Course MA2224: Hilary Term 2019.
Solutions to Assignment 1.

We set out below some definitions to which we will make reference to within this assignment.

Definition *Let* $f\colon [a,b] \to \mathbb{R}$ *be a real-valued function defined on a closed bounded interval* $[a,b]$*. We say that the function* f *is* continuous and piecewise linear *if there exists a partition* P *of the interval* $[a,b]$ *with division points* $u_0, u_1, \ldots, u_N$*, where*

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b$$

such that

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ *and for all real numbers* x *satisfying* $u_{i-1} \leq x \leq u_i$*.*

Definition *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous and piecewise linear function defined on an interval* $[a,b]$*. Let us say that a real number* u *satisfying* $a < u < b$ *is a* joint *of the function if the gradient of the function differs on both sides of* u*.*

Note that if u *is a joint of a continuous and piecewise linear function* f *on* $[a,b]$*, then the derivative of the function is not defined at* u*. On the other hand, if* u *is not a joint of the function then the derivative of the function is defined and is constant throughout some sufficiently small neighbourhood of the value* u*. Note also that if* P *is a partition of* $[a,b]$ *with division points* $u_0, u_1, \ldots, u_N$*, where*

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b,$$

and if

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ *and for all real numbers* x *satisfying* $u_{i-1} \leq x \leq u_i$ *(as in the definition of continuous and piecewise linear functions given above), then the joints of* f *must necessarily be division points of the partition* P*.*

Note also that the sum of a finite number of continuous and piecewise linear functions on the interval $[a,b]$ *must itself be a continuous and piecewise linear function on that interval. Indeed let* $f_1, f_2, \ldots, f_s$ *be continuous and* *piecewise linear functions on the interval* $[a,b]$. *Then there exists a partition* P *of* $[a,b]$ *that includes as division points all the joints of the functions* $f_1, f_2, \ldots, f_s$. *Then*

$$f_k(x) = \frac{u_i - x}{u_i - u_{i-1}} f_k(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f_k(u_i)$$

for $k = 1, 2, \ldots, s$, $i = 1, 2, \ldots, N$ *and for all real numbers* x *satisfying* $u_{i-1} \leq x \leq u_i$. *Let* $f\colon [a,b] \to \mathbb{R}$ *be defined such that* $f(x) = \sum\limits_{k=1}^{s} f_k(x)$ *for all* $x \in [a,b]$. *Then, on summing the identities satisfied by the functions* f_k *for* $k = 1, 2, \ldots, s$, *we find that*

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ *and for all real numbers* x *satisfying* $u_{i-1} \leq x \leq u_i$, *and thus the function* f *is indeed continuous and piecewise linear.*

1. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a real-valued function on a closed bounded interval that is continuous and piecewise linear on that interval, and let* P *be a partition of that interval whose division points* $u_0, u_1, \ldots, u_N$ *include all the joints of the function* f, *so that*

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ *and for all real numbers* x *satisfying* $u_{i-1} \leq x \leq u_i$. *Let*

$$S_P = \tfrac{1}{2} \sum_{i=1}^{N} (f(u_{i-1}) + f(u_i))(u_i - u_{i-1}).$$

Let S_Q *be the real number defined in the same fashion for the partition* Q, *where the partition* Q *is obtained from the partition* P *by adding an extra division point* w *in the interior of the subinterval* $[u_{k-1}, u_k]$ *of the partition* P. *Verify by direct calculation that* $S_Q = S_P$.

The value $f(w)$ of the function f at w satisfies

$$(u_k - u_{k-1}) f(w) = (u_k - w) f(u_{k-1}) + (w - u_{k-1}) f(u_k).$$

Therefore

$$\begin{aligned} S_Q - S_P &= \tfrac{1}{2}\Big((f(u_{k-1}) + f(w))(w - u_{k-1}) \\ &\qquad + (f(w) + f(u_k))(u_k - w)) \end{aligned}$$

$$
\begin{aligned}
& \qquad - (f(u_{k-1}) + f(u_k))(u_k - u_{k-1})\Big) \\
= {} & \tfrac{1}{2}\Big(f(w)(u_k - u_{k-1}) - f(u_k)(w - u_{k-1}) \\
& \qquad - f(u_{k-1})(u_k - w)\Big) \\
= {} & 0,
\end{aligned}
$$

and thus $S_Q = S_P$ as required.

Remarks. *Note that it follows from the result of (a) above that if $f\colon [a,b] \to \mathbb{R}$ is a continuous and piecewise linear function on the interval $[a,b]$, if P is a partition which includes all the joints of the function f, and if R is a refinement of P, and if the quantities S_P and S_R are defined for each of these partitions in the manner specified, then $S_R = S_P$. This follows directly from the observation that one can pass from the partition P to the refinement R by successively adding a finite number of extra division points, and the addition of each division point preserves the value of the associated quantity. Now any two partitions P and Q of the interval $[a,b]$ have a common refinement R. Thus if the partitions P and Q each contain all the joints of the function f then $S_P = S_R = S_Q$.*

This motivates and justifies the following definition.

Definition *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on a closed bounded interval $[a,b]$. We define the* formal integral $\mathbf{FI}(f; a, b)$ *of the function f on the interval $[a,b]$ to be the value of the quantity*

$$
S_P = \tfrac{1}{2} \sum_{i=1}^{N} (f(u_{i-1}) + f(u_i))(u_i - u_{i-1}),
$$

where $u_0, u_1, u_2, \ldots, u_N$ satisfy

$$
a = u_0 < u_1 < u_2 < \cdots < u_N = b
$$

and where the list $u_0, u_1, u_2, \ldots, u_N$ includes all the joints of the function f.

It should be noted that the value $\mathbf{FI}(f; a, b)$ of this formal integral is equal to the value of the integral $\int_a^b f(x)\,dx$ of the continuous and piecewise linear function f, when this integral is evaluated by standard methods. However the definition given of the formal integral $\mathbf{FI}(f; a, b)$ is purely algebraic, and the given definition does not make use of any limiting or approximation process.

It should also be noted that if f and g are both continuous and piecewise linear real-valued functions on the interval $[a,b]$ then

$$
\mathbf{FI}((f+g); a, b) = \mathbf{FI}(f; a, b) + \mathbf{FI}(g; a, b).
$$

Moreover if $f(x) \leq g(x)$ *for all* $x \in [a,b]$ *then* $\mathbf{FI}(f;a,b) \leq \mathbf{FI}(g;a,b)$. *These results follow directly by evaluating the formal integrals with respect to a partition of* $[a,b]$ *that includes as division points the joints of both functions.*

2. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous and piecewise linear real-valued function on the closed bounded interval* $[a,b]$, *and let* K *be a real constant that exceeds the absolute values of the slopes of the function* f, *and let* P *be a partition of* $[a,b]$ *that includes as division points the joints of* $[a,b]$. *Moreover let* $P = \{u_0, u_1, \ldots, u_N\}$, *where*

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

(The conditions listed above ensure that $|f(u_i) - f(u_{i-1})| \leq K(u_i - u_{i-1})$ *for* $i = 1, 2, \ldots, N$.*) Prove that*

$$\mathbf{FI}(f;a,b) = \tfrac{1}{2}(L(P,f) + U(P,f)),$$

where $L(P,f)$ *and* $U(P,f)$ *denote the Darboux lower sum and upper sum for the partition* P *and the function* f.

Also let δ *be a positive real number, and let the partition* P *be chosen such that* $u_i - u_{i-1} < \delta$ *for* $i = 1, 2, \ldots, N$. *Prove that*

$$U(P,f) - L(P,f) \leq K(b-a)\delta$$

(On each subinterval $[u_{i-1}, u_i]$ *of the partition, you should explicitly consider both the case where* $f(x_{i-1}) \leq f(x_i)$ *and also the case where* $f(x_{i-1}) \geq f(x_i)$.*)*

Let i be an integer between 1 and N. If $f(u_{i-1}) \leq f(u_i)$ then $m_i = f(u_{i-1})$ and $M_i = f(u_i)$. On the other hand, if $f(u_{i-1}) \geq f(u_i)$ then $m_i = f(u_i)$ and $M_i = f(u_{i-1})$. In both cases

$$\tfrac{1}{2}(f(u_{i-1}) + f(u_i)) = \tfrac{1}{2}(m_i + M_i).$$

Summing over all values of i between 1 and N, we find that

$$\begin{aligned}
\mathbf{FI}(f;a,b) &= \tfrac{1}{2}\sum_{i=1}^{N}(f(u_{i-1}) + f(u_i))(u_i - u_{i-1}) \\
&= \tfrac{1}{2}\sum_{i=1}^{N}(m_i + M_i)(u_i - u_{i-1}) \\
&= \tfrac{1}{2}(L(P,f) + U(P,f)).
\end{aligned}$$

Also, for each integer i between 1 and N,

$$M_i - m_{i-1} = |f(u_i) - f(u_{u-1})| \leq K(u_i - u_{i-1}) \leq K\delta,$$

and therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{N} (M_i - m_i)(u_i - u_{i-1}) \\ &\leq K\delta \sum_{i=1}^{N} (u_i - u_{i-1}) = K(b-a)\delta. \end{aligned}$$

3. *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded real-valued function on the closed bounded interval $[a, b]$ and let $p\colon [a, b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on $[a, b]$ that satisfies $p(x) \leq f(x)$ for all $x \in [a, b]$. Using the result of the previous question, or otherwise, prove that, given any real number ε satisfying $\varepsilon > 0$, there exists a partition P of $[a, b]$ with the property that $L(P, f) > \mathbf{FI}(p; a, b) - \varepsilon$. Then use this result to prove that*

$$\mathbf{FI}(p; a, b) \leq \mathcal{L} \int_a^b f(x)\, dx,$$

where $\mathcal{L} \int_a^b f(x)\, dx$ denotes the lower Riemann integral *of the function f on $[a, b]$.*

Let K be a constant with the properties specified in Question 2, let $\delta > 0$ be chosen such that $K(b-a)\delta < \varepsilon$ and let P be a partition with division points $u_0, u_1, \ldots, u_N$ (listed in increasing order) chosen such that P includes all the joints of p and $u_i - u_{i-1} < \delta$ for $i = 1, 2, \ldots, N$. Then $L(P, p) \leq \mathbf{FI}(p; a, b) \leq U(P, p)$ and $U(P, p) - L(P, p) < \varepsilon$, and therefore $\mathbf{FI}(p, a, b) < L(P, p) + \varepsilon$. Also $L(P, p) \leq L(P, f)$, because $p(x) \leq f(x)$ for all $x \in X$. It follows that

$$\mathbf{FI}(p, a, b) - \varepsilon < L(P, p) \leq L(P, f).$$

Now $L(P, f) \leq \mathcal{L} \int_a^b f(x)\, dx$. It follows that

$$\mathbf{FI}(p, a, b) \leq \mathcal{L} \int_a^b f(x)\, dx + \varepsilon$$

for all positive real numbers ε. Now the values of $\mathbf{FI}(p, a, b)$ and $\mathcal{L} \int_a^b f(x)\, dx$ do not depend on the value of the positive real number ε. Therefore

$$\mathbf{FI}(p, a, b) \leq \mathcal{L} \int_a^b f(x)\, dx,$$

as required.

Remarks. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a bounded real-valued function on the closed bounded interval* $[a,b]$ *and let* $q\colon [a,b] \to \mathbb{R}$ *be a continuous and piecewise linear real-valued function on* $[a,b]$ *that satisfies* $q(x) \geq f(x)$ *for all* $x \in [a,b]$. *Let* $g\colon [a,b] \to \mathbb{R}$ *and* $p\colon [a,b] \to \mathbb{R}$ *be defined such that* $g(x) = -f(x)$ *and* $p(x) = -q(x)$ *for all* $x \in [a,b]$. *It follows from the inequality proved in the previous question that*

$$\mathbf{FI}(q; a, b) = -\mathbf{FI}(p; a, b) \geq -\mathcal{L} \int_a^b g(x)\,dx = \mathcal{U} \int_a^b f(x)\,dx.$$

Thus $\mathbf{FI}(q; a, b) \geq \mathcal{U} \int_a^b f(x)\,dx$ *for all continuous and piecewise linear functions* $q\colon [a,b] \to \mathbb{R}$ *satisfying* $q(x) \geq f(x)$ *for all* $x \in [a,b]$.

We now set up some notation for the next question.

Let $f\colon [a,b] \to \mathbb{R}$ *be a bounded real-valued function on the closed bounded interval* $[a,b]$ *and let let* m *and* M *be real constants detetermined so that* $m \leq f(x) \leq M$ *for all* $x \in [a,b]$. *Let* P *be a partition of the interval* $[a,b]$, *and let* $P = \{u_0, u_1, \ldots u_N\}$, *where*

$$a = u_0 < u_1 < \cdots < u_N = b.$$

For each integer i *between* 1 *and* r *let*

$$m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\} \quad \textit{and} \quad M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}.$$

Let z *be a real number satisfying* $0 < z < \frac{1}{2}(u_i - u_{i-1})$ *for* $i = 1, 2, \ldots, N$, *and let* $p_z\colon [a,b] \to \mathbb{R}$ *and* $q_z\colon [a,b] \to \mathbb{R}$ *be the continuous and piecewise linear functions defined so that*

$$p_z(x) = \frac{(u_{i-1} + z - x)m + (x - u_{i-1})m_i}{z} \quad (\textit{whenever } u_{i-1} \leq x \leq u_{i-1} + z)),$$

$$q_z(x) = \frac{(u_{i-1} + z - x)M + (x - u_{i-1})M_i}{z} \quad (\textit{whenever } u_{i-1} \leq x \leq u_{i-1} + z)),$$

whenever $u_{i-1} \leq x \leq u_{i-1} + z$,

$$p_z(x) = m_i \quad (\textit{whenever } u_{i-1} + z \leq x \leq u_i - z)),$$

$$q_z(x) = M_i \quad (\textit{whenever } u_{i-1} + z \leq x \leq u_i - z)),$$

$$p_z(x) = \frac{(u_i - x)m_i + (x - u_i + z)m}{z} \quad (\textit{whenever } u_i - z \leq x \leq u_i),$$

$$q_z(x) = \frac{(u_i - x)M_i + (x - u_i + z)M}{z} \quad (\textit{whenever } u_i - z \leq x \leq u_i).$$

Note that these functions p_z, q_z are indeed continuous and piecewise linear and satisfy $p_z(u_i) = m$ and $q_z(u_i) = M$ for $i = 0, 1, 2, \ldots, N$. Also $p_z(u_{i-1} + z) = p_z(u_i - z) = m_i$ and $q_z(u_{i-1} + z) = q_z(u_i - z) = M_i$ for $i = 1, 2, \ldots, N$. Moreover $p_z(x) \leq f(x) \leq q_z(x)$ for all $x \in [a, b]$.

4. *In the context just described, calculate the values of the formal integrals $\mathbf{FI}(p_z; a, b)$ and $\mathbf{FI}(q_z; a, b)$ for the functions p_z and q_z on the interval $[a, b]$, and prove that*

$$\lim_{z \to 0^+} \mathbf{FI}(p_z; a, b) = L(P, f) \quad \textit{and} \quad \lim_{z \to 0^+} \mathbf{FI}(q_z; a, b) = U(P, f).$$

$$\begin{aligned} \mathbf{FI}(p_z; a, b) &= 2 \times \tfrac{1}{2} \times \sum_{i=1}^{N} (m + m_i) z + \sum_{i=1}^{N} m_i (u_i - u_{i-1} - 2z) \\ &= \sum_{i=1}^{N} m_i (u_i - u_{i-1}) - z \sum_{i=1}^{N} (m_i - m) \\ &= L(P, f) - z \sum_{i=1}^{N} (m_i - m) \end{aligned}$$

and

$$\begin{aligned} \mathbf{FI}(q_z; a, b) &= 2 \times \tfrac{1}{2} \times \sum_{i=1}^{N} (M + M_i) z + \sum_{i=1}^{N} M_i (u_i - u_{i-1} - 2z) \\ &= \sum_{i=1}^{N} M_i (u_i - u_{i-1}) + z \sum_{i=1}^{N} (M - M_i) \\ &= U(P, f) + z \sum_{i=1}^{N} (M - M_i). \end{aligned}$$

Therefore $\lim_{z \to 0^+} \mathbf{FI}(p_z; a, b) = L(P, f)$ and $\lim_{z \to 0^+} \mathbf{FI}(q_z; a, b) = U(P, f)$.

5. *Using the results proved in previous questions, prove that the lower Riemann integral $\mathcal{L} \int_a^b f(x)\, dx$ is the least upper bound of the formal integrals $\mathbf{FI}(p; a, b)$ as p ranges over all continuous and piecewise-linear functions on the interval $[a, b]$ that satisfy $p(x) \leq f(x)$ for all $x \in [a, b]$. Similarly prove that the upper Riemann integral $\mathcal{U} \int_a^b f(x)\, dx$ is the greatest lower bound of the formal integrals $\mathbf{FI}(q; a, b)$ as q ranges over all continuous and piecewise-linear functions on the interval $[a, b]$ that satisfy $q(x) \geq f(x)$ for all $x \in [a, b]$.*

It follows from Question 3 above that $\mathcal{L}\int_a^b f(x)\,dx$ is an upper bound on the values of $\mathbf{FI}(p,a,b)$ as p ranges over all continuous and piecewise-linear functions on the interval $[a,b]$ that satisfy $p(x) \leq f(x)$ for all $x \in [a,b]$. But it follows from Question 4 above that, given any real number c satisfying $c < \mathcal{L}\int_a^b f(x)\,dx$ there exists a positive real number z small enough to ensure that $\mathbf{FI}(p_z;a,b) > c$. Therefore $\mathcal{L}\int_a^b f(x)\,dx$ is the least upper bound of the quantities $\mathbf{FI}(p;a,b)$ with p as above.

Similarly $\mathcal{U}\int_a^b f(x)\,dx$ is an upper bound on the values of $\mathbf{FI}(q,a,b)$ as p ranges over all continuous and piecewise-linear functions on the interval $[a,b]$ that satisfy $q(x) \geq f(x)$ for all $x \in [a,b]$. But it follows from Question 4 above that, given any real number C satisfying $C > \mathcal{L}\int_a^b f(x)\,dx$ there exists a positive real number z small enough to ensure that $\mathbf{FI}(q_z;a,b) < C$. Therefore $\mathcal{U}\int_a^b f(x)\,dx$ is the greatest lower bound of the quantities $\mathbf{FI}(q;a,b)$ with q as above.