Course MA2224: Hilary Term 2019.

Solutions to Assignment II.

1. *Throughout this question, let a be a positive real number, and let $s\colon \mathbb{R} \to \mathbb{R}$ be an integrable simple function with the following properties:*

   - $s(x) \geq 0$ *for all* $x \in \mathbb{R}$*;*
   - $s(x) = 0$ *whenever* $x < a$*;* $s(x) \leq 1/x^2$ *whenever* $x \geq a$*.*

   Also let $c_1, c_2, \ldots, c_m$ *be the non-zero values taken on by the function* s*, where*

   $$a^{-2} \geq c_1 > c_2 > \cdots > c_m > 0,$$

   and let $b_j = 1/\sqrt{c_j}$ *for* $j = 1, 2, \ldots, n$*. Let*

   $$E_j = \{x \in \mathbb{R} : s(x) = c_j\}$$

   for $j = 1, 2, \ldots, m$*. Then the sets* $E_1, E_2, \ldots, E_m$ *are pairwise disjoint Lebesgue-measurable subsets of* $\mathbb{R}$ *that satisfy* $\mu(E_j) < +\infty$*, where* $\mu(E_j)$ *denotes the Lebesgue measure of the set* E_j *for* $j = 1, 2, \ldots, m$*. Also* $a \leq b_1 < b_2 < \cdots < \cdots b_m$*. The objective of this question is to show that*

   $$\int_{\mathbb{R}} s(x)\,dx \leq \frac{1}{a}.$$

   (a) *Explain why* $s(x) = 0$ *for all* $x \in \mathbb{R}$ *satisfying* $x > b_m$*.*

   If $x > b_m$ then $0 \leq s(x) < 1/b_m^2 = c_m$. But the function s does not take on any values y satisfying $0 < y < c_m$. It follows that $s(x) = 0$.

   Let

   $$s_1(x) = \begin{cases} s(x) & \text{if } a \leq x \leq b_1, \\ 0 & \text{otherwise,} \end{cases}$$

   and, for $j = 1, 2, \ldots, m$*, let*

   $$s_j(x) = \begin{cases} s(x) & \text{if } b_{j-1} < x \leq b_j; \\ 0 & \text{otherwise.} \end{cases}$$

   Note that $s_1 + s_2 + \cdots + s_m = s$*.*

   (b) *Explain why* $s_j(x) \leq b_j^{-2}$ *for* $j = 1, 2, \ldots, m$*.*

Now $s(x) \leq c_1 = 1/b_1^2$ for all real numbers x, and therefore $s_1(x) \leq 1/b_1^2$ for all $x \in \mathbb{R}$.

If $s_j(x) > 0$, where $j > 1$, then $x > b_{j-1}$ and therefore $s(x) < c_{j-1}$. But the function s does not take on any value y satisfying $c_j < y < c_{j-1}$. It follows that if $s_j(x) > 0$ then $s_j(x) \leq c_j = b_j^{-2}$. The result follows.

(c) *Explain why*

$$\int_{\mathbb{R}} s(x)\,dx \leq \frac{b_1 - a}{b_1^2} + \sum_{j=2}^{m} \frac{b_j - b_{j-1}}{b_j^2}$$

$$\begin{aligned}
\int_{\mathbb{R}} s(x)\,dx &= \sum_{j=1}^{m} \int_{\mathbb{R}} s_j(x)\,dx \\
&= \int_{[a,b_1]} s_1(x)\,dx + \sum_{j=2}^{m} \int_{(b_{j-1},b_j]} s_j(x)\,dx \\
&\leq \frac{b_1 - a}{b_1^2} + \sum_{j=2}^{m} \frac{b_j - b_{j-1}}{b_j^2}.
\end{aligned}$$

(d) *Explain why*

$$\frac{b_1 - a}{b_1^2} \leq \frac{1}{a} - \frac{1}{b_1}$$

and

$$\frac{b_j - b_{j-1}}{b_j^2} \leq \frac{1}{b_{j-1}} - \frac{1}{b_j}$$

for $j = 2, 3, \ldots, m$.

Now $b_1 > a$ and therefore

$$\frac{b_1 - a}{b_1^2} \leq \frac{b_1 - a}{b_1 a} \leq= \frac{1}{a} - \frac{1}{b_1}.$$

Similarly $b_j > b_{j-1}$, when $j > 1$, and therefore

$$\frac{b_j - b_{j-1}}{b_j^2} \leq \frac{b_j - b_{j-1}}{b_j b_{j-1}} \leq \frac{1}{b_{j-1}} - \frac{1}{b_j}.$$

(e) *Hence prove that*

$$\int_{\mathbb{R}} s(x)\,dx \leq \frac{1}{a}.$$

Adding these inequalities

$$\begin{aligned}
\int_{\mathbb{R}} s(x)\,dx &\leq \frac{b_1 - a}{b_1^2} + \sum_{j=2}^{m} \frac{b_j - b_{j-1}}{b_j^2} \\
&\leq \frac{1}{a} - \frac{1}{b_1} + \sum_{j=2}^{m} \left(\frac{1}{b_{j-1}} - \frac{1}{b_j} \right) \\
&= \frac{1}{a} - \frac{1}{b_m} < \frac{1}{a}.
\end{aligned}$$

2. *Let a be a positive real number, let r be a real number satisfying $r > 1$, let N be an integer greater than one, and let $t_{r,N}\colon \mathbb{R} \to \mathbb{R}$ be the integrable simple function defined such that $t_{r,N}(x) = 0$ whenver $x < a$ or $x > ar^N$, $t_{r,N}(a) = a^{-2}$ and $t_{r,N}(x) = (ar^j)^{-2}$ whenever $ar^{j-1} < x \leq ar^j$ for some integer j satisfying $1 \leq j \leq N$. Determine the value of $\int_{\mathbb{R}} t_{r,N}(x)\,dx$.*

$$\begin{aligned}
\int_{\mathbb{R}} t_{r,N}(x)\,dx &= \sum_{j=1}^{N} \int_{(ar^{j-1}, ar^j]} t_{r,N}(x)\,dx \\
&= \sum_{j=1}^{N} \frac{1}{a^2 r^{2j}} \times ar^{j-1}(r-1) \\
&= \frac{r-1}{ra} \sum_{j=1}^{N} \frac{1}{r^j} \\
&= \frac{r-1}{ra} \frac{\dfrac{1}{r} - \dfrac{1}{r^{N+1}}}{1 - \dfrac{1}{r}} \\
&= \frac{1}{a} \left(\frac{1}{r} - \frac{1}{r^{N+1}} \right).
\end{aligned}$$

3. *Making use of the results established in the two previous question, as appropriate, explain why, for any positive real number a, the quantity a^{-1} is the least upper bound of $\int_{\mathbb{R}} s(x)\,dx$ taken over all integrable simple functions $s\colon \mathbb{R} \to \mathbb{R}$ with the properties that $s(x) = 0$ whenever $x < a$ and $s(x) \leq x^{-2}$ whenever $x \geq a$.*

It follows from question 1 that $\dfrac{1}{a}$ is an upper bound on the values of the integrals of integrable simple function s satisfying the stated conditions. Now the integrable simple function $t_{r,N}$ satisfies the stated conditions for all real numbers r satisfying $r > 1$ and for all positive integers N. Let c be a real number satisyfing $0 < c < \dfrac{1}{a}$. We may choose $r > 1$ close enough to 1 to ensure that $\dfrac{1}{ar} > c$. We may then choose the positive integer N large enough to ensure that

$$\frac{1}{a}\left(\frac{1}{r} - \frac{1}{r^{N+1}}\right) > c.$$

But then

$$\int_{\mathbb{R}} t_{r,N}(x)\,dx > c.$$

This shows that $\dfrac{1}{a}$ is indeed the least upper bound of the integrals of all integrable simple functions satisfying the stated conditions.

Alternatively Let L be the least upper bound of the integrals of integrable simple functions s satisfying the stated conditions. It follows from question 1 that $L \leq 1/a$. It follows from question 2 that

$$L \geq \frac{1}{a}\left(\frac{1}{r} - \frac{1}{r^{N+1}}\right)$$

for all real numbers r satisfying $r > 1$ and for all positive integers N. Taking the limit of the right hand side as $N \to +\infty$, we find that $L \geq 1/(ar)$. Then taking the limit of the right hand side as r tends to 1 from above, we find that $L \geq 1/a$. From the inequalities $L \leq 1/a$ and $L \geq 1/a$, we deduce that $L = 1/a$, as required.

4. *Let E be the set of all irrational numbers x satisfying $0 < x < 1$.*

(a) *Is the subset E of the real line Lebesgue-measurable?*

Yes. Any subset of $\mathbb{R}$ consisting of a single point is Lebesgue-measurable. Any countable subset of $\mathbb{R}$ is a countable union of one-point sets and is therefore Lebesgue-measurable. The set of all rational numbers between 0 and 1 is a countable set, and is therefore Lebesgue-measurable. Therefore the given set is the difference of two Lebesgue-measurable sets, and is therefore Lebesgue-measurable.

(b) *What is the value of $\mu^*(E)$, where $\mu^*(E)$ denotes the Lebesgue outer measure of the set E?*

Here $\mu^*(E) = 1$. Note that $E \subset [0, 1]$, and therefore $\mu^*(E) \leq \mu^*([0, 1]) = 1$. Let G be the set of all rational numbers in the interval $[0, 1]$. Then G is a countable set, and therefore $\mu^*(G) = 0$. But

$$1 = \mu^*([0, 1]) \leq \mu^*(E) + \mu^*(G) = \mu * (E).$$

We have thus shown that $\mu^*(E) \leq 1$ and $\mu^*(E) \geq 1$. Therefore $\mu^*(E) = 1$.

(c) *Does there exist a subset F of E that is a countable union of intervals and that also satisfies $\mu^*(F) > 0$? [Justify your answer.]*

No. No interval of positive length is contained in E, since any such interval would contain a rational number. Therefore any subset F of E that is a countable union of intervals would be a countable union of sets of outer measure zero and therefore would itself have outer measure zero.