. Therefore

$$\begin{aligned} U(R,f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, v(K_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} M_{R,\beta}\, v(K_{R,\beta}) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}) \\ &= U(P,f). \end{aligned}$$

This completes the proof. ∎

Lemma 6.5 *Let $f\colon K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, and let P and Q be partitions of K. Then then the Darboux sums of the function f for the partitions P and Q satisfy $L(P,f) \leq U(Q,f)$.*

Proof There exists a partition R of K that is a common refinement of the partitions P and Q of K. (Lemma 6.3.) Moreover $L(R,f) \geq L(P,f)$ and $U(R,f) \leq U(Q,f)$ (Lemma 6.4). It follows that

$$L(P,f) \leq L(R,f) \leq U(R,f) \leq U(Q,f),$$

as required. ∎

6.3 The Multidimensional Riemann-Darboux Integral

Definition Let K be an n-dimensional rectangle in $\mathbb{R}^n$, and let $f\colon K \to \mathbb{R}$ be a bounded real-valued function on K. The *lower Riemann integral* and the *upper Riemann integral*, denoted by

$$\mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \quad \text{and} \quad \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n$$

respectively, are defined such that

$$\begin{aligned}
&\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad = \sup\{L(P,f) : P \text{ is a partition of } K\}, \\
&\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad = \inf\{U(P,f) : P \text{ is a partition of } K\}.
\end{aligned}$$

Lemma 6.6 *Let f be a bounded real-valued function on an n-dimensional rectangle K in $\mathbb{R}^n$. Then*

$$\mathcal{L}\int_K f(\mathbf{x})\,dx \leq \mathcal{U}\int_K f(\mathbf{x})\,dx.$$

$$\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \leq \mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.$$

Proof It follows from Lemma 6.5 that $L(P,f) \leq L(Q,f)$ for all partitions P and Q of K. It follows that, for a fixed partition Q, the upper sum $U(Q,f)$ is an upper bound on all the lower sums $L(P,f)$, and therefore

$$\mathcal{L}\int_K f(\mathbf{x})\,dx \leq U(Q,f).$$

The lower Riemann integral is then a lower bound on all the upper sums, and therefore

$$\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \leq \mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.$$

as required. ∎

Definition A bounded function $f\colon K \to \mathbb{R}$ on a closed n-dimensional rectangle K in $\mathbb{R}^n$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on K if

$$\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n = \mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n,$$

in which case the *Riemann integral* $\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$ (or *Darboux integral*) of f on X is defined to be the common value of $\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$ and $\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$.

Lemma 6.7 *Let $f\colon K \to \mathbb{R}$ be a bounded function on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned} \mathcal{U}\int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n &= -\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n, \\ \mathcal{L}\int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n &= -\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n. \end{aligned}$$

Proof Let P be a partition of K, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $K_{P,\alpha}$. Then the lower and upper sums of f for the partition P satisfy the equations

$$L(P,f) = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\,v(K_{P,\alpha}), \quad U(P,f) = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\,v(K_{P,\alpha}),$$

where

$$\begin{aligned} m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\ M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}. \end{aligned}$$

Now

$$\begin{aligned} &\sup\{-f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} \\ &\qquad = -\inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} = -m_{P,\alpha}, \\ &\inf\{-f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} \\ &\qquad = -\sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} = -M_{P,\alpha} \end{aligned}$$

It follows that

$$\begin{aligned} U(P,-f) &= \sum_{\alpha\in\Omega(P)} (-m_{P,\alpha})v(K_{P,\alpha}) = -L(P,f), \\ L(P,-f) &= \sum_{\alpha\in\Omega(P)} (-M_{P,\alpha})v(K_{P,\alpha}) = -U(P,f). \end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval K. Applying the definition of the upper and lower integrals, we see that

$$\mathcal{U}\int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n$$

$$
\begin{aligned}
&= \inf\{U(P,-f) : P \text{ is a partition of } K\} \\
&= \inf\{-L(P,f) : P \text{ is a partition of } K\} \\
&= -\sup\{L(P,f) : P \text{ is a partition of } K\} \\
&= -\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\mathcal{L}\int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad = \sup\{L(P,-f) : P \text{ is a partition of } K\} \\
&\qquad = \sup\{-U(P,f) : P \text{ is a partition of } K\} \\
&\qquad = -\inf\{U(P,f) : P \text{ is a partition of } K\} \\
&\qquad = -\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}
$$

This completes the proof. ∎

Lemma 6.8 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded functions on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P,f) + L(P,g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P,f) + U(P,g).$$

Proof Let P be a partition of K, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $K_{P,\alpha}$. Then

$$
\begin{aligned}
L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}), \\
L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g) v(K_{P,\alpha}), \\
L(P,f+g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f+g) v(K_{P,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
m_{P,\alpha}(f+g) &= \inf\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}
\end{aligned}
$$

for $\alpha \in \Omega(P)$.

Now

$$f(\mathbf{x}) \geq m_{P,\alpha}(f) \quad \text{and} \quad g(\mathbf{x}) \geq m_{P,\alpha}(g).$$

for all $\mathbf{x} \in K_{P,\alpha}$. Adding, we see that

$$f(\mathbf{x}) + g(\mathbf{x}) \geq m_{P,\alpha}(f) + m_{P,\alpha}(g)$$

for all $\mathbf{x} \in K_{P,\alpha}$, and therefore $m_{P,\alpha}(f) + m_{P,\alpha}(g)$ is a lower bound for the set

$$\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

The greatest lower bound for this set is $m_{P,\alpha}(f + g)$. Therefore

$$m_{P,\alpha}(f + g) \geq m_{P,\alpha}(f) + m_{P,\alpha}(g).$$

It follows that

$$\begin{aligned}
&L(P, f + g) \\
&\quad = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f + g) v(K_{P,\alpha}) \\
&\quad \geq \sum_{\alpha \in \Omega(P)} (m_{P,\alpha}(f) + m_{P,\alpha}(g)) v(K_{P,\alpha}) \\
&\quad = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}) + \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(g) v(K_{P,\alpha}) \\
&\quad = L(P, f) + L(P, g).
\end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned}
U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\
U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}), \\
U(P, f + g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f + g) v(K_{P,\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
M_{P,\alpha}(f + g) &= \sup\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}
\end{aligned}$$

for $\alpha \in \Omega(P)$.

Now

$$f(\mathbf{x}) \leq M_{P,\alpha}(f) \quad \text{and} \quad g(\mathbf{x}) \leq M_{P,\alpha}(g).$$

for all $\mathbf{x} \in K_{P,\alpha}$. Adding, we see that

$$f(\mathbf{x}) + g(\mathbf{x}) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g)$$

for all $\mathbf{x} \in K_{P,\alpha}$, and therefore $M_{P,\alpha}(f) + M_{P,\alpha}(g)$ is an upper bound for the set

$$\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

The least upper bound for this set is $M_{P,\alpha}(f+g)$. Therefore

$$M_{P,\alpha}(f+g) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g).$$

It follows that

$$\begin{aligned} & U(P, f+g) \\ & \quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f+g) v(K_{P,\alpha}) \\ & \quad \leq \sum_{\alpha \in \Omega(P)} (M_{P,\alpha}(f) + M_{P,\alpha}(g)) v(K_{P,\alpha}) \\ & \quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}) + \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}) \\ & \quad = U(P, f) + U(P, g). \end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 6.9 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-rectangle K. Then the functions $f + g$ and $f - g$ are Riemann-integrable on K, and moreover*

$$\begin{aligned} & \int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n \\ & \qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n, \end{aligned}$$

and

$$\int_K (f(\mathbf{x}) - g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n$$
$$= \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of K for which

$$L(P, f) > \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad L(R, g) \geq L(P, g).$$

Applying Lemma 6.8, and the definition of the lower Riemann integral, we see that

$$\begin{aligned}
\mathcal{L}\int_K & (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&\geq L(R, f + g) \geq L(R, f) + L(R, g) \\
&\geq L(P, f) + L(Q, g) \\
&> \left(\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \tfrac{1}{2}\varepsilon\right) \\
&\qquad + \left(\int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \tfrac{1}{2}\varepsilon\right) \\
&> \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \varepsilon
\end{aligned}$$

We have now shown that

$$\begin{aligned}
\mathcal{L}\int_K & (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&> \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \varepsilon
\end{aligned}$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n, \quad \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

and

$$\int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$$

have values that have no dependence whatsoever on the value of ε.

It follows that

$$\begin{aligned}
&\mathcal{L}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\qquad \geq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

We can deduce a corresponding inequality involving the upper integral of $f+g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 6.7). We find that

$$\begin{aligned}
&\mathcal{L}\int_K (-f(\mathbf{x})-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad \geq \int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n + \int_K (-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad = -\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n - \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathcal{U}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\qquad = -\mathcal{L}\int_K (-f(\mathbf{x})-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\qquad \leq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}
&\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad \leq \mathcal{L}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad \leq \mathcal{U}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad \leq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}
&\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n \\
&\qquad = \mathcal{U}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n.
\end{aligned}$$

Thus the function $f + g$ is Riemann-integrable on K, and

$$\begin{aligned}
&\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n.
\end{aligned}$$

Then, replacing g by $-g$, we find that

$$\begin{aligned}
&\int_K (f(\mathbf{x}) - g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n.
\end{aligned}$$

as required. ∎

Proposition 6.10 *Let $f\colon K \to \mathbb{R}$ be a bounded function on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the function f is Riemann-integrable on K if and only if, given any positive real number ε, there exists a partition P of K with the property that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof First suppose that $f\colon K \to \mathbb{R}$ is Riemann-integrable on K. Let some positive real number ε be given. Then

$$\int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n$$

is equal to the common value of the lower and upper integrals of the function f on K, and therefore there exist partitions Q and R of K for which

$$L(Q, f) > \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 6.4). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f\colon K \to \mathbb{R}$ is a bounded function on K with the property that, given any positive real number ε, there exists a partition P of K for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of K for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$\begin{aligned} L(P, f) &\leq \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n \\ &\leq \ \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n \leq U(P, f), \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n \\ < \ U(P, f) - L(P, f) < \varepsilon. \end{aligned}$$

Thus the difference between the values of the upper and lower integrals of f on K must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n = \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n.$$

This completes the proof. ∎

Lemma 6.11 *Let $f\colon K \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-dimensional rectangle K in $\mathbb{R}^n$, let $|f|\colon K \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in K$, and let P be a partition of the n-rectangle K. Then the Darboux sums $U(P, f)$ and $L(P, f)$ of the function f on K and the Darboux sums $U(P, |f|)$ and $L(P, |f|)$ of the function $|f|$ on K satisfy the inequality*

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f).$$

Proof Let P be a partition of K, let $\Omega(P)$ be a set that indexes the cells of the partition P of K, and let

$$\begin{aligned} M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\ M_{P,\alpha}(|f|) &= \sup\{|f(\mathbf{x})| : \mathbf{x} \in K_{P,\alpha}\}, \\ m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\ m_{P,\alpha}(|f|) &= \inf\{|f(\mathbf{x})| : \mathbf{x} \in K_{P,\alpha}\} \end{aligned}$$

for $\alpha \in \Omega(P)$. It follows from Lemma 3.8 that

$$M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|) \leq M_{P,\alpha}(f) - m_{P,\alpha}(f)$$

for $\alpha \in \Omega(P)$. Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}), \\ L(P, |f|) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(|f|) v(K_{P,\alpha}), \\ U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\ U(P, |f|) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(|f|) v(K_{P,\alpha}). \end{aligned}$$

It follows that

$$\begin{aligned} U(P, |f|) - L(P, |f|) &= \sum_{\alpha \in \Omega(P)} (M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|)) v(K_{P,\alpha}) \\ &\leq \sum_{\alpha \in \Omega(P)} (M_{P,\alpha}(f) - m_{P,\alpha}(f)) v(K_{P,\alpha}) \\ &= U(P, f) - L(P, f), \end{aligned}$$

as required. ∎

Proposition 6.12 *Let $f \colon K \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-dimensional rectangle K in $\mathbb{R}^n$, and let $|f| \colon K \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in K$. Then the function $|f|$ is Riemann-integrable on K, and*

$$\left| \int_a^b f(\mathbf{x}) \, dx_1 \, dx_2 \, \cdots \, dx_n \right| \leq \int_a^b |f(\mathbf{x})| \, dx_1 \, dx_2 \, \cdots \, dx_n.$$

Proof Let some positive real number ε be given. It follows from Proposition 6.10 that there exists a partition P of K such that

$$U(P, f) - L(P, f) < \varepsilon.$$

It then follows from Lemma 6.11 that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 6.10 then ensures that the function $|f|$ is Riemann-integrable on K.

Now $-|f(\mathbf{x})| \leq f(\mathbf{x}) \leq |f(\mathbf{x})|$ for all $\mathbf{x} \in K$. It follows that

$$\begin{aligned} -\int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \cdots dx_n &\leq \int_a^b f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \\ &\leq \int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \cdots dx_n. \end{aligned}$$

It follows that

$$\left| \int_a^b f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \right| \leq \int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \cdots dx_n,$$

as required. ∎

Lemma 6.13 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-dimensional rectangle K in $\mathbb{R}^n$, let B be a positive real number with the property that $|f(\mathbf{x})| \leq B$ and $|g(\mathbf{x})| \leq B$ for all $\mathbf{x} \in K$, and let P be a partition of the n-rectangle K. Then the Darboux sums $U(P, f)$, $U(P, g)$, $U(P, f \cdot g)$, $L(P, f)$, $L(P, g)$ and $L(P, f \cdot g)$ of the functions f, g and $f \cdot g$ on K satisfy the inequality*

$$\begin{aligned} &U(P, f \cdot g) - L(P, f \cdot g) \\ &\quad \leq B\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big). \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned} U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\ U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}), \\ U(P, f \cdot g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f \cdot g) v(K_{P,\alpha}), \end{aligned}$$

$$
\begin{aligned}
L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)v(K_{P,\alpha}),\\
L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g)v(K_{P,\alpha}),\\
L(P,f\cdot g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f\cdot g)v(K_{P,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\},\\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\},\\
M_{P,\alpha}(f\cdot g) &= \sup\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}\\
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\},\\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\},\\
m_{P,\alpha}(f\cdot g) &= \inf\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}.
\end{aligned}
$$

for $\alpha\in\Omega(P)$.

Now it follows from Lemma 3.11 that

$$
M_{P,\alpha}(f\cdot g) - m_{P,\alpha}(f\cdot g) \le B\Big(M_{P,\alpha}(f) - m_{P,\alpha}(f) + M_{P,\alpha}(g) - m_{P,\alpha}(g)\Big).
$$

for $\alpha\in\Omega(P)$. The required inequality therefore holds on multiplying both sides of the inequality above by $v(K_{P,\alpha})$ and summing over all integers between 1 and n. ∎

Proposition 6.14 *Let $f\colon K\to\mathbb{R}$ and $g\colon K\to\mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded n-dimensional rectangle K in $\mathbb{R}^n$. Then the function $f\cdot g$ is Riemann-integrable on K, where $(f\cdot g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x}\in K$.*

Proof The functions f and g are bounded on K, and therefore there exists some positive real number B with the property that $|f(\mathbf{x})|\le B$ and $|g(\mathbf{x})|\le B$ for all $\mathbf{x}\in K$.

Let some positive real number ε be given. It follows from Proposition 6.10 that there exist partitions Q and R of the closed n-rectangle K for which

$$
U(Q,f) - L(Q,f) < \frac{\varepsilon}{2B}
$$

and

$$
U(R,g) - L(R,g) < \frac{\varepsilon}{2B}.
$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 6.4 that

$$U(P,f) - L(P,f) \leq U(Q,f) - L(Q,f) < \frac{\varepsilon}{2B}$$

and

$$U(P,g) - L(P,g) \leq U(R,g) - L(R,g) < \frac{\varepsilon}{2B}.$$

It then follows from Proposition 6.13 that

$$\begin{aligned} & U(P, f\cdot g) - L(P, f \cdot g) \\ \leq\; & B\Big(U(P,f) - L(P,f) + U(P,g) - L(P,g)\Big) \\ <\; & \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed n-dimensional rectangle K with the property that

$$U(P, f\cdot g) - L(P, f\cdot g) < \varepsilon.$$

It follows from Proposition 6.10 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎

6.4 Integrability of Continuous functions

Theorem 6.15 *Let K be a closed n-dimensional rectangle in $\mathbb{R}^n$. Then any continuous real-valued function on K is Riemann-integrable.*

Proof Let $f\colon K \to \mathbb{R}$ be a continuous real-valued function on K. Then f is bounded above and below on K, and moreover $f\colon K \to \mathbb{R}$ is uniformly continuous on K. (These results follow from Theorem 5.5 and Theorem 5.6.) Therefore there exists some strictly positive real number δ such that $|f(\mathbf{u}) - f(\mathbf{w})| < \varepsilon$ whenever $\mathbf{u}, \mathbf{w} \in K$ satisfy $|\mathbf{u} - \mathbf{w}| < \delta$.

Choose a partition P of the n-rectangle K such that each cell in the partition has diameter less than δ. Let $\Omega(P)$ be an index set which indexes the cells of the partition P and, for each $\alpha \in \Omega(P)$ let $K_{P,\alpha}$ be the corresponding cell of the partition P of K. Also let $\mathbf{p}_\alpha$ be a point of $K_{P,\alpha}$ for all $\alpha \in \Omega(P)$. Then $|\mathbf{x} - \mathbf{p}_\alpha| < \delta$ for all $\mathbf{x} \in K_{P,\alpha}$. Thus if

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

then

$$f(\mathbf{p}_\alpha) - \varepsilon \leq m_{P,\alpha} \leq M_{P,\alpha} \leq f(\mathbf{p}_\alpha) + \varepsilon$$

for all $\alpha \in \Omega(P)$. It follows that

$$\begin{aligned} & \sum_{i=1}^{n} f(\mathbf{p}_\alpha) v(K_{P,\alpha}) - \varepsilon v(K) \\ \leq \quad & L(P, f) \leq U(P, f) \\ \leq \quad & \sum_{i=1}^{n} f(\mathbf{p}_\alpha) v(K_{P,\alpha}) + \varepsilon v(K), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$\begin{aligned} 0 \quad \leq \quad & \mathcal{U} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n - \mathcal{L} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n \\ \leq \quad & U(P, f) - L(P, f) \leq 2\varepsilon v(K). \end{aligned}$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n = \mathcal{L} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n,$$

and thus the function f is Riemann-integrable on K. ∎

6.5 Repeated Integration

Let K be an n-rectangle in $\mathbb{R}^n$, given by

$$\begin{aligned} K \quad &= \quad \prod_{i=1}^{n} [a_i, b_i] \\ &= \quad \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}, \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers which satisfy $a_i \leq b_i$ for each i. Given any continuous real-valued function f on K, let us denote by $\mathcal{I}_K(f)$ the repeated integral of f over the n-rectangle K whose value is

$$\int_{x_n=a_n}^{b_n} \Bigl(\cdots \int_{x_2=a_2}^{b_2} \Bigl(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1 \Bigr)\, dx_2 \ldots \Bigr)\, dx_n.$$

(Thus $\mathcal{I}_K(f)$ is obtained by integrating the function f first over the coordinate x_1, then over the coordinate x_2, and so on).

Note that if $m \leq f(\mathbf{x}) \leq M$ on K for some constants m and M then

$$m\, v(K) \leq \mathcal{I}_K(f) \leq M\, v(K).$$

We shall use this fact to show that if f is a continuous function on some n-rectangle K in $\mathbb{R}^n$ then

$$\mathcal{I}_K(f) = \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

(i.e., $\mathcal{I}_K(f)$ is equal to the Riemann integral of f over K).

Theorem 6.16 *Let f be a continuous real-valued function defined on some n-rectangle K in $\mathbb{R}^n$, where*

$$K = \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

Then the Riemann integral

$$\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

of f over K is equal to the repeated integral

$$\int_{x_n=a_n}^{b_n} \left(\cdots \int_{x_2=a_2}^{b_2} \left(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1 \right) dx_2 \ldots \right) dx_n.$$

Proof Given a partition P of the n-rectangle K, we denote by $L(P, f)$ and $U(P, f)$ the quantities so that

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, v(K_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, v(K_{P,\alpha})$$

where $\Omega(P)$ is an indexing set that indexes the cells of the partition P, and where, for all $\alpha \in \Omega(P)$, $v(K_{P,\alpha})$ is the volume of the cell $K_{P,\alpha}$ of the partition P indexed by α,

$$m_{P,\alpha}(f) = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\},$$

and

$$M_{P,\alpha}(f) = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

Now

$$m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f)$$

for all $\alpha \in \Omega(P)$ and $\mathbf{x} \in K_{P,\alpha}$, and therefore

$$m_{P,\alpha}(f)\, v(K_{P,\alpha}) \leq \mathcal{I}_{K,\alpha}(f) \leq M_{P,\alpha}(f)\, v(K_{P,\alpha})$$

for all $\alpha \in \Omega(P)$. Summing these inequalities as α ranges over the indexing set $\Omega(P)$, we find that

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, v(K_{P,\alpha}) \\ &\leq \sum_{\alpha \in \Omega(P)} \mathcal{I}_{K,\alpha}(f) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, v(K_{P,\alpha}) \\ &= U(P, f). \end{aligned}$$

But

$$\sum_{\alpha \in \Omega(P)} \mathcal{I}_{K,\alpha}(f) = \mathcal{I}_K(f).$$

It follows that

$$L(P, f) \leq \mathcal{I}_K(f) \leq U(P, f).$$

The Riemann integral of f is equal to the supremum of the quantities $L(P, f)$ as P ranges over all partitions of the n-rectangle K, hence

$$\int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n \leq \mathcal{I}_K(f).$$

Similarly the Riemann integral of f is equal to the infimum of the quantities $U(P, f)$ as P ranges over all partitions of the n-rectangle K, hence

$$\mathcal{I}_K(f) \leq \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n.$$

Hence

$$\mathcal{I}_K(f) = \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n,$$

as required. ∎

Note that the order in which the integrations are performed in the repeated integral plays no role in the above proof. We may therefore deduce the following important corollary.

Corollary 6.17 *Let f be a continuous real-valued function defined over some closed rectangle K in $\mathbb{R}^2$, where*

$$K = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\}.$$

Then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right)\, dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right)\, dy.$$

Proof It follows directly from Theorem 6.16 that the repeated integrals

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right)\, dx \text{ and } \int_c^d \left(\int_a^b f(x, y)\, dx \right)\, dy$$

are both equal to the Riemann integral of the function f over the rectangle K. Therefore these repeated integrals must be equal. ∎