Module MA2224: Lebesgue Integral
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Section 2: Measure

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2 Measure Spaces

2.1 Blocks

Definition We define an *n-dimensional block* to be a subset of $\mathbb{R}^n$ that is a Cartesian product of bounded intervals.

Let B be an n-dimensional block. Then there exist bounded intervals $I_1, I_2, \ldots, I_n$ such that $B = I_1 \times I_2 \times \cdots \times I_n$. Let a_i and b_i denote the endpoints of the interval I_i for $i = 1, 2, \ldots, n$, where $a_i \leq b_i$. Then the interval I_i must coincide with one of the intervals (a_i, b_i), $(a_i, b_i]$, $[a_i, b_i)$ and $[a_i, b_i]$ determined by its endpoints, where

$$(a_i, b_i) = \{x \in \mathbb{R} : a_i < x < b_i\}, \quad (a_i, b_i] = \{x \in \mathbb{R} : a_i < x \leq b_i\}$$

$$[a_i, b_i) = \{x \in \mathbb{R} : a_i \leq x < b_i\}, \quad [a_i, b_i] = \{x \in \mathbb{R} : a_i \leq x \leq b_i\}.$$

We say that the block B is *open* if $I_i = (a_i, b_i)$ for $i = 1, 2, \ldots, n$. Similarly we say that the block B is *closed* if $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, n$.

Definition Let B be an n-dimensional block that is the Cartesian product $I_1 \times I_2 \times \cdots \times I_n$ of bounded intervals $I_1, I_2, \ldots, I_n$, and let a_i and b_i denote the endpoints of the interval I_i, where $a_i \leq b_i$. The *content* $m(B)$ of the block B is then defined to be the product $\prod_{i=1}^{n} (b_i - a_i)$ of the lengths of the intervals $I_1, I_2, \ldots, I_n$.

Note that a one-dimensional block is a bounded interval in the real line, and the content of the block is the length of the interval. A two-dimensional block is a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes, and the content of the block is the area of the rectangle. The content of a three-dimensional block is the volume of that block.

Let B be an n-dimensional block, and let $B_1, B_2, \ldots, B_s$ be a finite collection of n-dimensional blocks. We shall show that if $B \subset \bigcup_{k=1}^{s} B_k$ then $m(B) \leq \sum_{k=1}^{s} m(B_k)$. We shall also show that if the interiors of the blocks $B_1, B_2, \ldots, B_s$ are disjoint and are contained in B then $m(B) \geq \sum_{k=1}^{s} m(B_k)$. These results are of course fairly intuitive, and may at first sight seem to be obvious.

Suppose that we are given a finite list $B_1, B_2, \ldots, B_s$ of n-dimensional blocks in $\mathbb{R}^n$. Then

$$B_k = I_{k,1} \times I_{k,2} \times \cdots \times I_{k,n}$$

for $k = 1, 2, \ldots, s$, where $I_{k,1}, I_{k,2}, \ldots, I_{k,n}$ are bounded intervals in $\mathbb{R}$. Let $P_1, P_2, \ldots, P_n$ be finite subsets of the set of real numbers, each with at least two elements, chosen so that the endpoints of the intervals $I_{k,i}$ both belong to P_i for $k = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, n$.

For each integer i between 1 and n let

$$P_i = \{u_{i,0}, u_{i,1}, \ldots, u_{i,m(i)}\},$$

where

$$u_{i,0} < u_{i,1} < \cdots < u_{i,m(i)},$$

and let J denote the set of n-tuples $(j_1, j_2, \ldots, j_n)$ of integers in which $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$. For each $(j_1, j_2, \ldots, j_n) \in J$, let

$$C_{j_1,j_2,\ldots,j_n} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_{i,j_i-1} \leq x_i \leq u_{i,j_i} \text{ for } i = 1, 2, \ldots, n\}.$$

Then

$$\begin{aligned} m(C_{j_1,j_2,\ldots,j_n}) &= \prod_{i=1}^{n} (u_{i,j_i} - u_{i,j_i-1}) \\ &= (u_{1,j_1} - u_{1,j_1-1})(u_{2,j_2} - u_{2,j_2-1}) \cdots (u_{n,j_n} - u_{n,j_n-1}) \end{aligned}$$

for all $(j_1, j_2, \ldots, j_n) \in C_{j_1,j_2,\ldots,j_n}$.

Now, the block B_k is a product of intervals of the form

$$I_{k,1} \times I_{k,2} \times \cdots \times I_{k,n}$$

in which each interval $I_{k,i}$ has endpoints belonging to the set P_i. It follows that the endpoints of the ith interval $I_{k,i}$ are $u_{c(k,i)}$ and $u_{d(k,i)}$, where $c(k,i)$ and $d(k,i)$ are integers satisfying the inequalities $1 \leq c(k,i) < d(k,i) \leq m(i)$ for $i = 1, 2, \ldots, n$. Then the content $m(B_k)$ of the block B_k satisfies

$$m(B_k) = \prod_{i=1}^{n} (u_{d(k,i)} - u_{c(k,i)}).$$

Moreover

$$u_{d(k,i)} - u_{c(k,i)} = \sum_{j_i=c(k,i)+1}^{d(k,i)} (u_{i,j_i} - u_{i,j_i-1}).$$

Applying the Distributive Law relating multiplication and addition in the real number system, we find that $m(B_k)$ is the sum of the quantities $\prod_{i=1}^{n}(u_{i,j_i} -$

$u_{i,j_i-1})$ taken over all n-tuples $(j_1, j_2, \ldots, j_n)$ of integers that satisfy $c(k,i) < j_i \leq d(k,i)$ for $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned} m(B_k) &= \sum_{j_1=c(k,1)+1}^{d(k,1)} \sum_{j_2=c(k,2)+1}^{d(k,2)} \cdots \sum_{j_n=c(k,n)+1}^{d(k,n)} m(C_{j_1,j_2,\ldots,j_n}) \\ &= \sum_{(j_1,j_2,\ldots,j_n)\in J} \sigma_{j_1,j_2,\ldots,j_n}(B_k) m(C_{j_1,j_2,\ldots,j_n}), \end{aligned}$$

where $\sigma_{j_1,j_2,\ldots,j_n}(B_k) = 1$ in cases in which $c(k,i) < j_i \leq d(k,i)$ for every integer i between 1 and n, and where $\sigma_{j_1,j_2,\ldots,j_n}(B_k) = 0$ in all other cases.

Let $(j_1, j_2, \ldots, j_n) \in J$. Then $j_1, j_2, \ldots, j_n$ are integers such that $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$. The definition of the quantities $\sigma_{j_1,j_2,\ldots,j_n}(B_k)$ then ensures that $\sigma_{j_1,j_2,\ldots,j_n}(B_k) = 1$ if and only if $\mathrm{int}(C_{j_1,j_2,\ldots,j_n}) \subset B_k$, where $\mathrm{int}(C_{j_1,j_2,\ldots,j_n})$ denotes the interior of the block $C_{j_1,j_2,\ldots,j_n}$, defined so that

$$\begin{aligned} &\mathrm{int}(C_{j_1,j_2,\ldots,j_n}) \\ &\qquad = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_{i,j_i-1} < x_i < u_{i,j_i} \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Now the collection of blocks $C_{j_1,j_2,\ldots,j_n}$ whose interiors are contained in at least one of the blocks $B_1, B_2, \ldots, B_s$ is a finite collection of blocks. The blocks occurring in this finite collection can therefore be enumerated as a finite list $D_1, D_2, \ldots, D_q$. We have therefore established the validity of the following proposition.

Proposition 2.1 *Let $B_1, B_2, \ldots, B_s$ be a finite list whose members are n-dimensional blocks in $\mathbb{R}^n$. Then there exists a finite list $D_1, D_2, \ldots, D_q$ of closed n-dimensional blocks in $\mathbb{R}^n$ such that the interiors of the blocks $D_1, D_2, \ldots, D_q$ are pairwise disjoint and such that, for $k = 1, 2, \ldots, s$, the closure $\overline{B}_k$ of each block B_k is the union of those blocks in the list $D_1, D_2, \ldots, D_q$ whose interiors are contained in B_k. Moreover the content $m(B_k)$ is equal to the sum of the contents $m(D_j)$ of those blocks D_j in the list $D_1, D_2, \ldots, D_q$ for which* $\mathrm{int}(D_j) \subset B_k$.

Proposition 2.2 *Let B be a block in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of blocks in $\mathbb{R}^n$. Suppose that $B \subset \bigcup_{k=1}^{s} B_k$. Then $m(B) \leq \sum_{k=1}^{s} m(B_k)$.*

Proof The result stated in Proposition 2.1, ensures that there is a finite list $D_1, D_2, \ldots, D_q$ of closed n-dimensional blocks in $\mathbb{R}^n$ such that the interiors of the blocks $D_1, D_2, \ldots, D_q$ are pairwise disjoint and such that the closure

$\overline{B}$ of the block B and also the closures $\overline{B}_k$ of the blocks B_k for $k = 1, 2, \ldots, s$ are expressible as unions of blocks in the list $D_1, D_2, \ldots, D_q$.

Let us define $\sigma_j(B)$ for $j = 1, 2, \ldots, s$ so that $\sigma_j(B) = 1$ whenever $\text{int}(D_j) \subset B$ and $\sigma_j(B) = 0$ in all other cases. Similarly, for each integer k between 1 and s, let us define $\sigma_j(B_k)$ so that $\sigma_j(B_k) = 1$ whenever $\text{int}(D_j) \subset B_k$ and $\sigma_j(B_k) = 0$ in all other cases. Then $\overline{B}$ is the union of those D_j for which $\sigma_j(B) = 1$, and therefore the content $m(B)$ of B satisfies the identity

$$m(B) = \sum_{j=1}^{q} \sigma_j(B) m(D_j)$$

(see Proposition 2.1). Similarly

$$m(B_k) = \sum_{j=1}^{q} \sigma_j(B_k) m(D_j)$$

for $k = 1, 2, \ldots, s$.

Now if $\sigma_j(B) = 1$ then $\sigma_j(B_k) = 1$ for at least one value of k between 1 and s, because $B \subset \bigcup_{k=1}^{s} B_k$. It follows that

$$\sigma_j(B) \leq \sum_{k=1}^{s} \sigma_j(B_k)$$

for $j = 1, 2, \ldots, q$, and therefore

$$m(B) = \sum_{j=1}^{q} \sigma_j(B) m(D_j) \leq \sum_{k=1}^{s} \sum_{j=1}^{q} \sigma_j(B_k) m(D_j) = \sum_{k=1}^{s} m(B_k),$$

as required. ∎

Proposition 2.3 *Let B be a block in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of blocks in $\mathbb{R}^n$. Suppose that the interiors of the blocks $B_1, B_2, \ldots, B_s$ are disjoint and are contained in B. Then $\sum_{k=1}^{s} m(B_k) \leq m(B)$.*

Proof The result stated in Proposition 2.1, ensures that there is a finite list $D_1, D_2, \ldots, D_q$ of closed n-dimensional blocks in $\mathbb{R}^n$ such that the interiors of the blocks $D_1, D_2, \ldots, D_q$ are pairwise disjoint and such that the closure $\overline{B}$ of the block B and also the closures $\overline{B}_k$ of the blocks B_k for $k = 1, 2, \ldots, s$ are expressible as unions of blocks in the list $D_1, D_2, \ldots, D_q$.

Let us define $\sigma_j(B)$ for $j = 1, 2, \ldots, s$ so that $\sigma_j(B) = 1$ whenever $\text{int}(D_j) \subset B$ and $\sigma_j(B) = 0$ in all other cases. Similarly, for each integer k between 1 and s, let us define $\sigma_j(B_k)$ so that $\sigma_j(B_k) = 1$ whenever $\text{int}(D_j) \subset B_k$ and $\sigma_j(B_k) = 0$ in all other cases. Then $\overline{B}$ is the union of those D_j for which $\sigma_j(B) = 1$, and therefore the content $m(B)$ of B satisfies the identity

$$m(B) = \sum_{j=1}^{q} \sigma_j(B) m(D_j)$$

(see Proposition 2.1). Similarly

$$m(B_k) = \sum_{j=1}^{q} \sigma_j(B_k) m(D_j)$$

for $k = 1, 2, \ldots, s$.

In this case, for each integer j between 1 and q, there is at most one block B_k in the list $B_1, B_2, \ldots, B_s$ for which $\text{int}(D_j) \subset B_k$, because the interiors of the blocks $B_1, B_2, \ldots, B_s$ are pairwise disjoint. It follows that

$$\sum_{k=1}^{s} \sigma_j(B_k) \leq 1$$

for $j = 1, 2, \ldots, q$. Moreover, given any integer k between 1 and s, the identity $\sigma_j(B) = 1$ is satisfied by those integers j between 1 and q for which $\sigma_j(B_k) = 1$. It follows that

$$\sum_{k=1}^{s} \sigma_j(B_k) \leq \sigma_j(B)$$

for $j = 1, 2, \ldots, q$, and therefore

$$\sum_{k=1}^{s} m(B_k) = \sum_{k=1}^{s} \sum_{j=1}^{q} \sigma_j(B_k) m(D_j) \leq \sum_{j=1}^{q} \sigma_j(B) m(D_j) = m(B),$$

as required. ▮

. The following corollary follows immediately from the inequalities proved above.

Corollary 2.4 *Let B be a block in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of blocks in $\mathbb{R}^n$. Suppose that the interiors of the blocks $B_1, B_2, \ldots, B_s$ are disjoint and $B = \bigcup_{k=1}^{s} B_k$. Then $m(B) = \sum_{k=1}^{s} m(B_k)$.*

Lemma 2.5 *Let B be an block in $\mathbb{R}^n$, and let ε be any positive real number. Then there exist a closed block F and and open block V such that $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$.*

Proof Suppose that $B = I_1 \times I_2 \times \cdots \times I_n$, where $I_1, I_2, \ldots, I_n$ are bounded intervals. Now

$$\lim_{h \to 0} \prod_{i=1}^{n} (m(I_i) + h) = \prod_{i=1}^{n} m(I_i) = m(B).$$

It follows that, given any positive real number ε, we can choose the positive real number δ small enough to ensure that

$$\prod_{i=1}^{n} (m(I_i) - \delta) > m(B) - \varepsilon, \quad \prod_{i=1}^{n} (m(I_i) + \delta) < m(B) + \varepsilon.$$

Let $F = J_1 \times J_2 \times \cdots \times J_n$ and $V = K_1 \times K_2 \times \cdots \times K_n$, where $J_1, J_2, \ldots, J_n$ are closed bounded intervals chosen such that $J_i \subset I_i$ and $m(J_i) > m(I_i) - \delta$ for $i = 1, 2, \ldots, n$, and $K_1, K_2, \ldots, K_n$ are open bounded intervals chosen such that $I_i \subset K_i$ and $m(K_i) < m(I_i) + \delta$ for $i = 1, 2, \ldots, n$. Then F is a closed block, V is an open block, $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$, as required. ∎

Any closed n-dimensional block F is a compact subset of $\mathbb{R}^n$. This means that, given any collection $\mathcal{V}$ of open sets in $\mathbb{R}^n$ that covers F (so that each point of F belongs to at least one of the open sets in the collection), there exists some finite collection $V_1, V_2, \ldots, V_s$ of open sets belonging to the collection $\mathcal{V}$ such that

$$F \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

We shall use this property of closed blocks in order to generalize Proposition 2.2 to countable infinite unions of blocks.

Proposition 2.6 *Let A be a block in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathcal{C}$ be a countable collection of blocks in $\mathbb{R}^n$. Suppose that $A \subset \bigcup_{B \in \mathcal{C}} B$. Then $m(A) \leq \sum_{B \in \mathcal{C}} m(B)$.*

Proof There is nothing to prove if $\sum_{B \in \mathcal{C}} m(B) = +\infty$. We may therefore restrict our attention to the case where $\sum_{B \in \mathcal{C}} m(B) < +\infty$. Moreover the result is an immediate consequence of Proposition 2.2 if the collection $\mathcal{C}$ is finite. It therefore only remains to prove the result in the case where the collection $\mathcal{C}$ is infinite, but countable. In that case there exists an infinite sequence $B_1, B_2, B_3, \ldots$ of blocks with the property that each block in the collection $\mathcal{C}$ occurs exactly once in the sequence. Let some positive real number ε be given. It follows from Lemma 2.5 that there exists a closed block F such that $F \subset A$ and $m(F) \geq m(A) - \varepsilon$. Also, for each $k \in \mathbb{N}$, there exists an open block V_k such that $B_k \subset V_k$ and $m(V_k) < m(B_k) + 2^{-k}\varepsilon$. Then $F \subset \bigcup_{k=1}^{+\infty} V_k$, and thus $\{V_1, V_2, V_3, \ldots\}$ is a collection of open sets in $\mathbb{R}^n$ which covers the closed bounded set F. It follows from the compactness of F that there exists a finite collection $k_1, k_2, \ldots, k_s$ of positive integers such that $F \subset V_{k_1} \cup V_{k_2} \cup \cdots \cup V_{k_s}$. It then follows from Proposition 2.2 that

$$m(F) \leq m(V_{k_1}) + m(V_{k_2}) + \cdots + m(V_{k_s}).$$

Now

$$\frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \cdots + \frac{1}{2^{k_s}} \leq \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1,$$

and therefore

$$\begin{aligned} m(F) &\leq m(V_{k_1}) + m(V_{k_2}) + \cdots + m(V_{k_s}) \\ &\leq m(B_{k_1}) + m(B_{k_2}) + \cdots + m(B_{k_s}) + \varepsilon \\ &\leq \sum_{k=1}^{+\infty} m(B_k) + \varepsilon. \end{aligned}$$

Also $m(A) < m(F) + \varepsilon$. It follows that

$$m(A) \leq \sum_{k=1}^{+\infty} m(B_k) + 2\varepsilon.$$

Moreover this inequality holds no matter how small the value of the positive real number ε. It follows that

$$m(A) \leq \sum_{k=1}^{+\infty} m(B_k),$$

as required. ∎

2.2 Lebesgue Outer Measure

We say that a collection $\mathcal{C}$ of n-dimensional blocks *covers* a subset E of $\mathbb{R}^n$ if $E \subset \bigcup_{B \in \mathcal{C}} B$, (where $\bigcup_{B \in \mathcal{C}} B$ denotes the union of all the blocks belonging to the collection $\mathcal{C}$). Given any subset E of $\mathbb{R}^n$, we shall denote by $\mathbf{CCB}_n(E)$ the set of all countable collections of n-dimensional blocks that cover the set E.

Definition Let E be a subset of $\mathbb{R}^n$. We define the *Lebesgue outer measure* $\mu^*(E)$ of E to be the infimum, or greatest lower bound, of the quantities $\sum_{B\in\mathcal{C}} m(B)$, where this infimum is taken over all countable collections $\mathcal{C}$ of n-dimensional blocks that cover the set E. Thus

$$\mu^*(E) = \inf \left\{ \sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(E) \right\}.$$

The Lebesgue outer measure $\mu^*(E)$ of a subset E of $\mathbb{R}^n$ is thus the greatest extended real number l with the property that $l \leq \sum_{B\in\mathcal{C}} m(B)$ for any countable collection $\mathcal{C}$ of n-dimensional blocks that covers the set E. In particular, $\mu^*(E) = +\infty$ if and only if $\sum_{B\in\mathcal{C}} m(B) = +\infty$ for every countable collection $\mathcal{C}$ of n-dimensional blocks that covers the set E.

Note that $\mu^*(E) \geq 0$ for all subsets E of $\mathbb{R}^n$.

Lemma 2.7 *Let E be a block in $\mathbb{R}^n$. Then $\mu^*(E) = m(E)$, where $m(E)$ is the content of the block E.*

Proof It follows from Proposition 2.6 that $m(E) \leq \sum_{B\in\mathcal{C}} m(B)$ for any countable collection of n-dimensional blocks that covers the block E. Therefore $m(E) \leq \mu^*(E)$. But the collection $\{E\}$ consisting of the single block E is itself a countable collection of blocks covering E, and therefore $\mu^*(E) \leq m(E)$. It follows that $\mu^*(E) = m(E)$, as required. ∎

Lemma 2.8 *Let E and F be subsets of $\mathbb{R}^n$. Suppose that $E \subset F$. Then $\mu^*(E) \leq \mu^*(F)$.*

Proof Any countable collection of n-dimensional blocks that covers the set F will also cover the set E, and therefore $\mathbf{CCB}_n(F) \subset \mathbf{CCB}_n(E)$. It follows that

$$\begin{aligned} \mu^*(F) &= \inf \left\{ \sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(F) \right\} \\ &\geq \inf \left\{ \sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(E) \right\} = \mu^*(E), \end{aligned}$$

as required. ∎

Proposition 2.9 *Let $\mathcal{E}$ be a countable collection of subsets of $\mathbb{R}^n$. Then*

$$\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{E\in\mathcal{E}} \mu^*(E).$$

Proof Let $K = \mathbb{N}$ in the case where the countable collection $\mathcal{E}$ is infinite, and let $K = \{1, 2, \ldots, m\}$ in the case where the collection $\mathcal{E}$ is finite and has m elements. Then there exists a bijective function $\varphi\colon K \to \mathcal{E}$. We define $E_k = \varphi(k)$ for all $k \in K$. Then $\mathcal{E} = \{E_k : k \in K\}$, and any subset of $\mathbb{R}^n$ belonging to the collection $\mathcal{E}$ is of the form E_k for exactly one element k of the indexing set K.

Let some positive real number ε be given. Then corresponding to each element k of K there exists a countable collection $\mathcal{C}_k$ of n-dimensional blocks covering the set E_k for which

$$\sum_{B\in\mathcal{C}_k} m(B) < \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Let $\mathcal{C} = \bigcup_{k\in K} \mathcal{C}_k$. Then $\mathcal{C}$ is a collection of n-dimensional blocks that covers the union $\bigcup_{E\in\mathcal{E}} E$ of all the sets in the collection $\mathcal{E}$. Moreover every block belonging to the collection $\mathcal{C}$ belongs to at least one of the collections $\mathcal{C}_k$, and therefore belongs to exactly one of the collections $\mathcal{D}_k$, where $\mathcal{D}_k = \mathcal{C}_k \setminus \bigcup_{j<k} \mathcal{C}_j$. It follows that

$$\begin{aligned}
\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) &\leq \sum_{B\in\mathcal{C}} m(B) = \sum_{k\in K} \sum_{B\in\mathcal{D}_k} m(B) \\
&\leq \sum_{k\in K} \sum_{B\in\mathcal{C}_k} m(B) \leq \sum_{k\in K} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k}\right) \\
&\leq \sum_{k\in K} \mu^*(E_k) + \varepsilon
\end{aligned}$$

Thus $\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum\limits_{k\in K} \mu^*(E_k) + \varepsilon$, no matter how small the value of ε. It follows that $\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum\limits_{k\in K} \mu^*(E_k)$, as required. ∎

Proposition 2.10 *Let B be a closed n-dimensional block in $\mathbb{R}^n$. Then*

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B)$$

for all subsets A of $\mathbb{R}^n$.

Proof First we deal with the case when $\mu^*(A) = +\infty$, and this case either $\mu^*(A \cap B) = +\infty$ or else $\mu^*(A \setminus B) = +\infty$ because otherwise the subadditivity of Lebesgue outer measure (Proposition 2.9) would ensure that $\mu^*(A)$, being non-negative and less than the sum of two finite quantities, would itself be a finite quantity. The stated result is thus valid in cases where $\mu^*(A) = +\infty$.

Now suppose that $\mu^*(A) < +\infty$. Let some positive real number ε be given. It then follows from the definition of Lebesgue outer measure that there exists a collection $(C_i : i \in I)$ of closed n-dimensional blocks indexed by a countable set I for which

$$\sum_{i \in I} m(C_i) < \mu^*(A) + \varepsilon.$$

Then, for each $i \in I$, Proposition 2.1 guarantees the existence of a finite list $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ of closed n-dimensional blocks satisfying the following conditions:

- the interiors of the blocks $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ are pairwise disjoint;
- C_i is the union of all the blocks $D_{i,k}$ for which $1 \leq k \leq q(i)$;
- $C_i \cap B$ is the union of those blocks $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap B$.

For each $i \in I$, let $L(i)$ denote the set of integers between 1 and $q(i)$ for which $D_{i,k} \not\subset C_i \cap B$. and let I_0 denote the subset of I consisting of those $i \in I$ for which $L(i)$ is non-empty. Then

$$C_i \setminus B \subset \bigcup\nolimits_{k \in L(i)} D_{i,k}$$

for all $i \in I_0$, and

$$A \setminus B \subset \bigcup\nolimits_{i \in I_0} (C_i \setminus B),$$

and therefore

$$A \setminus B \subset \bigcup\nolimits_{i \in I_0} \bigcup\nolimits_{k \in L(i)} D_{i,k}$$

It then follows from the definition of Lebesgue outer measure that

$$\mu^*(A \setminus B) \leq \sum_{i \in I_0} \sum_{k \in L(i)} m(D_{i,k}),$$

where $m(D_{i,k})$ denotes the content of the block $D_{i,k}$ for all $i \in I$ and for all integers k in the range $1 \leq k \leq q(i)$. But, for each $i \in I_0$, the content $m(C_i)$ of the block C_i is equal to the sum of the contents $m(D_{i,k})$ of the blocks $D_{i,k}$

for all integer values of k satisfying $1 \leq k \leq q(i)$ (see Corollary 2.4), whilst the content $m(C_i \cap B)$ of the block $C_i \cap B$ is equal to the sum of the contents $m(D_{i,k})$ of those blocks $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap B$. It follows that, for all $i \in I_0$,

$$\sum_{k \in L(i)} m(D_{i,k}) = m(C_i) - m(C_i \cap B).$$

Also $m(C_i) = m(C_i \cap B)$ for all $i \in I \setminus I_0$. It follows that

$$\begin{aligned} \mu^*(A \setminus B) &\leq \sum_{i \in I_0} \sum_{k \in L(i)} m(D_{i,k}) \\ &= \sum_{i \in I_0} (m(C_i) - m(C_i \cap B)) \\ &= \sum_{i \in I} (m(C_i) - m(C_i \cap B)). \end{aligned}$$

The definition of definition of Lebesgue outer measure also ensures that

$$\mu^*(A \cap B) \leq \sum_{i \in I} m(C_i \cap B).$$

Adding these two inequalities, we find that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) \leq \sum_{i \in I} \mu(C_i) < \mu^*(A) + \varepsilon.$$

We have now shown that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) < \mu^*(A) + \varepsilon$$

for all strictly positive numbers ε. It follows that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) \leq \mu^*(A).$$

The reverse inequality

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \setminus B),$$

is a consequence of Proposition 2.9. It follows that

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B),$$

as required. ∎

2.3 Outer Measures

Definition Let X be a set, and let $\mathcal{P}(X)$ be the collection of all subsets of X. An *outer measure* $\lambda\colon \mathcal{P}(X) \to [0, +\infty]$ on X is a function, mapping subsets of X to non-negative extended real numbers, which has the following properties:

(i) $\lambda(\emptyset) = 0$;

(ii) $\lambda(E) \leq \lambda(F)$ for all subsets E and F of X that satisfy $E \subset F$;

(iii) $\lambda\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum\limits_{E\in\mathcal{E}} \lambda(E)$ for any countable collection $\mathcal{E}$ of subsets of X.

Lebesgue outer measure is an outer measure on the set $\mathbb{R}^n$. (This follows directly from the definition of Lebesgue outer measure, and from Lemma 2.8 and Proposition 2.9.)

We shall prove that any outer measure on a set X determines a collection of subsets of X with particular properties. The subsets belonging to this collection are known as *measurable sets*. Any countable union or intersection of measurable sets is itself a measurable set. Also any difference of measurable sets is itself a measurable set. We shall also prove that if $\mathcal{C}$ is any countable collection of pairwise disjoint measurable sets then $\lambda\left(\bigcup_{E\in\mathcal{E}} E\right) = \sum\limits_{E\in\mathcal{E}} \lambda(E)$. These results are fundamental to the branch of mathematics known as *measure theory*. Moreover the existence of such collections of measurable sets underlies the powerful and very general theory of integration introduced into mathematics by Lebesgue.

Definition Let λ be an outer measure on a set X. A subset E of X is said to be λ-*measurable* if $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$ for all subsets A of X.

The above definition of measurable sets may seem at first somewhat strange and unmotivated. Nevertheless it serves to characterize a collection of subsets of X with convenient properties, as we shall see.

Proposition 2.11 *Let λ be an outer measure on a set X. Then the empty set $\emptyset$ and the whole set X are λ-measurable. Moreover the complement $X \setminus E$ of E, and the union $E \cup F$, intersection $E \cap F$ and difference $E \setminus F$ of E and F are λ-measurable for all λ-measurable subsets E and F of X.*

Proof It follows directly from the definition of λ-measurability that $\emptyset$ and X are λ-measurable.

For each subset E of X, let us denote the complement $X \setminus E$ of E in X by E^c. Then $A \setminus E = A \cap E^c$ for all subsets A and E of X, and thus a subset E of X is λ-measurable if and only if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$$

for all subsets A of X. Now $(E^c)^c = E$. It follows that if a subset E of X is λ-measurable, then so is E^c. Thus $X \setminus E$ is λ-measurable for all measurable subsets E of X.

Let E and F be λ-measurable subsets of X, and let A be an arbitrary subset of X. Then

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

Also

$$\lambda(A \cap E) = \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c)$$

and

$$\lambda(A \cap E^c) = \lambda(A \cap E^c \cap F) + \lambda(A \cap E^c \cap F^c).$$

It follows that

$$\begin{aligned} \lambda(A) &= \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) \\ &\quad + \lambda(A \cap E^c \cap F) + \lambda(A \cap E^c \cap F^c). \end{aligned}$$

Now, replacing A by $A \cap (E \cup F)$, we find that

$$\begin{aligned} \lambda(A \cap (E \cup F)) &= \lambda(A \cap (E \cup F) \cap E \cap F) \\ &\quad + \lambda(A \cap (E \cup F) \cap E \cap F^c) \\ &\quad + \lambda(A \cap (E \cup F) \cap E^c \cap F) \\ &\quad + \lambda(A \cap (E \cup F) \cap E^c \cap F^c). \end{aligned}$$

But

$$\begin{aligned} A \cap (E \cup F) \cap E \cap F &= A \cap E \cap F, \\ A \cap (E \cup F) \cap E \cap F^c &= A \cap E \cap F^c, \\ A \cap (E \cup F) \cap E^c \cap F &= A \cap E^c \cap F, \\ A \cap (E \cup F) \cap E^c \cap F^c &= \emptyset. \end{aligned}$$

It follows therefore that

$$\begin{aligned} \lambda(A \cap (E \cup F)) &= \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) \\ &\quad + \lambda(A \cap E^c \cap F). \end{aligned}$$

Also $A \cap (E \cup F)^c = A \cap E^c \cap E^c$. It follows that

$$\lambda(A) = \lambda(A \cap (E \cup F)) + \lambda(A \cap (E \cup F)^c),$$

for all subsets A of X, and thus the subset $E \cup F$ of X is λ-measurable.

Also if E and F are λ-measurable subsets of X then so are E^c and F^c, and therefore $E^c \cup F^c$ is a λ-measurable subset of X. But $E^c \cup F^c = (E \cap F)^c$. It follows that $E \cap F$ is λ-measurable for all λ-measurable subsets E and F of X. Moreover $E \setminus F = E \cap F^c$, and therefore $E \setminus F$ is λ-measurable for all λ-measurable subsets E and F of X. This completes the proof. ∎

It follows from the above proposition that any finite union or intersection of measurable sets is measurable.

We say that the sets in some collection are *pairwise disjoint* if the intersection of any two distinct sets belonging to this collection is the empty set.

Lemma 2.12 *Let λ be an outer measure on a set X, let A be a subset of X, and let $E_1, E_2, \ldots, E_m$ be pairwise disjoint λ-measurable sets. Then*

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \sum_{k=1}^{m} \lambda(A \cap E_k).$$

Proof There is nothing to prove if $m = 1$. Suppose that $m > 1$. It follows from the definition of measurable sets that

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \lambda\left(\left(A \cap \bigcup_{k=1}^{m} E_k\right) \setminus E_m\right) + \lambda\left(\left(A \cap \bigcup_{k=1}^{m} E_k\right) \cap E_m\right).$$

But $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \setminus E_m = A \cap \bigcup_{k=1}^{m-1} E_k$ and $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \cap E_m = A \cap E_m$, because the sets $E_1, E_2, \ldots, E_m$ are pairwise disjoint. Therefore

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \lambda\left(A \cap \bigcup_{k=1}^{m-1} E_k\right) + \lambda(A \cap E_m).$$

The required result therefore follows by induction on m. ∎

Proposition 2.13 *Let λ be an outer measure on a set X. Then the union of any countable collection of λ-measurable subsets of X is λ-measurable.*

Proof The union of any two λ-measurable sets is λ-measurable (Proposition 2.11). It follows from this that the union of any finite collection of λ-measurable sets is λ-measurable.

Now let $E_1, E_2, E_3, \ldots$ be an infinite sequence of pairwise disjoint λ-measurable subsets of X. We shall prove that the union of these sets is λ-measurable. Let A be a subset of X. Now $\bigcup_{k=1}^{m} E_k$ is a λ-measurable set for each positive integer m, because any finite union of λ-measurable sets is λ-measurable, and therefore

$$\lambda(A) = \lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{m} E_k\right)$$

for all positive integers m. Moreover it follows from Lemma 2.12 that

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \sum_{k=1}^{m} \lambda(A \cap E_k).$$

Also

$$A \setminus \bigcup_{k=1}^{+\infty} E_k \subset A \setminus \bigcup_{k=1}^{m} E_k,$$

and therefore

$$\lambda\left(A \setminus \bigcup_{k=1}^{m} E_k\right) \geq \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

It follows that

$$\lambda(A) \geq \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right),$$

and therefore

$$\begin{aligned} \lambda(A) &\geq \lim_{m \to +\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right) \\ &= \sum_{k=1}^{+\infty} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right). \end{aligned}$$

However it follows from the definition of outer measures that

$$\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \lambda\left(\bigcup_{k=1}^{+\infty} (A \cap E_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(A \cap E_k).$$

Therefore

$$\lambda(A) \geq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

But the set A is the union of the sets $A \cap \bigcup_{k=1}^{+\infty} E_k$ and $A \setminus \bigcup_{k=1}^{+\infty} E_k$, and therefore

$$\lambda(A) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

We conclude therefore that

$$\lambda(A) = \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right)$$

for all subsets A of X. We conclude from this that the union of any pairwise disjoint sequence of λ-measurable subsets of X. is itself λ-measurable.

Now let $E_1, E_2, E_3, \ldots$ be a countable sequence of (not necessarily pairwise disjoint) λ-measurable sets. Then $\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k$, where $F_1 = E_1$, and $F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$ for all integers k satisfying $k > 1$. Now we have proved that any finite union of λ-measurable sets is λ-measurable, and any difference of λ-measurable sets is λ-measurable. It follows that the sets $F_1, F_2, F_3, \ldots$ are all λ-measurable. These sets are also pairwise disjoint. We conclude that the union of the sets $F_1, F_2, F_3, \ldots$ is λ-measurable, and therefore the union of the sets $E_1, E_2, E_3, \ldots$ is λ-measurable.

We have now shown that the union of any finite collection of λ-measurable sets is λ-measurable, and the union of any infinite sequence of λ-measurable sets is λ-measurable. We conclude that the union of any countable collection of λ-measurable sets is λ-measurable, as required. ∎

Corollary 2.14 *Let λ be an outer measure on a set X. Then the intersection of any countable collection of λ-measurable subsets of X is λ-measurable.*

Proof Let $\mathcal{C}$ be a countable collection of λ-measurable subsets of X. Then $X \setminus \bigcap_{E \in \mathcal{C}} E = \bigcup_{E \in \mathcal{C}} (X \setminus E)$ (i.e., the complement of the intersection of the sets in the collection is the union of the complements of those sets.) Now $X \setminus E$ is λ-measurable for every $E \in \mathcal{C}$. Therefore the complement $X \setminus \bigcap_{E \in \mathcal{C}} E$ of $\bigcap_{E \in \mathcal{C}} E$ is a union of λ-measurable sets, and is thus itself λ-measurable. It follows that intersection $\bigcap_{E \in \mathcal{C}} E$ of the sets in the collection is λ-measurable, as required. ∎

Proposition 2.15 *Let λ be an outer measure on a set X, let A be a subset of X, and let $\mathcal{C}$ be a countable collection of pairwise disjoint λ-measurable sets. Then*

$$\lambda\left(A \cap \bigcup_{E\in\mathcal{C}} E\right) = \sum_{E\in\mathcal{C}} \lambda(A \cap E).$$

Proof It follows from Lemma 2.12 that the required identity holds for any finite collection of pairwise disjoint λ-measurable sets.

Now let $E_1, E_2, E_3, \ldots$ be an infinite sequence of pairwise disjoint λ-measurable subsets of X. Then

$$\sum_{k=1}^{m} \lambda(A \cap E_k) = \lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right)$$

for all positive integers m. It follows that

$$\sum_{k=1}^{+\infty} \lambda(A \cap E_k) = \lim_{m\to+\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right).$$

But the definition of outer measures ensures that

$$\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \lambda\left(\bigcup_{k=1}^{+\infty} (A \cap E_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(A \cap E_k)$$

We conclude therefore that $\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \sum_{k=1}^{+\infty} \lambda(A \cap E_k)$ for any infinite sequence $E_1, E_2, E_3, \ldots$ of pairwise disjoint λ-measurable subsets of X. Thus the required identity holds for any countable collection of pairwise disjoint λ-measurable subsets of X, as required. ∎

2.4 Measure Spaces

Definition Let X be a set. A collection $\mathcal{A}$ of subsets of X is said to a σ-*algebra* (or *sigma-algebra*) of subsets of X if it has the following properties:

(i) the empty set $\emptyset$ is a member of $\mathcal{A}$;

(ii) the complement $X \setminus E$ of any member E of $\mathcal{A}$ is itself a member of $\mathcal{A}$;

(iii) the union of any countable collection of members of $\mathcal{A}$ is itself a member of $\mathcal{A}$.

Lemma 2.16 *Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. Then the intersection of any countable collection of members of the σ-algebra $\mathcal{A}$ is itself a member of $\mathcal{A}$.*

Proof Let $\mathcal{C}$ be a countable collection of sets belonging to $\mathcal{A}$. Then $X \setminus E \in \mathcal{A}$ for all $E \in \mathcal{C}$, and therefore $\bigcup_{E\in\mathcal{C}} (X \setminus E) \in \mathcal{A}$. But $\bigcup_{E\in\mathcal{C}} (X \setminus E) = X \setminus \bigcap_{E\in\mathcal{C}} E$. It follows that the complement of the intersection $\bigcap_{E\in\mathcal{C}} E$ of the sets in the collection $\mathcal{C}$ is itself a member of $\mathcal{A}$, and therefore the intersection $\bigcap_{E\in\mathcal{C}} E$ of those sets is a member of the σ-algebra $\mathcal{A}$, as required. ∎

Let X be a set, and let $\mathcal{C}$ be a collection of subsets of X. The collection of all subsets of X is a σ-algebra. Also the intersection of any collection of σ-algebras of subsets of X is itself a σ-algebra. Let $\mathcal{A}$ be the intersection of all σ-algebras $\mathcal{B}$ of subsets of X that have the property that $\mathcal{C} \subset \mathcal{B}$. Then $\mathcal{A}$ is a σ-algebra, and $\mathcal{C} \subset \mathcal{A}$. Moreover if $\mathcal{B}$ is a σ-algebra of subsets of X, and if $\mathcal{C} \subset \mathcal{B}$ then $\mathcal{A} \subset \mathcal{B}$. The σ-algebra $\mathcal{A}$ may therefore be regarded as the smallest σ-algebra of subsets of X for which $\mathcal{C} \subset \mathcal{A}$. We shall refer to this σ-algebra $\mathcal{A}$ as the σ-algebra of subsets of X *generated* by $\mathcal{C}$. We see therefore that any collection of subsets of a set X generates a σ-algebra of subsets of X which is the smallest σ-algebra of subsets of X that contains the given collection of subsets.

Definition Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. A *measure* on $\mathcal{A}$ is a function $\mu\colon \mathcal{A} \to [0, +\infty]$, taking values in the set $[0, +\infty]$ of non-negative extended real numbers, which has the property that

$$\mu\left(\bigcup_{E\in\mathcal{C}} E\right) = \sum_{E\in\mathcal{C}} \mu(E)$$

for any countable collection $\mathcal{C}$ of pairwise disjoint sets belonging to the σ-algebra $\mathcal{A}$.

Definition A *measure space* $(X, \mathcal{A}, \mu)$ consists of a set X, a σ-algebra $\mathcal{A}$ of subsets of X, and a measure $\mu\colon \mathcal{A} \to [0, +\infty]$ defined on this σ-algebra $\mathcal{A}$. A subset E of a measure space $(X, \mathcal{A}, \mu)$ is said to be *measurable* (or *μ-measurable*) if it belongs to the σ-algebra $\mathcal{A}$.

Theorem 2.17 *Let λ be an outer measure on a set X. Then the collection $\mathcal{A}_\lambda$ of all λ-measurable subsets of X is a σ-algebra. The members of this σ-algebra are those subsets E of X with the property that $\lambda(A) = \lambda(A \cap$*

$E) + \lambda(A \setminus E)$ for any subset A of X. Moreover the restriction of the outer measure λ to the λ-measurable sets defines a measure μ on the σ-algebra $\mathcal{A}_\lambda$. Thus $(X, \mathcal{A}, \mu)$ is a measure space.

Proof Immediate from Propositions 2.11, 2.13 and 2.15. ∎

Definition A measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if, given any measurable subset E of X satisfying $\mu(E) = 0$, and given any subset F of E, the subset F is also measurable. The measure μ on $\mathcal{A}$ is then said to be *complete*.

Lemma 2.18 *Let λ be an outer measure on a set X, let $\mathcal{A}$ be the σ-algebra consisting of the λ-measurable subsets of X, and let μ be the measure on $\mathcal{A}$ obtained by restricting the outer measure λ to the members of $\mathcal{A}$. Then $(X, \mathcal{A}, \mu)$ is a complete measure space.*

Proof Let E be a measurable set in X satisfying $\mu(E) = 0$, let F be a subset of E, and let A be a subset of X. Then $A \cap F \subset A \cap E$ and $A \setminus E \subset A \setminus F \subset A$, and therefore $0 \leq \lambda(A \cap F) \leq \lambda(A \cap E)$ and $\lambda(A \setminus E) \leq \lambda(A \setminus F) \leq \lambda(A)$. Now it follows from the definition of measurable sets in X that $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$. Moreover $0 \leq \lambda(A \cap E) \leq \lambda(E) = \mu(E) = 0$. It follows that $\lambda(A \cap E) = 0$ and $\lambda(A \setminus E) = \lambda(A)$. The inequalities above then ensure that $\lambda(A \cap F) = 0$ and $\lambda(A \setminus F) = \lambda(A)$. But then $\lambda(A) = \lambda(A \cap F) + \lambda(A \setminus F)$, and thus F is λ-measurable, as required. ∎

2.5 Lebesgue Measure on Euclidean Spaces

We are now in a position to give the definition of *Lebesgue measure* on n-dimensional Euclidean space $\mathbb{R}^n$. We have already defined an outer measure μ^* on $\mathbb{R}^n$ known as *Lebesgue outer measure*. We defined a *block* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ that is a Cartesian product of n bounded intervals. The product of the lengths of those intervals is the *content* of the block. Then, given any subset E of $\mathbb{R}^n$, we defined the *Lebesgue outer measure* $\mu^*(E)$ of the set E to be the infimum of the quantities $\sum\limits_{B \in \mathcal{C}} m(B)$, where the infimum is taken over all countable collections of blocks in $\mathbb{R}^n$ that cover the set E, and where $m(B)$ denotes the content of a block B in such a collection. Thus

$$\sum_{B \in \mathcal{C}} m(B) \geq \mu^*(E)$$

for every countable collection $\mathcal{C}$ of blocks in $\mathbb{R}^n$ that covers E; and, moreover, given any positive real number ε, there exists a countable collection $\mathcal{C}$ of blocks in $\mathbb{R}^n$ covering E for which

$$\mu^*(E) \leq \sum_{B \in \mathcal{C}} m(B) \leq \mu^*(E) + \varepsilon.$$

These properties characterize the Lebesgue outer measure $\mu^*(E)$ of the set E.

We say that a subset E of $\mathbb{R}^n$ is *Lebesgue-measurable* if and only if it is μ^*-measurable, where μ^* denotes Lebesgue outer measure on $\mathbb{R}^n$. Thus a subset E of $\mathbb{R}^n$ is Lebesgue-measurable if and only if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ for all subsets A of $\mathbb{R}^n$. The collection $\mathcal{L}_n$ of all Lebesgue-measurable sets is a σ-algebra of subsets of $\mathbb{R}^n$, and therefore the difference of any two Lebesgue-measurable subsets of $\mathbb{R}^n$ is Lebesgue-measurable, and any countable union or intersection of Lebesgue-measurable sets is Lebesgue-measurable. The *Lebesgue measure* $\mu(E)$ of a Lebesgue-measurable subset E of $\mathbb{R}^n$ is defined to be the Lebesgue outer measure $\mu^*(E)$ of that set. Thus Lebesgue measure μ is the restriction of Lebesgue outer measure μ^* to the σ-algebra $\mathcal{L}_n$ of Lebesgue-measurable subsets of $\mathbb{R}^n$.

It follows from Lemma 2.18 that Lebesgue measure is a complete measure on $\mathbb{R}^n$.

Remark The Lebesgue measure $\mu(E)$ of a subset E of $\mathbb{R}^2$ may be regarded as the area of that set. It is not possible to assign an area to every subset of $\mathbb{R}^2$ in such a way that the areas assigned to such subsets have all the properties that one would expect from a well-defined notion of area. One might at first sight expect that Lebesgue outer measure would provide a natural definition of area, applicable to all subsets of the plane, that would have the properties that one would expect of a well-defined notion of area. One would expect in particular that the area of a disjoint union of two subsets of the plane would be the sum of the areas of those sets. However it is possible to construct examples of disjoint subsets E and F in the plane which interpenetrate one another to such an extent as to ensure that $\mu^*(E \cup F) < \mu^*(E) + \mu^*(F)$, where μ^* denotes Lebesgue outer measure on $\mathbb{R}^2$. The σ-algebra $\mathcal{L}_2$ consisting of the Lebesgue-measurable subsets of the plane $\mathbb{R}^2$ is in fact that largest collection of subsets of the plane for which the sets in the collection have a well-defined area; the Lebesgue measure of a Lebesgue-measurable subset of the plane can be regarded as the area of that set. Similarly the σ-algebra $\mathcal{L}_3$ of Lebesgue-measurable subsets of three-dimensional Euclidean space $\mathbb{R}^3$ is the largest collection of subsets of $\mathbb{R}^3$ for which the sets in the collection have a well-defined volume.

Proposition 2.19 *Every closed n-dimensional block in $\mathbb{R}^n$ is Lebesgue-measurable.*

Proof Proposition 2.10, ensures that closed blocks have the property that characterizes Lebesgue-measurable subsets of $\mathbb{R}^n$. ∎

Proposition 2.20 *Every open set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Proof Let $\mathcal{W}$ be the collection of all open blocks in $\mathbb{R}^n$ that are Cartesian products of intervals whose endpoints are rational numbers. Now the set $\mathcal{I}$ of all open intervals in $\mathbb{R}^n$ whose endpoints are rational numbers is a countable set, as the function that sends such an interval to its endpoints defines an injective function from $\mathcal{I}$ to the countable set $\mathbb{Q} \times \mathbb{Q}$. Moreover there is a bijection from the countable set $\mathcal{I}^n$ to $\mathcal{W}$ that sends each ordered n-tuple $(I_1, I_2, \ldots, I_n)$ of open intervals to the open block $I_1 \times I_2 \times \cdots \times I_n$. It follows that the collection $\mathcal{W}$ is countable.

Let V be an open set in $\mathbb{R}^n$, and let $\mathbf{v}$ be a point of V. Then there exists some positive real number δ such that $B(\mathbf{v}, \delta) \subset V$, where $B(\mathbf{v}, \delta) \subset V$ denotes the open ball of radius δ centred on $\mathbf{v}$. Moreover there exist open blocks W belonging to $\mathcal{W}$ for which $\mathbf{v} \in W$ and $W \subset B(\mathbf{v}, \delta)$. It follows that the open set V is the union of the countable collection

$$\{W \in \mathcal{W} : W \subset V\}$$

of open blocks. Now each open block is a Lebesgue-measurable set, and any countable union of Lebesgue-measurable sets is itself a Lebesgue-measurable set. Therefore the open set V is a Lebesgue-measurable set, as required. ∎

Corollary 2.21 *Every closed set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Proof This follows immediately from Proposition 2.20, since the complement of any Lebesgue-measurable set is itself Lebesgue measurable set. ∎

Definition A subset of $\mathbb{R}^n$ is said to be a *Borel set* if it belongs to the σ-algebra generated by the collection of open sets in $\mathbb{R}^n$.

All open sets and closed sets in $\mathbb{R}^n$ are Borel sets. The collection of all Borel sets is a σ-algebra in $\mathbb{R}^n$ and is the smallest such σ-algebra containing all open subsets of $\mathbb{R}^n$.

Definition A measure defined on a σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}^n$ is said to be a *Borel measure* if the σ-algebra $\mathcal{A}$ contains all the open sets in $\mathbb{R}^n$.

Corollary 2.22 *Lebesgue measure on $\mathbb{R}^n$ is a Borel measure, and thus every Borel set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Remark The definitions of Borel sets and Borel measures generalize in the obvious fashion to arbitrary topological spaces. The collection of Borel sets in a topological space X is the σ-algebra generated by the open subsets of X. A measure defined on a σ-ring of subsets of X is said to be a Borel measure if every Borel set is measurable.

2.6 Basic Properties of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the measure μ is defined on the σ-algebra $\mathcal{A}$ of measurable subsets of X, and takes values in the set $[0, +\infty]$, where $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$. Thus $\mu(E)$ is defined for each measurable subset E of X, and is either a non-negative real number, or else has the value $+\infty$. The measure μ is by definition *countably additive*, so that

$$\mu\left(\bigcup_{E \in \mathcal{C}} E\right) = \sum_{E \in \mathcal{C}} \mu(E)$$

for every countable collection $\mathcal{C}$ of pairwise disjoint measurable subsets of X. In particular μ is *finitely additive*, so that if $E_1, E_2, \ldots, E_r$ are measurable subsets of X that are pairwise disjoint, then

$$\mu(E_1 \cup E_2 \cup \cdots \cup E_r) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_r).$$

Also

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \mu(E_j)$$

for any infinite sequence $E_1, E_2, E_3, \ldots$ of pairwise disjoint measurable subsets of X.

Let E and F be measurable subsets of X. Then $E = (E \cap F) \cup (E \setminus F)$, and the sets $E \cap F$ and $E \setminus F$ are measurable and disjoint. It therefore follows from the finite additivity of the measure μ that $\mu(E) = \mu(E \cap F) + \mu(E \setminus F)$. Also $E \cup F$ is the disjoint union of E and $F \setminus E$. and therefore

$$\mu(E \cup F) = \mu(E) + \mu(F \setminus E) = \mu(E \cap F) + \mu(E \setminus F) + \mu(F \setminus E).$$

It follows that

$$\begin{aligned} &\mu(E \cup F) + \mu(E \cap F) \\ &\qquad = (\mu(E \cap F) + \mu(E \setminus F)) + (\mu(E \cap F) + \mu(F \setminus E)) \\ &\qquad = \mu(E) + \mu(F). \end{aligned}$$

Now let E and F be measurable subsets of X that satisfy $F \subset E$. Then $\mu(E) = \mu(F) + \mu(E \setminus F)$, and $\mu(E \setminus F) \geq 0$. It follows that $\mu(F) \leq \mu(E)$. Moreover $\mu(E \setminus F) = \mu(E) - \mu(F)$, provided that $\mu(E) < +\infty$.

Lemma 2.23 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X. Suppose that $E_j \subset E_{j+1}$ for all positive integers j. Then*

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $E = \bigcup_{j=1}^{+\infty} E_j$, let $F_1 = E_1$, and let $F_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k$ for all integers j satisfying $j > 1$. Then the sets $F_1, F_2, F_3, \ldots$ are pairwise disjoint, the set E_j is the disjoint union of the sets F_k for which $1 \leq k \leq j$, and the set E is the disjoint union of all of the sets F_k. It therefore follows from the countable (and finite) additivity of the measure μ that

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k), \quad \mu(E_j) = \sum_{k=1}^{j} \mu(F_k).$$

But then

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k) = \lim_{j \to +\infty} \sum_{k=1}^{j} \mu(F_k) = \lim_{j \to +\infty} \mu(E_j),$$

as required. ∎

Lemma 2.24 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X. Suppose that $E_{j+1} \subset E_j$ for all positive integers j, and that $\mu(E_1) < +\infty$. Then*

$$\mu\left(\bigcap_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $G_j = E_1 \setminus E_j$ for all positive integers j, let $E = \bigcap_{j=1}^{+\infty} E_j$, and let $G = \bigcup_{j=1}^{+\infty} G_j$. It then follows from Lemma 2.23 that $\mu(G) = \lim_{j \to +\infty} \mu(G_j)$. Now $E_j = E_1 \setminus G_j$ for all positive integers j, and $\mu(E_1) < \infty$. It follows that $\mu(E_j) = \mu(E_1) - \mu(G_j)$ for all positive integers j. Also $E = E_1 \setminus G$. Therefore

$$\mu(E) = \mu(E_1) - \mu(G) = \mu(E_1) - \lim_{j \to +\infty} \mu(G_j) = \lim_{j \to +\infty} \mu(E_j),$$

as required. ∎

2.7 The Existence of Non-Measurable Sets

Definition For each real number u, let $\tau_u \colon \mathbb{R} \to \mathbb{R}$ be the translation mapping the set $\mathbb{R}$ of real numbers onto itself defined so that $\tau_u(x) = x + u$ for all real numbers x. We say that an outer measure λ on $\mathbb{R}$ is *translation-invariant* if $\lambda(\tau_u(E)) = \lambda(E)$ for all subsets E of $\mathbb{R}$ and for all real numbers u.

Proposition 2.25 *Let λ be a translation-invariant outer measure on the set $\mathbb{R}$ of real numbers with the property that $[0,1)$ is λ-measurable and $\lambda([0,1)) = 1$. Then there exist subsets of $\mathbb{R}$ that are not λ-measurable.*

Proof Let $B = [0,1)$ and, for each real number u, let $\tau_u \colon \mathbb{R} \to \mathbb{R}$ and $\rho_u \colon B \to B$ be defined such that, for all $x \in B$, $\tau_u(x) = x + u$ and $\rho_u(x)$ is the unique element of B for which $x + u - \rho_u(x)$ is an integer.

Let $u \in B$. Then

$$\rho_u(x) = \begin{cases} x + u & \text{if } x < 1 - u; \\ x + u - 1 & \text{if } x \geq 1 - u. \end{cases}$$

Now the set B is λ-measurable. The translation-invariance of the outer measure λ then ensures that the set $\tau_{-u}(B)$ is λ-measurable. Indeed let A be a subset of $\mathbb{R}$. Then

$$\begin{aligned} \lambda(A) &= \lambda(\tau_u(A)) = \lambda(\tau_u(A) \cap B) + \lambda(\tau_u(A) \setminus B) \\ &= \lambda(\tau_{-u}(\tau_u(A) \cap B))) + \lambda(\tau_{-u}(\tau_u(A) \setminus B)) \\ &= \lambda(A \cap \tau_{-u}(B)) + \lambda(A \setminus \tau_{-u}(B)). \end{aligned}$$

Thus the set $\tau_{-u}(B)$ is λ-measurable, as claimed.

Next we show that $\lambda(\rho_u(E)) = \lambda(E)$ for all subsets E of B and for all $u \in B$. Now

$$B \cap \tau_{-u}(B) = \{x \in B : x < 1 - u\}$$

and

$$B \setminus \tau_{-u}(B) = \{x \in B : x \geq 1 - u\}.$$

Therefore $\rho_u(x) = \tau_u(x)$ for all $x \in B \cap \tau_{-u}(B)$ and $\rho_u(x) = \tau_{u-1}(x)$ for all $x \in B \setminus \tau_{-u}(B)$. It follows that

$$\begin{aligned} \lambda(\rho_u(E) \cap B) &= \lambda(\rho_u(E \cap \tau_{-u}(B))) = \lambda(\tau_u(E \cap \tau_{-u}(B))) \\ &= \lambda(E \cap \tau_{-u}(B)) \end{aligned}$$

and

$$\begin{aligned} \lambda(\rho_u(E) \setminus B) &= \lambda(\rho_u(E \setminus \tau_{-u}(B))) = \lambda(\tau_{u-1}(E \setminus \tau_{-u}(B))) \\ &= \lambda(E \setminus \tau_{-u}(B))). \end{aligned}$$

But

$$\lambda(\rho_u(E)) = \lambda(\rho_u(E) \cap B) + \lambda(\rho_u(E) \setminus B)$$

and

$$\lambda(E) = \lambda(E \cap \tau_{-u}(B)) + \lambda(E \setminus \tau_{-u}(B)),$$

because the sets B and $\tau_{-u}(B)$ are λ-measurable. It follows that $\lambda(\rho_u(E)) = \lambda(E)$ for all $u \in \mathbb{R}$.

Now let us define a relation $\sim$ on the interval B, where $B = [0, 1)$, where real numbers x and y belonging to B satisfy $x \sim y$ if and only if $x - y$ is a rational number. Clearly $x \sim x$ for all $x \in B$, and if $x, y \in B$ satisfy $x \sim y$ then they also satisfy $y \sim x$. And if $x, y, z \in B$ satisfy $x \sim y$ and $y \sim z$ then they also satisfy $x \sim z$. Thus the relation $\sim$ on B is reflexive, symmetric and transitive, and is therefore an equivalence relation. This equivalence relation then partitions the set B into equivalence classes: every real number in the set B belongs to a unique equivalence class; two real numbers in the set set B belong to the same equivalence class if and only if their difference is a rational number.

Now the Axiom of Choice in set theory guarantees the existence of a subset E of B that contains exactly one element from each equivalence class. Then, given any real number x in the set B, there exists exactly one element z of the set E for which $x - z$ is a rational number. If $x \geq z$ then $x = \rho_q(z)$ if and only if $q = x - z$. On the other hand if $x < z$ then $x = \rho_q(z)$ if and only if $q = x - z + 1$. It follows that, given any real number x in the set B, there exists a unique real number z belonging to E and a unique rational number q satisfying $0 \leq q < 1$ for which $x = \rho_q(z)$. We conclude from this that the set B is the union of the sets $\rho_q(E)$ as q ranges over the set T of all rational numbers q satisfying $0 \leq q < 1$. Moreover the sets $\rho_q(E)$ obtained as q ranges over the countable set T are pairwise disjoint.

But $\lambda(\rho_q(E)) = \lambda(E)$ for all $q \in T$. If it were the case that $\lambda(E) = 0$, it would then follow that $\lambda(B) = 0$, because λ is an outer measure. But $\lambda(B) = 1$. It then follows that the sum $\sum_{q \in T} \lambda(\rho_q(E))$ diverges, and therefore cannot equal $\lambda(B)$, though $B = \bigcup_{q \in T} \rho_q(E)$. If the set E were λ-measurable, then all the sets $\rho_q(E)$ would be λ-measurable, and the sum of the outer measures of these pairwise-disjoint sets would be equal to $\lambda(B)$. Because this is not the case, it follows that the set E cannot be λ-measurable. The result follows. ∎