Module MA2224: Lebesgue Integral
Hilary Term 2019
Section 1: Explorations and Examples in Real Analysis

D. R. Wilkins

Version of 26 March, 2019

Contents

1 Explorations and Examples in Real Analysis

1.1 Countable Sets

Definition A set X is said to be *countable* if there exists an injection $f\colon X \to \mathbb{N}$ mapping X into the set $\mathbb{N}$ of natural numbers.

Example The set $\mathbb{Z}$ of integers is countable. For there is a well-defined bijection $f\colon \mathbb{Z} \to \mathbb{N}$ defined such that $f(n) = 2n + 1$ when $n \geq 0$ and $f(n) = -2n$ when $n < 0$. This bijection maps the set of non-negative integers onto the set of odd natural numbers, and maps the set of negative integers onto the set of even natural numbers.

Lemma 1.1 *Any subset of a countable set is countable.*

Proof Let Y be a subset of a countable set X. Then there exists an injection $f\colon X \to \mathbb{N}$ from X to the set $\mathbb{N}$ of natural numbers. The restriction of this injection to the set Y gives an injection from Y to $\mathbb{N}$. ∎

Proposition 1.2 *A non-empty set X is countable if and only if there exists a surjective function $g\colon \mathbb{N} \to X$ mapping the set $\mathbb{N}$ of natural numbers onto X.*

Proof Suppose that X is a countable non-empty set. Then there exists an injection $f\colon X \to \mathbb{N}$ from X to $\mathbb{N}$. Let x_0 be some chosen element of the set X. Then there is a well-defined function $g\colon \mathbb{N} \to X$ defined such that $g(f(x)) = x$ for all $x \in X$, and $g(n) = x_0$ for natural numbers n that do not belong to the range $f(X)$ of the function f. (The definition of the function g relies on the fact that, given an element n of the range $f(X)$ of the injection f, there exists exactly one element x of the set X for which $f(x) = n$.) The function g is clearly a surjection, in view of the fact that $x = g(f(x))$ for all $x \in X$.

Conversely let X be a non-empty set, and let $g\colon \mathbb{N} \to X$ be a surjection from $\mathbb{N}$ to X. Given an element x, there exists at least one natural number n for which $g(n) = x$. It follows that there is a well-defined function $f\colon X \to \mathbb{N}$ such that, given any element x of X, $f(x)$ is the smallest natural number n for which $g(n) = x$. Then $g(f(x)) = x$ for all $x \in X$. It follows from this that if x_1 and x_2 are elements of X (not necessarily distinct), and if $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$. We conclude that distinct elements of the set X get mapped to distinct natural numbers. Thus the function $f\colon X \to \mathbb{N}$ is an injection, and therefore the set X is countable, as required. ∎

Corollary 1.3 *Let $h: X \to Y$ be a surjection. Suppose that the set X is countable. Then the set Y is countable.*

Proof There is nothing to prove if the set X is the empty set, since in that case the set Y must also be the empty set. Suppose therefore that the set X is non-empty and countable. It follows from Proposition 1.2 that there exists a surjection $g: \mathbb{N} \to X$ from $\mathbb{N}$ to X. The composition $h \circ g: \mathbb{N} \to Y$ of g and h is then a surjection from $\mathbb{N}$ to Y (since the composition of two surjections is always a surjection). It then follows from Proposition 1.2 that the set Y is countable, as required. ∎

1.2 Cartesian Products and Unions of Countable Sets

Lemma 1.4 *There exists a bijection between the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.*

Proof Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined such that

$$f(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2) + k.$$

One can check that this function f is a bijection.

Note that, for each natural number m greater than one, this function f maps the set D_m into the set I_m, where $D_m = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j + k = m\}$ and

$$I_m = \{n \in \mathbb{N} : \tfrac{1}{2}(m-1)(m-2) < n \leq \tfrac{1}{2}m(m-1)\}.$$

Now, given any natural number n, there exists a unique natural number m greater than one such that $\frac{1}{2}(m-1)(m-2) < n \leq \frac{1}{2}m(m-1)$. It follows that each natural number belongs to exactly one of the sets $I_2, I_3, I_4 \ldots$. Moreover if n is a natural number, and if $n \in I_m$, where m is a natural number greater than one, then $n = f(m-k,k)$ where $k = n - \frac{1}{2}(m-1)(m-2)$. Moreover $(m-k,k)$ is the unique element of D_n satisfying $f(n-k,k) = n$. These facts ensure that, given any natural number n, there exists exactly one pair (j,k) of natural numbers satisfying $f(j,k) = n$. (These natural numbers j and k satisfy $j + k = m$, where m is the unique natural number greater than one that satisfies the inequalities $\frac{1}{2}(m-1)(m-2) < n \leq \frac{1}{2}m(m-1)$.) Therefore the function f is both injective and surjective, and is thus a bijection, as required. ∎

Remark The function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ used in the proof of Lemma 1.4 is constructed so that

$$f(1,1) = 1,$$

$$f(2,1) = 2, \quad f(1,2) = 3,$$

$$f(3,1) = 4, \quad f(2,2) = 5, \quad f(1,3) = 6,$$

$$f(4,1) = 7, \quad f(3,2) = 8, \quad f(2,3) = 9, \quad f(1,4) = 10, \quad \text{etc.}$$

These examples giving the value of (j,k) for small values of j and k should convey the basic scheme used to construct this function f.

Proposition 1.5 *Let X and Y be countable sets. Then the Cartesian product $X \times Y$ of X and Y is a countable set.*

Proof There exist injective functions $g: X \to \mathbb{N}$ and $h: Y \to \mathbb{N}$, because the sets X and Y are countable. Also there exists a bijection $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ (Lemma 1.4). Let $p: X \times Y \to \mathbb{N}$ be the function defined such that $p(x,y) = f(g(x), h(y))$ for all $x \in X$ and $y \in Y$. We claim that the function p is an injection.

Let x_1 and x_2 be elements of X (not necessarily distinct), and let y_1 and y_2 be elements of Y. Suppose that $p(x_1, y_1) = p(x_2, y_2)$. Then $(g(x_1), h(y_1)) = (g(x_2), h(y_2))$, because the function $f: \mathbb{N} \to \mathbb{N}$ is an injection, and therefore $g(x_1) = g(x_2)$ and $h(y_1) = h(y_2)$. But the functions g and h are injections. It follows that $x_1 = x_2$ and $y_1 = y_2$, and thus $(x_1, y_1) = (x_2, y_2)$. We have therefore shown that if the elements (x_1, y_1) and (x_2, y_2) of $X \times Y$ are such that $p(x_1, y_1) = p(x_2, y_2)$ then $(x_1, y_1) = (x_2, y_2)$. This shows that the function $p: X \times Y \to \mathbb{N}$ is an injection. The existence of such an injection guarantees that the set $X \times Y$ is countable, as required.

Corollary 1.6 *Let $X_1, X_2, \ldots, X_n$ be countable sets. Then the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of these sets is a countable set.*

Proof The result follows by induction on the number of sets forming the Cartesian product, because the set $X_1 \times X_2 \times \cdots \times X_n$ may be regarded as the Cartesian product of the sets $X_1 \times X_2 \times \cdots \times X_{n-1}$ and X_n whenever $n > 1$, and the Cartesian product of any two countable sets is countable (Proposition 1.5). ∎

Lemma 1.7 *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof The set $\mathbb{Z}$ of integers and the set $\mathbb{N}$ of natural numbers are countable sets, and therefore the Cartesian product $\mathbb{Z} \times \mathbb{N}$ is a countable set (Proposition 1.5). There is an obvious surjection $g: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, where $g(z,n) = z/n$ for all integers z and natural numbers n. The result therefore follows immediately on applying Corollary 1.3. ∎

Proposition 1.8 *Let $X_1, X_2, X_3, \ldots$ be a sequence of countable sets Then the union $\bigcup_{n=1}^{\infty} X_n$ of these countable sets is itself a countable set.*

Proof For each natural number n let $g_n: X_n \to \mathbb{N}$ be an injective function from X_n to the set $\mathbb{N}$ of natural numbers. (Such injective functions exist because each set X_n is countable.) We shall construct an injective function $h: X \to \mathbb{N} \times \mathbb{N}$ from X to $\mathbb{N}$, where $X = \bigcup_{n=1}^{\infty} X_n$.

Given any element x of X, let $h(x) = (n(x), g_{n(x)}(x))$, where $n(x)$ is the smallest natural number with the property that $x \in X_{n(x)}$. (Note that x belongs to at least one of the sets X_n, and therefore this natural number $n(x)$ is well-defined.)

Let x and y be elements of X satisfying $h(x) = h(y)$. We claim that $x = y$. Now if $h(x) = h(y)$ then $n(x) = n(y)$. It follows that $x \in X_n$ and $y \in X_n$, where $n = n(x) = n(y)$. Moreover $g_n(x) = g_n(y)$. But $g: X_n \to \mathbb{N}$ is an injective function. It follows that $x = y$. We conclude therefore that the function $h: X \to \mathbb{N} \times \mathbb{N}$ is injective.

Now Lemma 1.4 ensures that there exists a bijective function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. The composition function $f \circ h: X \to \mathbb{N}$ is then an injective function from X to $\mathbb{N}$. We conclude therefore that the set X is countable, as required. ∎

Corollary 1.9 *Let $(X_i : i \in I)$ be a collection of countable sets, indexed by a countable set I. Then the union $\bigcup_{i \in I} X_i$ of the sets in this countable collection is a countable set.*

Proof The indexing set I is a countable set. Therefore there exists an injective function $g: I \to \mathbb{N}$. It follows that, for each natural number n, there exists at most one element i of the indexing set such that $g(i) = n$. If there exists some element i of I such that $g(i) = n$, let $Y_n = X_i$; otherwise let $Y_n = \emptyset$. Then $Y_1, Y_2, Y_3, \ldots$ is an infinite sequence of countable sets, and clearly $\bigcup_{i \in I} X_i = \bigcup_{n=1}^{\infty} Y_n$. It follows immediately from Proposition 1.8 $\bigcup_{i \in I} X_i$ is a countable set, as required. ∎

We define a *countable union* of sets to be a union of sets where the sets making up the collection can be indexed by some countable sets. Thus the union of a finite number of sets is a countable union of sets. Also the union of an infinite sequence $X_1, X_2, X_3, \ldots$ of sets is a countable union. The result of Corollary 1.9 may be summed up in the statement that *any countable union of countable sets is itself a countable sets.*

1.3 Uncountable Sets

A set that is not countable is said to be *uncountable.* Many sets occurring in mathematics are uncountable. These include the set of real numbers.

It follows directly from Lemma 1.1 that if a set X has an uncountable subset, then X must itself be uncountable.

It also follows directly from Corollary 1.3 that if $h: X \to Y$ is a surjection from a set X to a set Y, and if the set Y is uncountable, then the set X is uncountable.

1.4 Least Upper Bounds and Greatest Lower Bounds

Definition Let S be a set of real numbers which is bounded above. The *least upper bound*, or *supremum*, of the set S is the smallest real number that is greater than or equal to elements of the set S, and is denoted by $\sup S$.

Thus if S is a set of real numbers that is bounded above, then the least upper bound $\sup S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \leq \sup S$;
- if u is a real number, and if, for all $x \in S$, $x \leq u$ then $\sup S \leq u$.

The *Least Upper Bound Property* of the real number system guarantees that, given any non-empty set S of real numbers that is bounded above, there exists a least upper bound $\sup S$ for the set S.

Definition Let S be a set of real numbers which is bounded below. The *greatest lower bound*, or *infimum*, of the set S is the largest real number that is less than or equal to elements of the set S, and is denoted by $\inf S$.

Thus if S is a set of real numbers that is bounded below, then the greatest lower bound $\inf S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \geq \inf S$;
- if l is a real number, and if, for all $x \in S$, $x \geq l$ then $\inf S \geq l$.

Given any non-empty set S of real numbers that is bounded below, there exists a greatest lower bound $\inf S$ for the set S.

1.5 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.10 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.6 The Uncountability of the Real Numbers

Theorem 1.11 *The set $\mathbb{R}$ of real numbers is uncountable.*

Proof Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. We prove the existence of a real number that does not occur as a member of this sequence.

Let a_0 and b_0 be real numbers satisfying $a_0 < b_0$. We construct infinite sequences of real numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ in accordance with the prescription that follows.

Suppose that real numbers a_{j-1} and b_{j-1} have been determined for some positive integer j, where $a_{j-1} < b_{j-1}$. Let the closed interval $[a_{j-1}, b_{j-1}]$ be divided into three subintervals of equal length, with division points between the subintervals at g_{j-1} and h_{j-1}, where

$$g_{j-1} = \tfrac{2}{3}a_{j-1} + \tfrac{1}{3}b_{j-1} \quad \text{and} \quad h_{j-1} = \tfrac{1}{3}a_{j-1} + \tfrac{2}{3}b_{j-1}.$$

Note that $a_{j-1} \leq g_{j-1} \leq h_{j-1} \leq b_{j-1}$, and

$$g_{j-1} - a_{j-1} = h_{j-1} - g_{j-1} = b_{j-1} - h_{j-1} = \tfrac{1}{3}(b_{j-1} - a_{j-1}).$$

Let $a_j = a_{j-1}$ and $b_j = g_{j-1}$, provided that $x_j \notin [a_{j-1}, g_{j-1}]$. Otherwise let $a_j = g_{j-1}$ and $b_j = h_{j-1}$, provided that $x_j \notin [g_{j-1}, h_{j-1}]$. Otherwise let

$a_j = h_{j-1}$ and $b_j = b_{j-1}$. (Note that x_j cannot belong to all three of the closed intervals $[a_{j-1}, g_{j-1}]$, $[g_{j-1}, h_{j-1}]$ and $[h_{j-1}, b_{j-1}]$.) This ensures that a_j and b_j have been determined so as to ensure that $a_{j-1} \leq a_j \leq b_j \leq b_{j-1}$, $b_j - a_j = \frac{1}{3}(b_{j-1} - a_{j-1})$ and $x_j \notin [a_j, b_j]$.

The infinite sequence $a_1, a_2, a_3, \ldots$ is a non-decreasing sequence of real numbers that is bounded above by b_1. This infinite sequence therefore converges to some real number s, and indeed

$$s = \sup\{a_j : j \in \mathbb{N}\}.$$

Moreover

$$\lim_{j \to +\infty} b_j = \lim_{j \to +\infty} (b_j - a_j) + \lim_{j \to +\infty} a_j = 0 + s = s.$$

Also $a_j \leq s \leq b_j$ for all positive integers j, and thus $s \in [a_j, b_j]$ for all positive integers j. But $x_j \notin [a_j, b_j]$ for each positive integer j. It follows therefore that $s \neq x_j$ for all positive integers j, and thus s is a real number that is not a member of the infinite sequence $x_1, x_2, x_3, \ldots$.

If there were to exist a surjective function $g\colon \mathbb{N} \to \mathbb{R}$ then, on setting $x_j = g(j)$ for all positive integers j, we would obtain an infinite sequence $x_1, x_2, x_3, \ldots$ whose members would include every real number. This is not possible. Therefore there cannot exist any surjective function $g\colon \mathbb{N} \to \mathbb{R}$ from $\mathbb{N}$ to $\mathbb{R}$, and thus the set $\mathbb{R}$ of real numbers is uncountable (see Proposition 1.2). ∎

1.7 Upper and Lower Limits

Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and, for each positive integer j, let

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

The sets $S_1, S_2, S_3, \ldots$ are all bounded. It follows that there exist well-defined infinite sequences $u_1, u_2, u_3, \ldots$ and $l_1, l_2, l_3, \ldots$ of real numbers, where $u_j = \sup S_j$ and $l_j = \inf S_j$ for all positive integers j. Now S_{j+1} is a subset of S_j for each positive integer j, and therefore $u_{j+1} \leq u_j$ and $l_{j+1} \geq l_j$ for each positive integer j. It follows that the bounded infinite sequence $(u_j : j \in \mathbb{N})$ is a non-increasing sequence, and is therefore convergent (Theorem 1.10). Similarly the bounded infinite sequence $(l_j : j \in \mathbb{N})$ is a non-decreasing sequence, and is therefore convergent. We define

$$\limsup_{j \to +\infty} a_j = \lim_{j \to +\infty} u_j = \lim_{j \to +\infty} \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$$

and

$$\liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j = \lim_{j\to+\infty} \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}.$$

The quantity $\limsup_{j\to+\infty} a_j$ is referred to as the *upper limit* of the sequence $a_1, a_2, a_3, \ldots$. The quantity $\liminf_{j\to+\infty} a_j$ is referred to as the *lower limit* of the sequence $a_1, a_2, a_3, \ldots$.

Note that every bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers has a well-defined upper limit $\limsup_{j\to+\infty} a_j$ and a well-defined lower limit $\liminf_{j\to+\infty} a_j$.

Proposition 1.12 *A bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers is convergent if and only if $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j$, in which case the limit of the sequence is equal to the common value of its upper and lower limits.*

Proof For each positive integer j, let $u_j = \sup S_j$ and $l_j = \inf S_j$, where

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

Then $\liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j$ and $\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j$.

Suppose that $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j = c$ for some real number c. Then, given any positive real number ε, there exist natural numbers N_1 and N_2 such that $c - \varepsilon < l_j \leq c$ whenever $j \geq N_1$, and $c \leq u_j < c + \varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $a_j \in S_N$, and therefore

$$c - \varepsilon < l_N \leq a_j \leq u_N < c + \varepsilon.$$

Thus $|a_j - c| < \varepsilon$ whenever $j \geq N$. This proves that the infinite sequence $a_1, a_2, a_3, \ldots$ converges to the limit c.

Conversely let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers that converges to some value c. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $c - \frac{1}{2}\varepsilon < a_j < c + \frac{1}{2}\varepsilon$ whenever $j \geq N$. It follows that $S_j \subset (c - \frac{1}{2}\varepsilon, c + \frac{1}{2}\varepsilon)$ whenever $j \geq N$. But then

$$c - \tfrac{1}{2}\varepsilon \leq l_j \leq u_j \leq c + \tfrac{1}{2}\varepsilon$$

whenever $j \geq N$, where $u_j = \sup S_j$ and $l_j = \inf S_j$. We see from this that, given any positive real number ε, there exists some natural number N such that $|l_j - c| < \varepsilon$ and $|u_j - c| < \varepsilon$ whenever $j \geq N$. It follows from this that

$$\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j = c \text{ and } \liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j = c,$$

as required. ∎

1.8 Rearrangement of Infinite Series

Example Consider the infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

For each positive integer k, let p_k denote the kth partial sum of this infinite series, defined so that

$$p_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{j}.$$

The absolute values of the summands constitute a decreasing sequence, and accordingly examination of the form of the infinite series establishes that

$$p_1 > p_3 > p_5 > p_7 > \cdots$$

$$p_2 < p_4 < p_6 < p_8 < \cdots$$

Moreover $p_{2m} \leq p_{2m+1} \leq p_1$ and $p_{2m+1} \geq p_{2m} \geq p_2$ for all positive integers m. Thus p_1, p_3, p_5, p_7 is a decreasing sequence that is bounded below, and p_2, p_4, p_6, p_8 is an increasing sequence that is bounded above. A standard result of real analysis ensures that these bounded monotonic sequences are convergent. Moreover

$$\begin{aligned} \lim_{m \to +\infty} p_{2m+1} &= \lim_{m \to \infty} \left(p_{2m} + \frac{1}{2m+1} \right) \\ &= \lim_{m \to \infty} p_{2m} + \lim_{m \to +\infty} \frac{1}{2m+1} \\ &= \lim_{m \to \infty} p_{2m}. \end{aligned}$$

It then follows easily from examination of the definition of convergence that the infinite sequence $p_1, p_2, p_3, \ldots$ converges, and

$$\lim_{j \to +\infty} p_j = \lim_{m \to +\infty} p_{2m} = \lim_{m \to +\infty} p_{2m+1}.$$

Let $\alpha = \lim_{j \to +\infty} p_j$. Then $p_2 < \alpha < p_1$, and thus $\frac{1}{2} < \alpha < 1$.

Now consider the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots.$$

The individual summands are those of the infinite series previously considered, but they occur in a different order. This new infinite series is thus a *rearrangement* of the infinite series previously considered. Nevertheless the sum of this new infinite series may be represented as

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

and therefore the sum of the new infinite series is equal to that of the infinite series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots,$$

and is therefore equal to $\frac{1}{2}\alpha$. This example demonstrates that when the terms of an infinite series are rearranged, so that they are summed together in a different order, the sum of the rearranged series is not necessarily equal to that of the original series.

The example just discussed considers the behaviour of a particular infinite series that is convergent but not *absolutely convergent*. An infinite series $\sum\limits_{j=1}^{+\infty} a_j$ is said to be *absolutely convergent* if $\sum\limits_{j=1}^{+\infty} |a_j|$ is convergent. The following proposition and its corollaries ensure that any absolutely convergent infinite series may be rearranged at will without affecting convergence, and without changing the value of the sum of the series. In particular an infinite series whose summands are non-negative may be rearranged without affecting the value of the sum of that infinite series.

Proposition 1.13 *Let $\sum\limits_{j=1}^{+\infty} a_j$ be a convergent infinite series, where a_j is real and $a_j \geq 0$ for all positive integers j. Let Q be the subset of the real numbers consisting of the values of all sums of the form $\sum\limits_{j\in F} a_j$ obtained as F ranges over all the non-empty finite subsets of $\mathbb{N}$. Then*

$$\sum_{j=1}^{+\infty} a_j = \sup Q.$$

Proof For each positive integer k, let

$$p_k = \sum_{j=1}^{k} a_j.$$

This number p_k is referred to as the kth *partial sum* of the infinite series $a_1 + a_2 + a_3 + \cdots$. The definition of the sum of this infinite series then ensures that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k\to+\infty} p_k.$$

Moreover $p_1 \leq p_2 \leq p_3 \leq \cdots$, because $a_j \geq 0$ for all positive integers j, and therefore

$$\sum_{j=1}^{+\infty} a_j = \sup\{p_k : k \in \mathbb{N}\}.$$

For each non-empty finite subset F of the set $\mathbb{N}$ of positive integers, let

$$q_F = \sum_{j \in F} a_j.$$

If F and H are finite subsets of $\mathbb{N}$, and if $F \subset H$ then $q_F \leq q_H$, because the summand a_j is non-negative for all positive integers j.

Now, given any non-empty finite subset F of $\mathbb{N}$ there exists some positive integer k such that $F \subset J_k$, where $J_k = \{1, 2, \ldots, k\}$. But then

$$q_F \leq q_{J_k} = p_k \leq \sum_{j=1}^{+\infty} a_j.$$

Therefore the set Q consisting of the values of the sums q_F as F ranges over all the non-empty finite subsets F of $\mathbb{N}$ is bounded above. Moreover it is non-empty. The Least Upper Bound Principle then ensures that the set Q has a well-defined least upper bound $\sup Q$.

Let $s = \sup Q$. We have shown that $q_F \leq \sum\limits_{j=1}^{+\infty} a_j$ for each non-empty finite subset F of $\mathbb{N}$. It follows that $s \leq \sum\limits_{j=1}^{+\infty} a_j$. But $p_k \in Q$ for all positive integers k, because $p_k = q_{J_k}$, and therefore $p_k \leq s$. Taking limits as $k \to +\infty$, we find that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k \to +\infty} p_k \leq s.$$

The inequalities just obtained together ensure that

$$\sum_{j=1}^{+\infty} a_j = s = \sup Q,$$

as required. ∎

A *permutation* of the set $\mathbb{N}$ of positive integers is a function $\sigma\colon \mathbb{N} \to \mathbb{N}$ from the set $\mathbb{N}$ to itself that is bijective. A function $\sigma\colon \mathbb{N} \to \mathbb{N}$ is thus a permutation if and only if it has a well-defined inverse $\sigma^{-1}\colon \mathbb{N} \to \mathbb{N}$. This is the case if and only if, given any positive integer k, there exists a unique positive integer j for which $k = \sigma(j)$.

Definition An infinite sequence $b_1, b_2, b_3, \ldots$ of real numbers is said to be a *rearrangement* of an infinite sequence $a_1, a_2, a_3, \ldots$ if there exists a permutation σ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. In this situation we also say that the infinite series $\sum_{k=1}^{+\infty} b_k$ is a rearrangement of the infinite series $\sum_{j=1}^{+\infty} a_j$.

Corollary 1.14 *Let $\sum_{j=1}^{+\infty} a_j$ be a convergent infinite series, and let $\sum_{k=1}^{+\infty} b_k$ be a rearrangement of infinite series $\sum_{j=1}^{+\infty} a_j$. Suppose that $a_j \geq 0$ for all positive integers j. Then the infinite series $\sum_{k=1}^{+\infty} b_k$ is convergent, and $\sum_{k=1}^{+\infty} b_k = \sum_{j=1}^{+\infty} a_j$.*

Proof There exists a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $q_F = \sum_{j \in F} a_j$ for all non-empty finite subsets F of $\mathbb{N}$, and let $r_G = \sum_{k \in G} b_k$ for all non-empty finite subsets G of $\mathbb{N}$. Then

$$q_{\sigma(G)} = \sum_{j \in \sigma(G)} a_j = \sum_{k \in G} a_{\sigma(k)} = \sum_{k \in G} b_k = r_G$$

for all non-empty finite subsets G of $\mathbb{N}$, and accordingly $q_F = r_{\sigma^{-1}(F)}$ for all non-empty finite subsets F of $\mathbb{N}$. Moroever G is a non-empty finite subset of $\mathbb{N}$ if and only if $\sigma(G)$ is a non-empty finite subset of $\mathbb{N}$. It follows that the set Q consisting of all sums of the form q_F as F ranges over the non-empty finite subsets of $\mathbb{N}$ is also the set consisting of all sums of the form r_G as G ranges over the non-empty finite subsets of $\mathbb{N}$. It follows from Proposition 1.13 that

$$\sum_{j=1}^{+\infty} a_j = \sup Q = \sum_{k=1}^{+\infty} b_k,$$

as required. ∎

It follows from Corollary 1.14 that, given any collection $(c_\alpha : \alpha \in A)$ of *non-negative* real numbers c_α indexed by the members of a countable set A, we can form the sum $\sum_{\alpha \in A} c_\alpha$. If the countable indexing set A is infinite then there exists an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ in which each element of the set A occurs exactly once. Then

$$\sum_{\alpha \in A} c_\alpha = \sum_{j=1}^{+\infty} c_{\alpha_j}.$$

The requirement that $c_\alpha \geq 0$ for all $\alpha \in A$ ensures that the value of $\sum\limits_{j=1}^{+\infty} c_{\alpha_j}$ does not depend on the choice of infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ enumerating the elements of the indexing set A.

Let $c_1, c_2, c_3, \ldots$ be an infinite sequence of real numbers that are not necessarily all non-negative or all non-positive, and let $c_j^+ = \max(c_j, 0)$ and $c_j^- = \min(0, c_j)$ for all positive integers j. Then $c_j^+ \geq 0$, $c_j^- \leq 0$, $c_j = c_j^+ + c_j^-$ and $|c_j| = c_j^+ - c_j^- = c_j^+ + |c_j^-|$ for all positive integers j. Moreover, for each positive integer j, at most one of the numbers c_j^+ asnd c_j^- is non-zero. Now $0 \leq c_j^+ \leq |c_j|$ and $0 \leq |c_j^-| \leq |c_j|$ for all positive integers j. It follows from this that $\sum\limits_{j=1}^{+\infty} |c_j|$ is convergent if and only if both $\sum\limits_{j=1}^{+\infty} c_j^+$ and $\sum\limits_{j=1}^{+\infty} c_j^-$ convergent. In this case we say that the infinite series $\sum_{j=1}^{+\infty} c_j$ is *absolutely convergent*.

Corollary 1.15 *Let $\sum\limits_{j=1}^{+\infty} a_j$ be an absolutely convergent infinite series, and let $\sum\limits_{k=1}^{+\infty} b_k$ be a rearrangement of infinite series $\sum\limits_{j=1}^{+\infty} a_j$. Then the infinite series $\sum\limits_{k=1}^{+\infty} b_k$ is absolutely convergent, and $\sum\limits_{k=1}^{+\infty} b_k = \sum\limits_{j=1}^{+\infty} a_j$.*

Proof There exists a permutation $\sigma: \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers with the property that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $a_j^+ = \max(a_j, 0)$ and $a_j^- = \min(0, a_j)$ for all positive integers j and $b_k^+ = \max(b_k, 0)$ and $b_k^- = \min(0, b_k)$ for all positive integers k. The absolute convergence of $\sum\limits_{j=1}^{\infty} a_j$ then ensures that the infinite series $\sum\limits_{j=1}^{\infty} a_j^+$ and $\sum\limits_{j=1}^{\infty} a_j^-$ both converge. It then follows from Corollary 1.14 that

$$\sum_{j=1}^{+\infty} |a_j| = \sum_{j=1}^{+\infty} a_j^+ - \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ - \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} |b_k|$$

and

$$\sum_{j=1}^{+\infty} a_j = \sum_{j=1}^{+\infty} a_j^+ + \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ + \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} b_k.$$

The result follows. ∎

1.9 Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Definition A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$.

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$.

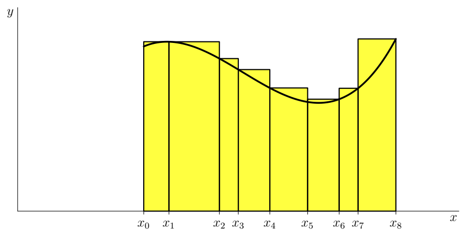
Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{n} (x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

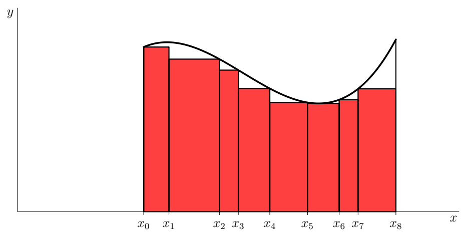
for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

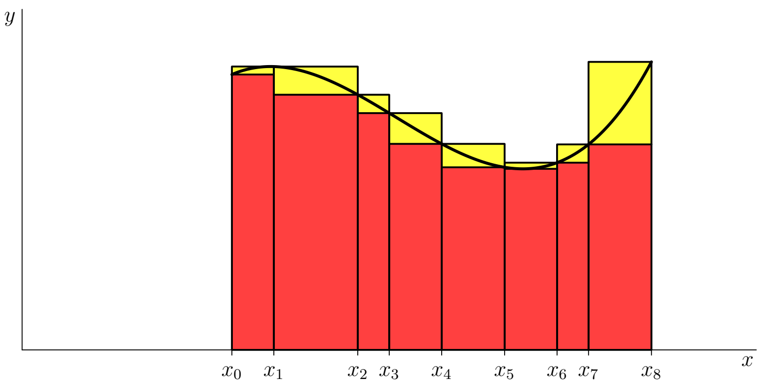
$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\left\{U(P, f) : P \text{ is a partition of } [a, b]\right\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\left\{L(P, f) : P \text{ is a partition of } [a, b]\right\}. \end{aligned}$$



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8

The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$, since $L(P,f) \le L(P,g)$ and $U(P,f) \le U(P,g)$ for all partitions P of $[a,b]$.

Definition Let P and R be partitions of $[a,b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 1.16 *Let R be a refinement of some partition P of $[a,b]$. Then*

$$L(R,f) \ge L(P,f) \quad \textit{and} \quad U(R,f) \le U(P,f)$$

for any bounded function $f\colon [a,b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$

given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 1.17 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 1.16 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$
\begin{aligned}
L(P_n, f) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{ci}{n} + d - \frac{c}{n} \right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{ci}{n} + d \right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}
$$

It follows that

$$\lim_{n \to +\infty} L(P_n, f) = \frac{c}{2} + d$$

and

$$\lim_{n \to +\infty} U(P_n, f) = \frac{c}{2} + d$$

Now $L(P_n, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_n, f)$ for all positive integers n. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

1.10 Interchanging Limits and Integrals

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges for all $x \in [a, b]$. We wish to determine whether or not

$$\lim_{j \to +\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0,1]$ defined by $f_j(x) = j(x^j - x^{2j})$. Now

$$\lim_{j\to+\infty} \int_0^1 f_j(x)\,dx = \lim_{j\to+\infty} \left(\frac{j}{j+1} - \frac{j}{2j+1}\right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim\limits_{j\to+\infty} f_j(x) = 0$ for all $x \in [0,1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $jx^j \to 0$ as $j \to +\infty$. Now

$$\lim_{j\to+\infty} \frac{j+1}{j} = 1.$$

It follows that

$$\lim_{j\to+\infty} \frac{(j+1)x}{j} = x < 1,$$

Let r be chosen so that $x < r < 1$. Then there exists some positive integer N large enough to ensure that

$$1 + \frac{1}{N} < \frac{r}{x}.$$

if $j \geq N$ then

$$\frac{(j+1)x^{j+1}}{jx^j} = \frac{(j+1)x}{j} < r.$$

It follows that

$$0 \leq jx^j \leq B$$

whenever $j \geq N$, where $B = Nx^N$, and therefore $jx^j \to 0$ as $j \to +\infty$. It follows that

$$\lim_{j\to+\infty} f_j(x) = \left(\lim_{j\to+\infty} jx^j\right)\left(\lim_{j\to+\infty} (1 - x^j)\right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_j(1) = 0$ for all positive integers j. We conclude that $\lim\limits_{j\to+\infty} f_j(x) = 0$ for all $x \in [0,1]$, which is what we set out to show.

1.11 Uniform Convergence

We now introduce the concept of *uniform convergence*. Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a, b]$, the identity

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a, b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_j) is said to converge *uniformly* to a function f on D as $j \to +\infty$ if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all positive integers j satisfying $j \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each positive integer j, let $M_j = \sup\{f_j(x) : x \in D\}$. We claim that $M_j \to 0$ as $j \to +\infty$.

To prove this, let some strictly positive real number ε be given. Then there exists some positive integer N such that $|f_j(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $j \geq N$. Thus if $j \geq N$ then $M_j \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_j \to 0$ as $j \to +\infty$, as claimed.

Example Let $(f_j : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. We have already shown (in an earlier example) that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. However a straightforward exercise in calculus shows that the maximum value attained by the function f_j is $j/4$ (which is attained at $x = 1/2^{\frac{1}{j}}$), and $j/4 \to +\infty$ as $j \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \ldots$ does not converge uniformly to the zero function on the interval $[0, 1]$.

Proposition 1.18 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let some strictly positive real number ε be given. If j is chosen sufficiently large then $|f(x) - f_j(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_j \to f$ uniformly on D as $j \to +\infty$. It then follows from the continuity of f_j that there exists some strictly positive real number δ such that $|f_j(x) - f_j(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$\begin{aligned} |f(x) - f(s)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(s)| + |f_j(s) - f(s)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|x - s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 1.19 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a, b]$ to some continuous real-valued function f. Then*

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b f(x)\, dx.$$

Proof Let some strictly positive real number ε. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b - a) < \varepsilon$. Then there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon_0$ for all $x \in [a, b]$ and $j \geq N$, since the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to f on $[a, b]$.

Now

$$f_j(x) - f(x) \leq |f_j(x) - f(x)| \quad \text{and} \quad -\left(f_j(x) - f(x)\right) \leq |f_j(x) - f(x)|$$

for all $x \in [a, b]$. It follows that

$$\int_a^b \left(f_j(x) - f(x)\right) dx \leq \int_a^b |f_j(x) - f(x)|\, dx$$

and

$$-\int_a^b \left(f_j(x) - f(x)\right) dx \leq \int_a^b |f_j(x) - f(x)|\, dx$$

and therefore

$$\left| \int_a^b \left(f_j(x) - f(x)\right) dx \right| \leq \int_a^b |f_j(x) - f(x)|\, dx$$

for all positive integers j. It follows that

$$\left| \int_a^b f_j(x)\, dx - \int_a^b f(x)\, dx \right| \leq \int_a^b |f_j(x) - f(x)|\, dx \leq \varepsilon_0(b - a) < \varepsilon$$

whenever $j \geq N$. The result follows. ∎

1.12 Compactness and the Heine-Borel Theorem

Definition Let K be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection $\mathcal{C}$ of open sets in $\mathbb{R}^n$ is said to *cover* K if

$$K = \bigcup_{V \in \mathcal{C}} V.$$

In other words, a collection $\mathcal{C}$ of open sets in $\mathbb{R}^n$ is said to cover K if and only if each point of K belongs to at least one open set belonging to the collection $\mathcal{C}$.

Definition A subset K of $\mathbb{R}^n$ is said to be *compact* if, given any collection of open sets in $\mathbb{R}^n$ which covers K, there exists some finite subcollection which also covers K.

Remark The definition of compactness given above is formulated for subsets of Euclidean spaces, but the given definition generalizes in the obvious fashion to metric spaces and, even more generally, to topological spaces. Some of the results below generalize so as to be applicable to compact subsets of topological spaces, and others generalize so as to be applicable to compact subsets of metric spaces. However, in what follows, we restrict the statements and proofs of the results to the context of subsets of Euclidean spaces.

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 1.20 (Heine-Borel in One Dimension) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{C}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{C}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{C}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{C}$, as required. ∎

Lemma 1.21 *Let F and K be subsets of $\mathbb{R}^n$ where F is closed, K is compact and $F \subset K$. Then F is compact.*

Proof Let $\mathcal{C}$ be any collection of open sets in $\mathbb{R}^n$ covering F. On adjoining the open set $\mathbb{R}^n \setminus F$ to $\mathcal{C}$, we obtain a collection of open sets which covers the compact set K. The compactness of K ensures that some finite subcollection of this collection covers K. The open sets in this subcollection that belong to $\mathcal{C}$ then constitute a finite subcollection of $\mathcal{C}$ that covers F. Thus F is compact, as required. ∎

Lemma 1.22 *Let $\varphi: \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function between Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, and let K be a compact subset of $\mathbb{R}^m$. Then $\varphi(K)$ is a compact subset of $\mathbb{R}^n$.*

Proof Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}^n$ which covers $\varphi(K)$. Then K is covered by the collection of all open sets of the form $\varphi^{-1}(V)$ for some $V \in \mathcal{C}$. It follows from the compactness of K that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{C}$ such that

$$K \subset \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \cup \cdots \cup \varphi^{-1}(V_k).$$

But then $\varphi(K) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $\varphi(K)$ is compact. ∎

Lemma 1.23 *Let $f: K \to \mathbb{R}$ be a continuous real-valued function on a compact subset K of $\mathbb{R}^n$. Then f is bounded above and below on K.*

Proof The range $f(K)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(K)$ is compact (Lemma 1.22) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(K) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on K, as required. ∎

Proposition 1.24 *Let $f: K \to \mathbb{R}$ be a continuous real-valued function on a compact subset K of $\mathbb{R}^n$. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of K such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in K$.*

Proof Let $m = \inf\{f(\mathbf{x}) : \mathbf{x} \in K\}$ and $M = \sup\{f(\mathbf{x}) : \mathbf{x} \in K\}$. There must exist $\mathbf{v} \in K$ satisfying $f(\mathbf{v}) = M$, for if $f(\mathbf{x}) < M$ for all $\mathbf{x} \in K$ then the function $\mathbf{x} \mapsto 1/(M - f(\mathbf{x}))$ would be a continuous real-valued function on K that was not bounded above, contradicting Lemma 1.23. Similarly there must exist $\mathbf{u} \in K$ satisfying $f(\mathbf{u}) = m$, since otherwise the function $\mathbf{x} \mapsto 1/(f(\mathbf{x}) - m)$ would be a continuous function on K that was not bounded above, again contradicting Lemma 1.23. But then $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in K$, as required. ∎

Proposition 1.25 *Let K be a compact subset of a Euclidean space $\mathbb{R}^n$. Then K is closed in $\mathbb{R}^n$.*

Proof Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to K, and let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{p}|$ for all $\mathbf{x} \in \mathbb{R}^n$. It follows from Proposition 1.24 that there is a point $\mathbf{q}$ of K such that $f(\mathbf{x}) \geq f(\mathbf{q})$ for all $\mathbf{x} \in K$, because K is compact. Now $f(\mathbf{q}) > 0$, since $\mathbf{q} \neq \mathbf{p}$. Let δ satisfy $0 < \delta \leq f(\mathbf{q})$. Then the open ball of radius δ about the point $\mathbf{p}$ is contained in the complement of K, because $f(\mathbf{x}) < f(\mathbf{q})$ for all points $\mathbf{x}$ of this open ball. It follows that the complement of K is an open set in $\mathbb{R}^n$, and thus K itself is closed in $\mathbb{R}^n$. ∎

Let F be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. For each $\mathbf{x} \in \mathbb{R}^n$, we denote by $d(\mathbf{x}, F)$ the (Euclidean) distance from the point $\mathbf{x}$ to the set F. This distance $d(\mathbf{x}, F)$ is defined so that

$$d(\mathbf{x}, F) = \inf\{|\mathbf{x} - \mathbf{w}| : \mathbf{w} \in F\}.$$

Lemma 1.26 *Let F be a subset of $\mathbb{R}^n$. Then*

$$|d(\mathbf{x}, F) - d(\mathbf{y}, F)| \leq |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in F$, and thus the function sending points $\mathbf{x}$ on $\mathbb{R}^n$ to their distance $d(\mathbf{x}, F)$ from the set F is a continuous real-valued function on $\mathbb{R}^n$.

Proof Let ε be a real number satisfying $\varepsilon > 0$, and let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$. Then there exists $\mathbf{z} \in F$ for which $|\mathbf{y} - \mathbf{z}| < d(\mathbf{y}, F) + \varepsilon$. It follows from the Triangle Inequality that

$$d(\mathbf{x}, F) \leq |\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}| < |\mathbf{x} - \mathbf{y}| + d(\mathbf{y}, F) + \varepsilon$$

and thus

$$d(\mathbf{x}, F) - d(\mathbf{y}, F) < |\mathbf{x} - \mathbf{y}| + \varepsilon.$$

Now the inequality just obtained must hold for all positive real numbers ε, and the left hand side of the inequality is independent of the value of ε. It must therefore be the case that

$$d(\mathbf{x}, F) - d(\mathbf{y}, F) \leq |\mathbf{x} - \mathbf{y}|.$$

Interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$, we see also that

$$d(\mathbf{y}, F) - d(\mathbf{x}, F) \leq |\mathbf{x} - \mathbf{y}|.$$

It follows that

$$|d(\mathbf{x}, F) - d(\mathbf{y}, F)| \leq |\mathbf{x} - \mathbf{y}|.$$

This inequality ensures that the function that sends points $\mathbf{x}$ of $\mathbb{R}^n$ to their distance $d(\mathbf{x}, F)$ from the set F is a continuous function on $\mathbb{R}^n$, as required. ∎

Given any subset F of $\mathbb{R}^n$, we denote by $B(F, \delta)$ the *δ-neighbourhood* of the set F in $\mathbb{R}^n$, defined so that

$$B(F, \delta) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, F) < \delta\}.$$

Proposition 1.27 *Let K and V be subsets of $\mathbb{R}^n$, where K is compact, V is open and $K \subset V$. Then there exists some positive real number δ for which $B(K, \delta) \subset V$.*

Proof Let $F = \mathbb{R}^n \setminus V$, and let $f(\mathbf{x}) = d(\mathbf{x}, F)$ for all $\mathbf{x} \in \mathbb{R}^n$, where $d(\mathbf{x}, F)$ denotes the distance from the point $\mathbf{x}$ to the set F. Now the function f is a continuous real-valued function on $\mathbb{R}^n$. Moreover $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in V$, and therefore $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$. It then follows from Proposition 1.24 that there exists some point $\mathbf{u}$ of K with the property that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in K$. Let $\delta = f(\mathbf{u})$. Then $|\mathbf{x} - \mathbf{z}| \geq \delta$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in F$. It follows that $B(\mathbf{x}, \delta) \subset V$ for all $\mathbf{x} \in K$, where $B(\mathbf{x}, \delta)$ denotes the open ball of radius δ centred on the point $\mathbf{x}$. Therefore $B(K, \delta) \subset V$, as required. ∎

Alternative Proof For each point $\mathbf{w}$ of K there exists some positive real number $\delta_{\mathbf{w}}$ such that $B(\mathbf{w}, 2\delta_{\mathbf{w}}) \subset V$ where $B(\mathbf{w}, 2\delta_{\mathbf{w}})$ denotes the open ball of radius $2\delta_{\mathbf{w}}$ centred on the point $\mathbf{w}$ for each $\mathbf{w} \in K$. Now the collection $(B(\mathbf{w}, \delta_{\mathbf{w}}) : \mathbf{w} \in K)$ of open balls constitutes an open cover of the compact set K. The definition of compactness therefore ensures that there exist points $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ (finite in number) such that

$$K \subset \bigcup_{j=1}^{m} B(\mathbf{w}_j, \delta_{\mathbf{w}_j}).$$

Let δ be the minimum of the positive real numbers $\delta_{\mathbf{w}_j}$ for $j = 1, 2, \ldots, m$. Then $\delta > 0$. Moreover the Triangle Inequality ensures that

$$B(\mathbf{z}, \delta) \subset B(\mathbf{w}_j, 2\delta_{\mathbf{w}_j}) \subset V$$

for all $\mathbf{z} \in B(\mathbf{w}_j, \delta_{\mathbf{w}_j})$, and therefore $\bigcup_{\mathbf{z}\in K} B(\mathbf{z}, \delta) \subset V$. But $\bigcup_{\mathbf{z}\in K} B(\mathbf{z}, \delta) = B(K, \delta)$, because a point $\mathbf{x}$ of $\mathbb{R}^n$ belongs to $B(K, \delta)$ if and only if $|\mathbf{x} - \mathbf{z}| < \delta$ for some $\mathbf{z} \in K$. Thus $B(K, \delta) \subset V$, as required. ∎

Definition We define a *closed n-dimensional block* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ that is a product of closed bounded intervals.

Thus a subset K of $\mathbb{R}^n$ is a closed n-dimensional block if and only if there exist real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and

$$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Proposition 1.28 *A closed n-dimensional block is a compact set.*

Proof We prove the result by induction on the dimension n of the Euclidean space. The result when $n = 1$ is the one-dimensional Heine-Borel Theorem (Theorem 1.20). Thus suppose as our induction hypothesis that $n > 1$ and that that every closed $(n-1)$-dimensional block in $\mathbb{R}^{n-1}$ is a compact set. Let K be an n-dimensional block in $\mathbb{R}^n$, and let

$$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers that satisfy $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. Let $p \colon \mathbb{R}^n \to \mathbb{R}$ be the projection function defined such that

$$p(x_1, x_2, \ldots, x_n) = x_n$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The induction hypothesis then ensures that K_z is a compact set for all $z \in [a_n, b_n]$, where

$$K_z = \{\mathbf{x} \in K : p(\mathbf{x}) = z\}.$$

Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}^n$ that covers K. The compactness of K_z ensures that, for each real number z satisfying $a_n \leq z \leq b_n$ there exists a finite subcollection $\mathcal{C}_z$ of $\mathcal{C}$ such that $K_z \subset \bigcup_{V\in\mathcal{C}_z} V$. Let $W_z = \bigcup_{V\in\mathcal{C}_z} V$. (The set W_z is thus the union of the open sets belonging to the finite subcollection $\mathcal{C}_z$ of $\mathcal{C}$.)

Now K_z is compact, W_z is open, and $K_z \subset W_z$. It follows that there exists some positive real number δ_z such that $B(K, \delta_z) \subset W_z$, where $B(K, \delta_z)$ denotes the δ-neighbourhood of K in $\mathbb{R}^n$ i.e., the subset of $\mathbb{R}^n$ consisting of those points of $\mathbb{R}^n$ that lie within a distance δ_z of the set K_z (see Proposition 1.27). But then

$$\{\mathbf{x} \in K : z - \delta_z < p(\mathbf{x}) < z + \delta_z\} \subset W_z$$

for all $z \in [a_n, b_n]$. Now the collection of all open intervals in $\mathbb{R}$ that are of the form $(z - \delta_z, z + \delta_z)$ constitute an open cover of the closed bounded interval $[a_n, b_n]$. It follows from the one-dimensional Heine-Borel Theorem (Theorem 1.20) that there exist $z_1, z_2, \ldots, z_m \in [a_n, b_n]$ such that

$$[a_n, b_n] \subset \bigcup_{j=1}^{m} (z_j - \delta_{z_j}, z_j + \delta_{z_j}).$$

But then

$$K \subset \bigcup_{j=1}^{n} W_{z_j}.$$

Moreover $\bigcup_{j=1}^{n} W_{z_j}$ is the union of all the open sets that belong to the collection $\mathcal{D}$ obtained by amalgamating the finite collections $\mathcal{C}_{z_1}, \mathcal{C}_{z_2}, \ldots, \mathcal{C}_{z_m}$. Then $\mathcal{D}$ is a finite subcollection of $\mathcal{C}$ which covers the n-dimensional block K. The result follows. ∎

Theorem 1.29 (Multidimensional Heine-Borel Theorem) *A subset of a Euclidean space is compact if and only if it is both closed and bounded.*

Proof Let K be a compact subset of n-dimensional Euclidean space. The function that maps each point $\mathbf{x}$ of $\mathbb{R}^n$ to its Euclidean distance $|\mathbf{x}|$ from the origin is then a bounded function on K (Lemma 1.23) and therefore K is a bounded set. Moreover it follows from Proposition 1.25 that K is closed in $\mathbb{R}^n$.

Conversely let K be a subset of $\mathbb{R}^n$ that is both closed and bounded. Then there exists some positive real number R large enough to ensure that $K \subset H$, where

$$H = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -R \le x_i \le R \text{ for } i = 1, 2, \ldots, n\}.$$

Now H is a closed n-dimensional block in $\mathbb{R}^n$. It follows from Proposition 1.28 that H is a compact subset of $\mathbb{R}^n$. Thus K is a closed subset of a compact set. It follows from Lemma 1.21 that K is a compact subset of $\mathbb{R}^n$, as required. ∎

1.13 The Extended Real Number System

It is sometimes convenient to make use of the *extended real line* $[-\infty, +\infty]$. This is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ obtained on adjoining to the real line $\mathbb{R}$ two extra elements $+\infty$ and $-\infty$ that represent points at 'positive infinity' and 'negative infinity' respectively. We define

$$c + (+\infty) = (+\infty) + c = +\infty$$

and

$$c + (-\infty) = (-\infty) + c = -\infty$$

for all real numbers c. We also define products of non-zero real numbers with these extra elements $\pm\infty$ so that

$$\begin{array}{rcll} c \times (+\infty) & = & (+\infty) \times c = +\infty & \text{when } c > 0, \\ c \times (-\infty) & = & (-\infty) \times c = -\infty & \text{when } c > 0, \\ c \times (+\infty) & = & (+\infty) \times c = -\infty & \text{when } c < 0, \\ c \times (-\infty) & = & (-\infty) \times c = +\infty & \text{when } c < 0, \end{array}$$

We also define

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0,$$

and

$$(+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty,$$
$$(+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty.$$

The sum of $+\infty$ and $-\infty$ is not defined. We define $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$. The difference $p - q$ of two extended real numbers is then defined by the formula $p - q = p + (-q)$, unless $p = q = +\infty$ or $p = q = -\infty$, in which cases the difference of the extended real numbers p and q is not defined.

We extend the definition of inequalities to the extended real line in the obvious fashion, so that $c < +\infty$ and $c > -\infty$ for all real numbers c, and $-\infty < +\infty$.

Given any real number c, we define

$$\begin{array}{rcl} [c, +\infty] & = & [c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p \geq c\}, \\ (c, +\infty] & = & (c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p > c\}, \\ [-\infty, c] & = & (-\infty, c] \cup \{-\infty\} = \{p \in [-\infty, \infty] : p \leq c\}, \\ [-\infty, c) & = & (-\infty, c) \cup \{-\infty\} = \{p \in [-\infty, \infty] : p < c\}. \end{array}$$

There is an order-preserving bijective function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ from the extended real line $[-\infty, +\infty]$ to the closed interval $[-1, 1]$ which is defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$, and $\varphi(c) = \dfrac{c}{1 + |c|}$ for all real numbers c. Let us define $\rho(p, q) = |\varphi(q) - \varphi(p)|$ for all extended real numbers p and q. Then the set $[-\infty, +\infty]$ becomes a metric space with distance function ρ. Moreover the function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is a homeomorphism from this metric space to the closed interval $[-1, 1]$. It follows directly from this that $[-\infty, +\infty]$ is a compact metric space. Moreover an infinite sequence $(p_j : j \in \mathbb{N})$ of extended real numbers is convergent if and only if the corresponding sequence $(\varphi(p_j) : j \in \mathbb{N})$ of real numbers is convergent.

Given any non-empty set S of extended real numbers, we can define $\sup S$ to be the least extended real number p with the property that $s \leq p$ for all $s \in S$. If the set S does not contain the extended real number $+\infty$, and if there exists some real number B such that $s \leq B$ for all $s \in S$, then $\sup S < +\infty$; otherwise $\sup S = +\infty$. Similarly we define $\inf S$ to be the greatest extended real number p with the property that $s \geq p$ for all $s \in S$. If the set S does not contain the extended real number $-\infty$, and if there exists some real number A such that $s \geq A$ for all $s \in S$, then $\inf S > +\infty$; otherwise $\inf S = -\infty$. Moreover

$$\varphi(\sup S) = \sup \varphi(S) \text{ and } \varphi(\inf S) = \inf \varphi(S),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$ and $\varphi(c) = c(1 + |c|)^{-1}$ for all real numbers c.

Given any sequence $(p_j : j \in \mathbb{N})$ of extended real numbers, we define the *upper limit* $\limsup\limits_{j \to +\infty} p_j$ and the *lower limit* $\liminf\limits_{j \to +\infty} p_j$ of the sequence so that

$$\limsup_{j \to +\infty} p_j = \lim_{j \to +\infty} \sup\{p_k : k \geq j\}$$

and

$$\liminf_{j \to +\infty} p_j = \lim_{j \to +\infty} \inf\{p_k : k \geq j\}.$$

Every sequence of extended real numbers has both an upper limit and a lower limit. Moreover an infinite sequence of extended real numbers converges to some extended real number if and only if the upper and lower limits of the sequence are equal. (These results follow easily from the corresponding results for bounded sequences of real numbers, on using the identities

$$\varphi(\limsup_{j \to +\infty} p_j) = \limsup_{j \to +\infty} \varphi(p_j), \quad \varphi(\liminf_{j \to +\infty} p_j) = \liminf_{j \to +\infty} \varphi(p_j),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined above.)

The function that sends a pair (p, q) of extended real numbers to the extended real number $p + q$ is not defined when $p = +\infty$ and $q = -\infty$, or when $p = -\infty$ and $q = +\infty$ but is continuous elsewhere. The function that sends a pair (p, q) of extended real numbers to the extended real number pq is defined everywhere. This function is discontinuous when $p = \pm\infty$ and $q = 0$, and when $p = 0$ and $q = \pm\infty$. It is continuous for all other values of the extended real numbers p and q.

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of extended real numbers which does not include both the values $+\infty$ and $-\infty$, and let $p_k = \sum_{j=0}^{k} a_j$ for all natural numbers k. If the infinite sequence $p_1, p_2, p_3, \ldots$ of extended real numbers converges in the extended real line $[-\infty, +\infty]$ to some extended real number p, then this value p is said to be the *sum* of the infinite series $\sum\limits_{j=1}^{+\infty} a_j$, and we write $\sum\limits_{j=1}^{+\infty} a_j = p$.

It follows easily from this definition that if $+\infty$ is one of the values of the infinite series $a_1, a_2, a_3, \ldots$, then $\sum\limits_{j=1}^{+\infty} a_j = +\infty$. Similarly if $-\infty$ is one of the values of this infinite series then then $\sum\limits_{j=1}^{+\infty} a_j = -\infty$. Suppose that the members of the sequence $a_1, a_2, a_3, \ldots$ are all real numbers. Then $\sum\limits_{j=1}^{+\infty} a_n = +\infty$ if and only if, given any real number B, there exists some real number N such that $\sum\limits_{j=1}^{k} a_n > B$ whenever $k \geq N$. Similarly $\sum\limits_{j=1}^{+\infty} a_j = -\infty$ if and only if, given any real number A, there exists some real number N such that $\sum\limits_{j=1}^{k} a_j < A$ whenever $k \geq N$.