Module MA2321: Analysis in Several Real Variables
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Section 3: The Riemann Integral in One Real Variable

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Contents

3 The Riemann Integral in One Real Variable

3.1 Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *lower sum* (or *lower Darboux sum*) $L(P, f)$ and the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ are defined by

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$.

Clearly $L(P, f) \le U(P, f)$. Moreover $\sum\limits_{i=1}^{n} (x_i - x_{i-1}) = b - a$, and therefore

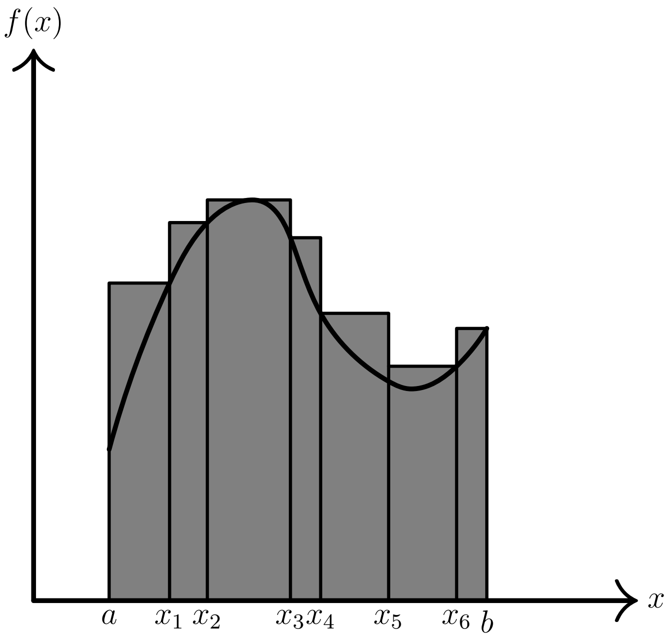
$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a),$$

for any real numbers m and M satisfying $m \le f(x) \le M$ for all $x \in [a, b]$.

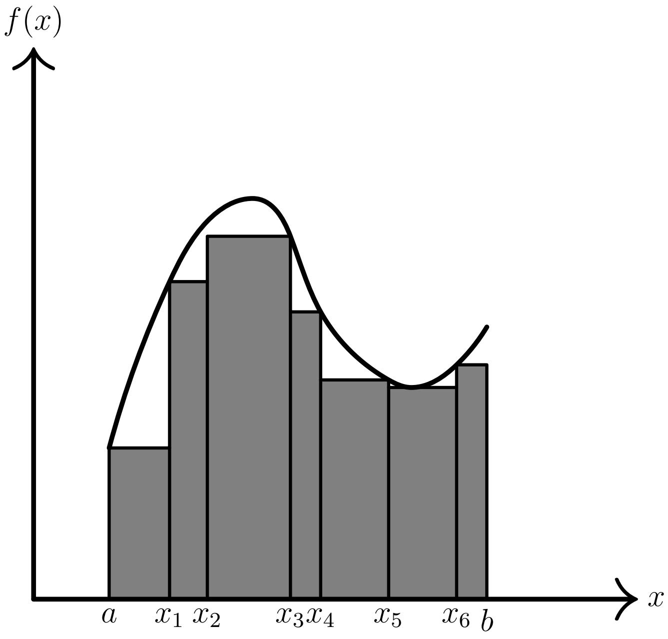
Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U} \int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L} \int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U} \int_a^b f(x)\,dx &= \inf\left\{U(P, f) : P \text{ is a partition of } [a, b]\right\}, \\ \mathcal{L} \int_a^b f(x)\,dx &= \sup\left\{L(P, f) : P \text{ is a partition of } [a, b]\right\}. \end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U} \int_a^b f(x)\,dx$ be the infimum of the values of $U(P, f)$ and that $\mathcal{L} \int_a^b f(x)\,dx$ be the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$.



The upper sum $U(P, f)$



The lower sum $L(P, f)$

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P,f) \leq L(P,g)$ and $U(P,f) \leq U(P,g)$ for all partitions P of $[a,b]$.

Definition Let P and R be partitions of $[a,b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 3.1 *Let R be a refinement of some partition P of $[a,b]$. Then*

$$L(R,f) \geq L(P,f) \quad \textit{and} \quad U(R,f) \leq U(P,f)$$

for any bounded function $f\colon [a,b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R,f) = \sum_{i=1}^{n} L(R_i,f)$ and $U(R,f) = \sum_{i=1}^{n} U(R_i,f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i,f) \leq U(R_i,f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P,f) \leq L(R,f) \leq U(R,f) \leq U(P,f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 3.2 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 3.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$\begin{aligned}
L(P_n, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d - \frac{c}{n}\right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d\right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}$$

It follows that

$$\lim_{n \to +\infty} L(P_n, f) = \frac{c}{2} + d$$

and

$$\lim_{n \to +\infty} U(P_n, f) = \frac{c}{2} + d$$

Now $L(P_n, f) \leq \mathcal{L} \int_a^b f(x)\, dx \leq \mathcal{U} \int_a^b f(x)\, dx \leq U(P_n, f)$ for all positive integers n. It follows that $\mathcal{L} \int_a^b f(x)\, dx = \frac{1}{2}c + d = \mathcal{U} \int_a^b f(x)\, dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\, dx = \frac{1}{2}c + d$.

Example Let $f\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L} \int_0^1 f(x)\, dx = 0$ and $\mathcal{U} \int_0^1 f(x)\, dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

3.2 Basic Properties of the Riemann Integral

Lemma 3.3 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned} \mathcal{U} \int_a^b (-f(x))\, dx &= -\mathcal{L} \int_a^b f(x)\, dx, \\ \mathcal{L} \int_a^b (-f(x))\, dx &= -\mathcal{U} \int_a^b f(x)\, dx. \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned} m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \end{aligned}$$

Then the lower and upper sums of f for the partition P are given by the formulae

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Now

$$\begin{aligned} \sup\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &= -\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = -m_i, \\ \inf\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &= -\sup\{f(x) : x_{i-1} \leq x \leq x_i\} = -M_i \end{aligned}$$

It follows that

$$\begin{aligned} U(P,-f) &= \sum_{i=1}^{n} (-m_i)(x_i - x_{i-1}) = -L(P,f), \\ L(P,-f) &= \sum_{i=1}^{n} (-M_i)(x_i - x_{i-1}) = -U(P,f). \end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval $[a,b]$. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= \inf\{U(P,-f) : P \text{ is a partition of } [a,b]\} \\ &= \inf\{-L(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\sup\{L(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\mathcal{L}\int_a^b f(x)\,dx \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P,-f) : P \text{ is a partition of } [a,b]\} \\ &= \sup\{-U(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\inf\{U(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

This completes the proof. ∎

Lemma 3.4 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, and let P be a partition of the interval $[a,b]$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P, f) + L(P, g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P, f) + U(P, g).$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned} L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\ L(P, g) &= \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}), \\ L(P, f+g) &= \sum_{i=1}^{n} m_i(f+g)(x_i - x_{i-1}), \end{aligned}$$

where

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(g) &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(f+g) &= \inf\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\} \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \geq m_i(f) \quad \text{and} \quad g(x) \geq m_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \geq m_i(f) + m_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $m_i(f) + m_i(g)$ is a lower bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The greatest lower bound for this set is $m_i(f+g)$. Therefore

$$m_i(f+g) \geq m_i(f) + m_i(g).$$

It follows that

$$\begin{aligned}
L(P, f+g) &= \sum_{i=1}^{n} m_i(f+g)(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{n} (m_i(f) + m_i(g))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) \\
&= L(P, f) + L(P, g).
\end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\
U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}),
\end{aligned}$$

where

$$\begin{aligned}
M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(g) &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(f+g) &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \leq M_i(f) \quad \text{and} \quad g(x) \leq M_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \leq M_i(f) + M_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $M_i(f) + M_i(g)$ is an upper bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The least upper bound for this set is $M_i(f+g)$. Therefore

$$M_i(f+g) \leq M_i(f) + M_i(g).$$

It follows that

$$\begin{aligned}
U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (M_i(f) + M_i(g))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}) \\
&= U(P, f) + U(P, g).
\end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 3.5 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of $[a,b]$ for which

$$L(P, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad L(R, g) \geq L(P, g).$$

Applying Lemma 3.4, and the definition of the lower Riemann integral, we see that

$$\begin{aligned}
&\mathcal{L}\int_a^b (f(x) + g(x))\,dx \\
&\qquad \geq\ L(R, f + g) \geq L(R, f) + L(R, g) \\
&\qquad \geq\ L(P, f) + L(Q, g) \\
&\qquad >\ \left(\int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon\right) + \left(\int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon\right) \\
&\qquad >\ \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon
\end{aligned}$$

We have now shown that

$$\mathcal{L}\int_a^b (f(x) + g(x))\,dx > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_a^b (f(x) + g(x))\,dx, \qquad \int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

have values that have no dependence whatsoever on the value of ε. It follows that

$$\mathcal{L}\int_a^b (f(x) + g(x))\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We can deduce a corresponding inequality involving the upper integral of $f + g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 3.3). We find that

$$\begin{aligned}
\mathcal{L}\int_a^b (-f(x) - g(x))\,dx &\geq \int_a^b (-f(x))\,dx + \int_a^b (-g(x))\,dx \\
&= -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathcal{U}\int_a^b (f(x) + g(x))\,dx &= -\mathcal{L}\int_a^b (-f(x) - g(x))\,dx \\
&\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}
&\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
&\qquad \le \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \le \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}
\mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

Thus the function $f+g$ is Riemann-integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Then, replacing g by $-g$, we find that

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

as required. ∎

Proposition 3.6 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \le b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof First suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a, b]$, and therefore there exist partitions Q and R of $[a, b]$ for which

$$L(Q, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 3.1). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f\colon [a, b] \to \mathbb{R}$ is a bounded function on $[a, b]$ with the property that, given any positive real number ε, there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P, f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P, f) - L(P, f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a, b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Let u and v be real numbers. Then

$$|u| \leq |u - v| + |v| \quad \text{and} \quad |v| \leq |u - v| + |u|,$$

and therefore $|u| - |v| \leq |u - v|$. Interchanging u and v, and using the identity $|u - v| = |v - u|$, we see that $|v| - |u| \leq |u - v|$. It follows from this that

$$\Bigl| |u| - |v| \Bigr| \leq |u - v|$$

for all real numbers u and v.

Lemma 3.7 *Let $f: X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}. \end{aligned}$$

Then

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

for all $u, v \in X$.

Proof Let $u, v \in X$. Then either $f(v) \geq f(u)$ or $f(u) \geq f(v)$. In the case where $f(v) \geq f(u)$ the inequalities $m_X(f) \leq f(u) \leq f(v) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. In the case where $f(u) \geq f(v)$ the inequalities $m_X(f) \leq f(v) \leq f(u) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. The result follows. ∎

Lemma 3.8 *Let $f: X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(|f|) &= \sup\{|f(x)| : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(|f|) &= \inf\{|f(x)| : x \in X\}. \end{aligned}$$

Then

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f).$$

Proof Let δ be a positive real number. Then there exist $u, v \in X$ such that

$$m_X(|f|) \leq |f(u)| < m_X(|f|) + \delta$$

and

$$M_X(|f|) - \delta < |f(v)| \leq M_X(|f|).$$

Then

$$|f(v)| - |f(u)| > M_X(|f|) - m_X(|f|) - 2\delta.$$

But

$$|f(v)| - |f(u)| \leq |f(v) - f(u)|,$$

and

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

(see Lemma 3.7). It follows that

$$\begin{array}{rcl} M_X(|f|) - m_X(|f|) - 2\delta & < & |f(v)| - |f(u)| \leq |f(v) - f(u)| \\ & \leq & M_X(f) - m_X(f). \end{array}$$

But the values of $M_X(|f|) - m_X(|f|)$ and $M_X(f) - m_X(f)$ are independent of δ, where $\delta > 0$. It follows that

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f),$$

as required. ■

Lemma 3.9 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, let $|f|\colon [a,b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a,b]$, and let P be a partition of the interval $[a,b]$. Then the Darboux sums $U(P,f)$ and $L(P,f)$ of the function f on $[a,b]$ and the Darboux sums $U(P,|f|)$ and $L(P,|f|)$ of the function $|f|$ on $[a,b]$ satisfy the inequality*

$$U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f).$$

Proof Let P be a partition of $[a,b]$, and let

$$P = \{x_0, x_1, x_2, \ldots, x_n\},$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

and let

$$\begin{array}{rcl} M_i(f) & = & \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i(|f|) & = & \sup\{|f(x)| : x_{i-1} \leq x \leq x_i\}, \\ m_i(f) & = & \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(|f|) & = & \inf\{|f(x)| : x_{i-1} \leq x \leq x_i\} \end{array}$$

for $i = 1, 2, \ldots, n$. It follows from Lemma 3.8 that

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

for $i = 1, 2, \ldots, n$. Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned} L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\ L(P, |f|) &= \sum_{i=1}^{n} m_i(|f|)(x_i - x_{i-1}), \\ U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\ U(P, |f|) &= \sum_{i=1}^{n} M_i(|f|)(x_i - x_{i-1}). \end{aligned}$$

It follows that

$$\begin{aligned} U(P, |f|) - L(P, |f|) &= \sum_{i=1}^{n} (M_i(|f|) - m_i(|f|))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) \\ &= U(P, f) - L(P, f), \end{aligned}$$

as required. ∎

Proposition 3.10 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, and let $|f|\colon [a, b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$. Then the function $|f|$ is Riemann-integrable on $[a, b]$, and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof Let some positive real number ε be given. It follows from Proposition 3.6 that there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

It then follows from Lemma 3.9 that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 3.6 then ensures that the function $|f|$ is Riemann-integrable on $[a, b]$.

Now $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx,$$

as required. ∎

Let X be a non-empty set, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X. We denote by $f \cdot g\colon X\mathbb{R}$ the product function defined such that We denote by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.

Lemma 3.11 *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be bounded real-valued functions defined on a non-empty set X, let K be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in X$, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(g) &= \sup\{g(x) : x \in X\}, \\ M_X(f \cdot g) &= \sup\{f(x)g(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(g) &= \inf\{g(x) : x \in X\}, \\ m_X(f \cdot g) &= \inf\{f(x)g(x) : x \in X\}. \end{aligned}$$

Then

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Proof Let u and v be elements of the set X. Then

$$f(v)g(v) - f(u)g(u) = (f(v) - f(u))g(v) + f(u)(g(v) - g(u)),$$

and therefore

$$\begin{aligned} &|f(v)g(v) - f(u)g(u)| \\ &\qquad \leq \ |f(v) - f(u)|\,|g(v)| + |f(u)|\,|g(v) - g(u)|, \\ &\qquad \leq \ K\Big(|f(v) - f(u)| + |g(v) - g(u)|\Big). \end{aligned}$$

Now $|f(v) - f(u)| \leq M_X(f) - m_X(f)$ and $|g(v) - g(u)| \leq M_X(g) - m_X(g)$ and (see Lemma 3.7). Therefore

$$|f(v)g(v) - f(u)g(u)| \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Now, given any positive real number δ, elements u and v of X can be chosen so that

$$m_X(f \cdot g) \leq f(u)g(u) < m_X(f \cdot g) + \delta$$

and

$$M_X(f \cdot g) - \delta < f(v)g(v) \leq M_X(f \cdot g).$$

Then

$$f(v)g(v) - f(u)g(u) > M_X(f \cdot g) - m_X(f \cdot g) - 2\delta.$$

It follows that

$$M_X(f \cdot g) - m_X(f \cdot g) - 2\delta < K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big)$$

for all positive real numbers δ, and therefore

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big),$$

as required. ∎

Lemma 3.12 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, let K be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a,b]$, and let P be a partition of the interval $[a,b]$. Then the Darboux sums $U(P,f)$, $U(P,g)$, $U(P,f \cdot g)$, $L(P,f)$, $L(P,g)$ and $L(P,f \cdot g)$ of the functions f, g and $f \cdot g$ on $[a,b]$ satisfy the inequality*

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \;\; K\Big(U(P,f) - L(P,f) + U(P,g) - L(P,g)\Big). \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\
U(P, f \cdot g) &= \sum_{i=1}^{n} M_i(f \cdot g)(x_i - x_{i-1}), \\
L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\
L(P, g) &= \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}), \\
L(P, f \cdot g) &= \sum_{i=1}^{n} m_i(f \cdot g)(x_i - x_{i-1}),
\end{aligned}$$

where

$$\begin{aligned}
M_i(f) &= \sup\{f(x) : x_{i-1} \le x \le x_i\}, \\
M_i(g) &= \sup\{g(x) : x_{i-1} \le x \le x_i\}, \\
M_i(f \cdot g) &= \sup\{f(x)g(x) : x_{i-1} \le x \le x_i\} \\
m_i(f) &= \inf\{f(x) : x_{i-1} \le x \le x_i\}, \\
m_i(g) &= \inf\{g(x) : x_{i-1} \le x \le x_i\}, \\
m_i(f \cdot g) &= \inf\{f(x)g(x) : x_{i-1} \le x \le x_i\}.
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now it follows from Lemma 3.11 that

$$M_i(f \cdot g) - m_i(f \cdot g) \le K\Big(M_i(f) - m_i(f) + M_i(g) - m_i(g)\Big).$$

for $i = 1, 2, \ldots, n$. The required inequality therefore holds on multiplying both sides of the inequality above by $x_i - x_{i-1}$ and summing over all integers between 1 and n. ∎

Proposition 3.13 *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \le b$. Then the function $f \cdot g$ is Riemann-integrable on $[a, b]$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [a, b]$.*

Proof The functions f and g are bounded on $[a, b]$, and therefore there exists some positive real number K with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a, b]$.

Let some positive real number ε be given. It follows from Proposition 3.6 that there exist partitions Q and R of the closed interval $[a, b]$ for which

$$U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 3.1 that

$$U(P, f) - L(P, f) \leq U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(P, g) - L(P, g) \leq U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

It then follows from Proposition 3.12 that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \quad K\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big) \\ & \quad < \quad \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed bounded interval $[a, b]$ with the property that

$$U(P, f \cdot g) - L(P, f \cdot g) < \varepsilon.$$

It follows from Proposition 3.6 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎

Proposition 3.14 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$

such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{u_0, u_1, \ldots, u_m\}$, where $u_0, u_1, \ldots, u_m$ are real numbers satisfying

$$a = u_0 < u_1 < u_2 < \cdots u_{m-1} < u_m = b,$$

and $R = \{v_0, v_1, \ldots, v_n\}$, where $v_0, v_1, \ldots, v_n$ are real numbers satisfying

$$b = v_0 < v_1 < v_2 < \cdots v_{n-1} < v_n = c.$$

Then

$$P = \{a, u_1, u_2, \ldots, u_{m-1}, b, v_1, v_2, \ldots, v_{n-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\, dx + \int_b^c f(x)\, dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\displaystyle\int_a^c f(x)\, dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\, dx > \int_a^b f(x)\, dx + \int_b^c f(x)\, dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\, dx \geq \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma 3.3). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$

It follows that

$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The result follows. ∎

3.3 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a,b] \to \mathbb{R}$ defined on the closed bounded interval $[a,b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a,b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a,b] \to \mathbb{R}$ is said to be *monotonic* on $[a,b]$ if either it is non-decreasing on $[a,b]$ or else it is non-increasing on $[a,b]$.

Proposition 3.15 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a,b]$ is Riemann-integrable on $[a,b]$.*

Proof Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \dots, n$.

The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \dots, n$. It follows that

$$\begin{aligned} & U(P, f) - L(P, f) \\ = \;& \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ < \;& \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f\colon [a, b] \to \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ∎

Corollary 3.16 *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \dots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof The result follows immediately on applying the results of Proposition 3.14 and Proposition 3.15. ∎

Remark The result and proof of Proposition 3.15 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

3.4 Integrability of Continuous functions

Theorem 3.17 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a, b]$. Then f is bounded above and below on the interval $[a, b]$, and moreover $f\colon [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$. (These results follow from Theorem 1.7 and Theorem 1.8.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) \\ &\qquad \leq \quad L(P, f) \leq U(P, f) \\ &\qquad \leq \quad \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

3.5 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 3.17). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 3.18 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$.

Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a, b]$. Also

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{f(x)}{h}\int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x+h$. It follows that

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt$$

whenever $x+h \in [a,b]$. But if $0 < |h| < \delta$ and $x+h \in [a,b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x+h$, and therefore

$$-\varepsilon_0|h| \leq \int_x^{x+h} (f(t)-f(x))\,dt \leq \varepsilon_0|h|.$$

It follows that

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq \varepsilon_0 < \varepsilon$$

whenever $x+h \in [a,b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \\ f'(b) &= \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \end{aligned}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h\to 0^+} \frac{F(a+h)-F(a)}{h} = f(a), \qquad \lim_{h\to 0^-} \frac{F(b+h)-F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 3.19 *Let f be a continuously differentiable real-valued function on the interval $[a,b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfying $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 2.2) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 3.20 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof This result follows from Corollary 3.19 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \quad ∎$$

Corollary 3.21 (Integration by Substitution) *Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\,dy,$$

and $H'(s) = f(s)$ for all $s \in [a, b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a, b)$. On applying the Mean Value Theorem (Theorem 2.2) to the function $F - G$ on the interval $[a, b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

3.6 Interchanging Limits and Integrals

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x)$ converges for all $x \in [a, b]$. We wish to determine whether or not

$$\lim_{j \to +\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. Now

$$\lim_{j \to +\infty} \int_0^1 f_j(x)\,dx = \lim_{j \to +\infty} \left(\frac{j}{j+1} - \frac{j}{2j+1} \right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $jx^j \to 0$ as $j \to +\infty$. Now

$$\lim_{j \to +\infty} \frac{j+1}{j} = 1.$$

It follows that

$$\lim_{j \to +\infty} \frac{(j+1)x}{j} = x < 1,$$

Let r be chosen so that $x < r < 1$. Then there exists some positive integer N such that

$$\frac{(j+1)x^{j+1}}{jx^j} = \frac{(j+1)x}{j} \leq r$$

whenever $j \geq N$. Then $0 \leq (j+1)x^{j+1} \leq rjx^j$ whenever $j \geq N$. Let $B = Nx^N$. Then $0 \leq jx^j \leq Br^{j-N}$ whenever $j \geq N$, and therefore $jx^j \to 0$ as $j \to +\infty$. It follows that

$$\lim_{j\to+\infty} f_j(x) = \left(\lim_{j\to+\infty} jx^j\right)\left(\lim_{j\to+\infty}(1-x^j)\right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_j(1) = 0$ for all positive integers j. We conclude that $\lim\limits_{j\to+\infty} f_j(x) = 0$ for all $x \in [0,1]$, which is what we set out to show.

3.7 Uniform Convergence

We now introduce the concept of *uniform convergence*. Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a,b]$, the identity

$$\lim_{j\to+\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j\to+\infty} f_j(x)\right)\,dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a,b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_j) is said to converge *uniformly* to a function f on D as $j \to +\infty$ if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all positive integers j satisfying $j \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each positive integer j, let $M_j = \sup\{f_j(x) : x \in D\}$. We claim that $M_j \to 0$ as $j \to +\infty$.

To prove this, let some strictly positive real number ε be given. Then there exists some positive integer N such that $|f_j(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $j \geq N$. Thus if $j \geq N$ then $M_j \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_j \to 0$ as $j \to +\infty$, as claimed.

Example Let $(f_j : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0,1]$ defined by $f_j(x) = j(x^j - x^{2j})$. We have already shown (in an earlier example) that $\lim_{j\to+\infty} f_j(x) = 0$ for all $x \in [0,1]$. However a straightforward exercise in calculus shows that the maximum value attained by the function f_j is $j/4$ (which is attained at $x = 1/2^{\frac{1}{j}}$), and $j/4 \to +\infty$ as $j \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \ldots$ does not converge uniformly to the zero function on the interval $[0,1]$.

Proposition 3.22 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let some strictly positive real number ε be given. If j is chosen sufficiently large then $|f(x) - f_j(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_j \to f$ uniformly on D as $j \to +\infty$. It then follows from the continuity of f_j that there exists some strictly positive real number δ such that $|f_j(x) - f_j(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$\begin{aligned} |f(x) - f(s)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(s)| + |f_j(s) - f(s)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|x - s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 3.23 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a,b]$ to some continuous real-valued function f. Then*

$$\lim_{j\to+\infty} \int_a^b f_j(x)\,dx = \int_a^b f(x)\,dx.$$

Proof Let some strictly positive real number ε. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b-a) < \varepsilon$. Then there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon_0$ for all $x \in [a,b]$ and $j \geq N$, since the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to f on $[a,b]$. Now

$$\left|\int_a^b (f_j(x) - f(x))\,dx\right| \leq \int_a^b |f_j(x) - f(x)|\,dx$$

for all positive integers j (see Proposition 3.10). It follows that

$$\left|\int_a^b f_j(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_j(x) - f(x)|\,dx \leq \varepsilon_0(b-a) < \varepsilon$$

whenever $j \geq N$. The result follows. ∎

3.8 Integrals over Unbounded Intervals

We define integrals over unbounded intervals by appropriate limiting processes. Given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $[a, +\infty)$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. Similarly, given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined.

If f is bounded and Riemann integrable over each closed bounded interval in $\mathbb{R}$ then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_a^b f(x)\,dx,$$

provided that this limit exists.

Remark Using techniques of complex analysis, it can be shown that

$$\lim_{b\to+\infty} \int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

However it can also be shown that

$$\int_0^b \frac{|\sin x|}{x}\,dx \to +\infty \text{ as } b \to +\infty.$$

Therefore, in the standard theory of the Riemann integral, the integral of the function $(\sin x)/x$ on the interval $[0, +\infty)$ is defined, and $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

There is an alternative theory of integration, due to Lebesgue, which is generally more powerful. All bounded Riemann-integrable functions on a closed bounded interval are Lebesgue-integrable on that interval. But a real-valued function f on a "measure space" is Lebesgue-integrable if and only if $|f|$ is Lebesgue-integrable on that measure space. Let $f\colon [0, +\infty) \to \mathbb{R}$ be the real-valued function defined such that $f(0) = 1$ and $f(x) = (\sin x)/x$ for all positive real numbers x. Then the function $|f|$ is neither Riemann-integrable nor Lebesgue-integrable on $[0, +\infty)$. It follows that the function f itself is not Lebesgue-integrable on $[0, +\infty)$. But, as we have remarked, the theory of the Riemann integral assigns a value of $\frac{\pi}{2}$ to $\int_0^{+\infty} f(x)\, dx$.