Module MA2321: Analysis in Several Real Variables
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Section 4: Analysis in Euclidean Spaces

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Contents

4 Continuous Functions of Several Real Variables

4.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 4.1 (Schwarz's Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda \mathbf{x} + \mu \mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda \mathbf{x} + \mu \mathbf{y}|^2 = (\lambda \mathbf{x} + \mu \mathbf{y}).(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2.$$

Therefore $\lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4 |\mathbf{x}|^2 - 2|\mathbf{y}|^2 (\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2 |\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right) |\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Corollary 4.2 (Triangle Inequality) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$. *Then* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Proof Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 4.2) that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact that the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

4.2 Convergence of Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{p}-\mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 4.3 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim\limits_{j\to+\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^n (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

4.3 Continuity of Functions of Several Real Variables

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 4.4 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f: X \to Y$ and $g: Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.5 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 4.6 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f: X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 4.4. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.7 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $m(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $m: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 4.8 *Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h: X \to \mathbb{R}^2$, $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 4.6, Lemma 4.7 and Lemma 4.4 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f: \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 4.8. It then follows from Proposition 4.6 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lemma 4.9 *Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|: X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

Proof Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

4.4 Open Sets in Euclidean Spaces

Definition Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a non-negative real number r, the *open ball* $B(\mathbf{p}, r)$ in $\mathbb{R}^n$ of *radius* r about $\mathbf{p}$ is defined to be the subset of $\mathbb{R}^n$ defined so that

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B(\mathbf{p}, r)$ is the set consisting of all points of $\mathbb{R}^n$ that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

The *open ball* $B(\mathbf{p}, r)$ of radius r about a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius r about $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = r\}.$$

Definition A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, \delta)$ is the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$, defined so that

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset $\mathbb{R}^3$ of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ about the origin. This open ball is an open set. Indeed let $\mathbf{x}$ be a point of V. Then $|\mathbf{x}| < 3$. Let $\delta = 3 - |\mathbf{x}|$. Then $\delta > 0$. Moreover if $\mathbf{y}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{y} - \mathbf{x}| < \delta$ then

$$|\mathbf{y}| = |\mathbf{x} + (\mathbf{y} - \mathbf{x})| \leq |\mathbf{x}| + |\mathbf{y} - \mathbf{x}| < |\mathbf{x}| + \delta = 3,$$

and therefore $\mathbf{y} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius about any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 4.10).

4.5 Open Sets in Subsets of Euclidean Spaces

Definition Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 4.10 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 4.11 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 4.12 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 4.13 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 4.10), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 4.12). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

4.6 Convergence of Sequences and Open Sets

Lemma 4.14 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 4.10. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

4.7 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 4.10 and Lemma 4.11 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcup_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcup_{S \in \mathcal{A}} S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \notin S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcap_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S \in \mathcal{A}} S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcup_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcap_{S\in\mathcal{A}} S = \bigcup_{S\in\mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 4.12.

Proposition 4.15 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Lemma 4.16 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 4.14 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

4.8 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 4.17 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $f: X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f: X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 4.10, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f: X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

4.9 Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p} \in \mathbb{R}^m$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

It follows easily from the definition of convergence of sequences of points in Euclidean space that if X is a subset of m-dimensional Euclidean space $\mathbb{R}^m$ and if $\mathbf{p}$ is a point of $\mathbb{R}^m$ then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Lemma 4.18 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f: X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g: Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = g(\mathbf{q}),$$

as required. ∎

Proposition 4.19 *Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}^n$ has the property that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. It therefore follows from Lemma 4.18 that

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

then

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < |f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 4.20 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Proof The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x} - \mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

Proposition 4.21 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional*

Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (\mathbf{q} + \mathbf{r})| &\leq |f(\mathbf{x}) - \mathbf{q}| + |g(\mathbf{x}) - \mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r},$$

as required. ∎

Definition Let $f\colon X \to \mathbb{R}^n$ be a function mapping some subset X of m-dimensional Euclidean space $\mathbb{R}^m$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. We say that $f(\mathbf{x})$ *remains bounded* as $\mathbf{x}$ tends to $\mathbf{p}$ in X if strictly positive constants C and δ can be determined so that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Proposition 4.22 *Let $f: X \to \mathbb{R}^m$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(h(\mathbf{x}) f(\mathbf{x}) \Big) = \mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined so that $|h(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|h(\mathbf{x})| \leq C$ and $|f(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x}) f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.23 *Let $f: X \to \mathbb{R}^m$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = 0$. Suppose also that $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x}) f(\mathbf{x})) = \mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|h(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| \leq C$ and $|h(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x}) f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.24 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{s}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(f(\mathbf{x}) \cdot g(\mathbf{x})\Bigr) = 0.$$

Proof Let some strictly positive real number ε be given. Now $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|g(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x}-\mathbf{p}| < \delta$ then $|f(\mathbf{x})| < \varepsilon_0$ and $|g(\mathbf{x})| \leq C$. It then follows from Schwarz's Inequality (Proposition 4.1) that

$$|f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{x})|\,|g(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.25 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of $\mathbb{R}^n$ and let s be a real number. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = s.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})f(\mathbf{x}) = s\mathbf{q}.$$

Proof The functions f and h satisfy the equation

$$h(\mathbf{x})f(\mathbf{x}) = h(\mathbf{x})\Bigl(f(\mathbf{x}) - \mathbf{q}\Bigr) + (h(\mathbf{x}) - s)\mathbf{q} + s\mathbf{q},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(f(\mathbf{x}) - \mathbf{q}\Bigr) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Bigl(h(\mathbf{x}) - s\Bigr) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that $|h(\mathbf{x}) - s| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x}-\mathbf{p}| < \delta_0$. But it then follows from the Triangle

Inequality that $|h(\mathbf{x})| < |s| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) = \mathbf{0}$$

(see Proposition 4.23). Similarly

$$\lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x}) - s)\, \mathbf{q} = \mathbf{0}.$$

It follows that

$$\begin{aligned} & \lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x}) f(\mathbf{x})) \\ = \; & \lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) + \lim_{\mathbf{x} \to \mathbf{p}} \Big(\Big(h(\mathbf{x}) - s \Big) \mathbf{q} \Big) + s\mathbf{q} \\ = \; & \mathbf{0} + s\mathbf{q}, \end{aligned}$$

as required. ∎

Proposition 4.26 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) = \mathbf{q} \cdot \mathbf{r}.$$

Proof The functions f and g satisfy the equation

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = (f(\mathbf{x}) - \mathbf{q}) \cdot g(\mathbf{x}) + \mathbf{q} \cdot (g(\mathbf{x}) - \mathbf{r}) + \mathbf{q} \cdot \mathbf{r},$$

where

$$\lim_{\mathbf{x} \to \mathbf{p}} \Big(f(\mathbf{x}) - \mathbf{q} \Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{p}} \Big(g(\mathbf{x}) - \mathbf{r} \Big) = \mathbf{0}.$$

Moreover there exists a strictly positive constant δ_0 such that $|g(\mathbf{x}) - \mathbf{r}| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|g(\mathbf{x})| < |\mathbf{r}| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} \Big(\Big(f(\mathbf{x}) - \mathbf{q} \Big) \cdot g(\mathbf{x}) \Big) = 0$$

(see Proposition 4.24). Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q} \cdot \Big(g(\mathbf{x}) - \mathbf{r}\Big)\Big) = 0.$$

It follows that

$$\begin{aligned}
& \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) \\
= \;& \lim_{\mathbf{x}\to\mathbf{p}} \Big(\Big(f(\mathbf{x}) - \mathbf{q}\Big) \cdot g(\mathbf{x})\Big) + \lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q} \cdot \Big(g(\mathbf{x}) - \mathbf{r}\Big)\Big) + \mathbf{q} \cdot \mathbf{r} \\
= \;& \mathbf{q} \cdot \mathbf{r},
\end{aligned}$$

as required. ∎

Proposition 4.27 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x}))$, $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x}))$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

First Proof It follows from Proposition 4.21 (applied in the case when the target space is one-dimensional) that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

It follows from Proposition 4.25 (applied in the case when the target space is one-dimensional), or alternatively from Proposition 4.26, that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) \neq 0$. Let $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the reciprocal function defined so that $r(t) = 1/t$ for all non-zero real numbers t. Then the reciprocal function r is continuous. Applying the result of Lemma 4.18, we find that

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{1}{g(\mathbf{x})} = \lim_{\mathbf{x} \to \mathbf{p}} r(g(\mathbf{x})) = r\left(\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})\right) = \frac{1}{\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})}.$$

It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})},$$

as required. ∎

Second Proof Let $l = \lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})$ and $m = \lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})$, and let $h\colon X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x}) = (l, m)$$

(see Proposition 4.19).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u, v) = u + v$ and $m(u, v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 4.7). Also $f + g = s \circ h$ and $f \cdot g = m \circ f$. It follows from this that

$$\begin{aligned}
\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x} \to \mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x} \to \mathbf{p}} s(h(\mathbf{x})) \\
&= s\left(\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x})\right) = s(l, m) = l + m,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) g(\mathbf{x})) &= \lim_{\mathbf{x} \to \mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x} \to \mathbf{p}} m(h(\mathbf{x})) \\
&= m\left(\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x})\right) = m(l, m) = lm
\end{aligned}$$

(see Lemma 4.18).

Also

$$\lim_{\mathbf{x} \to \mathbf{p}} (-g(\mathbf{x})) = -m.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = l - m.$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $r(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{m}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{l}{m},$$

as required. ∎

Proposition 4.28 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ and $g: Y \to \mathbb{R}^k$ be functions satisfying $f(X) \subset Y$. Let $\mathbf{p}$ be a limit point of X in $\mathbb{R}^m$, let $\mathbf{q}$ be a limit point of Y in $\mathbb{R}^n$ let $\mathbf{r}$ be a point of $\mathbb{R}^k$. Suppose that the following three conditions are satisfied:*

(i) $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$;

(ii) $\lim_{\mathbf{y}\to\mathbf{q}} g(\mathbf{y}) = \mathbf{r}$;

(iii) there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_0$.

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = \mathbf{r}.$$

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{y} \in Y$ satisfies $0 < |\mathbf{y} - \mathbf{q}| < \eta$. There then exists some positive real number δ_1 such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Also there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and $0 < |f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. But this then ensures that $|g(f(\mathbf{x})) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

4.10 Limits and Neighbourhoods

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. A subset N of X is said to be a *neighbourhood* of $\mathbf{p}$ in X if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N.$$

Lemma 4.29 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X that is not an isolated point of X. Let $f: X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in X such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$.

Proof This result follows directly from the definitions of limits and neighbourhoods. ∎

Remark Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a limit point of X that does not belong to X. Let $f: X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in $X \cup \{\mathbf{p}\}$ such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$. Thus the result of Lemma 4.29 can be extended so as to apply to limits of functions taken at limit points of the domain that do not belong to the domain of the function.