Course MA2224: Hilary Term 2019.

Assignment II.

To be handed in by Friday 29th March, 2019.

Students are reminded that they must comply with College policies with regard to plagiarism, which are published on the website located at the following URL:

`http://tcd-ie.libguides.com/plagiarism`

Please complete the attached cover sheet and attach it to your assignment, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear or logically confused will not gain substantial credit.

1. Throughout this question, let a be a positive real number, and let $s: \mathbb{R} \to \mathbb{R}$ be an integrable simple function with the following properties:

   - $s(x) \geq 0$ for all $x \in \mathbb{R}$;
   - $s(x) = 0$ whenever $x < a$; $s(x) \leq 1/x^2$ whenever $x \geq a$.

   Also let $c_1, c_2, \ldots, c_m$ be the non-zero values taken on by the function s, where
   $$a^{-2} \geq c_1 > c_2 > \cdots > c_m > 0,$$
   and let $b_j = 1/\sqrt{c_j}$ for $j = 1, 2, \ldots, n$. Let
   $$E_j = \{x \in \mathbb{R} : s(x) = c_j\}$$
   for $j = 1, 2, \ldots, m$. Then the sets $E_1, E_2, \ldots, E_m$ are pairwise disjoint Lebesgue-measurable subsets of $\mathbb{R}$ that satisfy $\mu(E_j) < +\infty$, where $\mu(E_j)$ denotes the Lebesgue measure of the set E_j for $j = 1, 2, \ldots, m$. Also $a \leq b_1 < b_2 < \cdots < \cdots b_m$. The objective of this question is to show that
   $$\int_{\mathbb{R}} s(x)\, dx \leq \frac{1}{a}.$$

   (a) Explain why $s(x) = 0$ for all $x \in \mathbb{R}$ satisfying $x > b_m$.

   Let
   $$s_1(x) = \begin{cases} s(x) & \text{if } a \leq x \leq b_1, \\ 0 & \text{otherwise,} \end{cases}$$
   and, for $j = 1, 2, \ldots, m$, let
   $$s_j(x) = \begin{cases} s(x) & \text{if } b_{j-1} < x \leq b_j; \\ 0 & \text{otherwise.} \end{cases}$$
   Note that $s_1 + s_2 + \cdots + s_m = s$.

   (b) Explain why $s_j(x) \leq b_j^{-2}$ for $j = 1, 2, \ldots, m$.

   (c) Explain why
   $$\int_{\mathbb{R}} s(x)\, dx \leq \frac{b_1 - a}{b_1^2} + \sum_{j=2}^{m} \frac{b_j - b_{j-1}}{b_j^2}$$

(d) Explain why

$$\frac{b_1 - a}{b_1^2} \leq \frac{1}{a} - \frac{1}{b_1}$$

and

$$\frac{b_j - b_{j-1}}{b_j^2} \leq \frac{1}{b_{j-1}} - \frac{1}{b_j}$$

for $j = 2, 3, \ldots, m$.

(e) Hence prove that

$$\int_{\mathbb{R}} s(x)\, dx \leq \frac{1}{a}.$$

2. Let a be a positive real number, let r be a real number satisfying $r > 1$, let N be an integer greater than one, and let $t_{r,N}\colon \mathbb{R} \to \mathbb{R}$ be the integrable simple function defined such that $t_{r,N}(x) = 0$ whenver $x < a$ or $x > ar^N$, $t_{r,N}(a) = a^{-2}$ and $t_{r,N}(x) = (ar^j)^{-2}$ whenever $ar^{j-1} < x \leq ar^j$ for some integer j satisfying $1 \leq j \leq N$. Determine the value of $\int_{\mathbb{R}} t_{r,N}(x)\, dx$.

3. Making use of the results established in the two previous question, as appropriate, explain why, for any positive real number a, the quantity a^{-1} is the least upper bound of $\int_{\mathbb{R}} s(x)\, dx$ taken over all integrable simple functions $s\colon \mathbb{R} \to \mathbb{R}$ with the properties that $s(x) = 0$ whenever $x < a$ and $s(x) \leq x^{-2}$ whenever $x \geq a$.

4. Let E be the set of all irrational numbers x satisfying $0 < x < 1$.

(a) Is the subset E of the real line Lebesgue-measurable?

(b) What is the value of $\mu^*(E)$, where $\mu^*(E)$ denotes the Lebesgue outer measure of the set E?

(c) Does there exist a subset F of E that is a countable union of intervals and that also satisfies $\mu^*(F) > 0$? [Justify your answer.]

Module MA2224—Lebesgue Integral,
Hilary Term 2019.
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Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at

`http://www.tcd.ie/calendar`

I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at

`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

Signed:

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