Course MA2224: Hilary Term 2019.

Assignment 1.

To be handed in by Friday 15th March, 2019.

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Please complete the attached cover sheet and attach it to your assignment, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear or logically confused will not gain substantial credit.

We set out below some definitions to which we will make reference to within this assignment.

Definition Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on a closed bounded interval $[a,b]$. We say that the function f is *continuous and piecewise linear* if there exists a partition P of the interval $[a,b]$ with division points $u_0, u_1, \ldots, u_N$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b$$

such that

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ and for all real numbers x satisfying $u_{i-1} \leq x \leq u_i$.

Definition Let $f\colon [a,b] \to \mathbb{R}$ be a continuous and piecewise linear function defined on an interval $[a,b]$. Let us say that a real number u satisfying $a < u < b$ is a *joint* of the function if the gradient of the function differs on both sides of u.

Note that if u is a joint of a continuous and piecewise linear function f on $[a,b]$, then the derivative of the function is not defined at u. On the other hand, if u is not a joint of the function then the derivative of the function is defined and is constant throughout some sufficiently small neighbourhood of the value u. Note also that if P is a partition of $[a,b]$ with division points $u_0, u_1, \ldots, u_N$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b,$$

and if

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ and for all real numbers x satisfying $u_{i-1} \leq x \leq u_i$ (as in the definition of continuous and piecewise linear functions given above), then the joints of f must necessarily be division points of the partition P.

Note also that the sum of a finite number of continuous and piecewise linear functions on the interval $[a,b]$ must itself be a continuous and piecewise linear function on that interval. Indeed let $f_1, f_2, \ldots, f_s$ be continuous and piecewise linear functions on the interval $[a,b]$. Then there exists a partition P of $[a,b]$ that includes as division points all the joints of the functions $f_1, f_2, \ldots, f_s$. Then

$$f_k(x) = \frac{u_i - x}{u_i - u_{i-1}} f_k(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f_k(u_i)$$

for $k = 1, 2, \ldots, s$, $i = 1, 2, \ldots, N$ and for all real numbers x satisfying $u_{i-1} \leq x \leq u_i$. Let $f\colon [a,b] \to \mathbb{R}$ be defined such that $f(x) = \sum_{k=1}^{s} f_k(x)$ for all $x \in [a,b]$. Then, on summing the identities satisfied by the functions f_k for $k = 1, 2, \ldots, s$, we find that

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ and for all real numbers x satisfying $u_{i-1} \leq x \leq u_i$, and thus the function f is indeed continuous and piecewise linear.

1. Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on a closed bounded interval that is continuous and piecewise linear on that interval, and let P be a partition of that interval whose division points $u_0, u_1, \ldots, u_N$ include all the joints of the function f, so that

$$f(x) = \frac{u_i - x}{u_i - u_{i-1}} f(u_{i-1}) + \frac{x - u_{i-1}}{u_i - u_{i-1}} f(u_i)$$

for $i = 1, 2, \ldots, N$ and for all real numbers x satisfying $u_{i-1} \leq x \leq u_i$. Let

$$S_P = \tfrac{1}{2} \sum_{i=1}^{N} (f(u_{i-1}) + f(u_i))(u_i - u_{i-1}).$$

Let S_Q be the real number defined in the same fashion for the partition Q, where the partition Q is obtained from the partition P by adding an extra division point w in the interior of the subinterval $[u_{k-1}, u_k]$ of the partition P. Verify by direct calculation that $S_Q = S_P$.

Remarks. Note that it follows from the result of (a) above that if $f\colon [a,b] \to \mathbb{R}$ is a continuous and piecewise linear function on the interval $[a,b]$, if P is a partition which includes all the joints of the function f, and if R is a refinement of P, and if the quantities S_P and S_R are defined for each of these partitions in the manner specified, then $S_R = S_P$. This follows directly from the observation that one can pass from the partition P to the refinement R by successively adding a finite number of extra division points, and the addition of each division point preserves the value of the associated quantity. Now any two partitions P and Q of the interval $[a,b]$ have a common refinement R. Thus if the partitions P and Q each contain all the joints of the function f then $S_P = S_R = S_Q$.

This motivates and justifies the following definition.

Definition Let $f\colon [a,b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on a closed bounded interval $[a,b]$. We define the *formal integral* $\mathbf{FI}(f; a, b)$ of the function f on the interval $[a,b]$ to be the value of the quantity

$$S_P = \tfrac{1}{2} \sum_{i=1}^{N} (f(u_{i-1}) + f(u_i))(u_i - u_{i-1}),$$

where $u_0, u_1, u_2, \ldots, u_N$ satisfy

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b$$

and where the list $u_0, u_1, u_2, \ldots, u_N$ includes all the joints of the function f.

It should be noted that the value $\mathbf{FI}(f; a, b)$ of this formal integral is equal to the value of the integral $\int_a^b f(x)\,dx$ of the continuous and piecewise linear function f, when this integral is evaluated by standard methods. However the definition given of the formal integral $\mathbf{FI}(f; a, b)$ is purely algebraic, and the given definition does not make use of any limiting or approximation process.

It should also be noted that if f and g are both continuous and piecewise linear real-valued functions on the interval $[a,b]$ then

$$\mathbf{FI}((f+g); a, b) = \mathbf{FI}(f; a, b) + \mathbf{FI}(g; a, b).$$

Moreover if $f(x) \leq g(x)$ for all $x \in [a,b]$ then $\mathbf{FI}(f; a, b) \leq \mathbf{FI}(g; a, b)$. These results follow directly by evaluating the formal integrals with respect to a partition of $[a,b]$ that includes as division points the joints of both functions.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on the closed bounded interval $[a,b]$, and let K be a real constant that exceeds the absolute values of the slopes of the function f, and let P be a partition of $[a,b]$ that includes as division points the joints of $[a,b]$. Moreover let $P = \{u_0, u_1, \ldots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

(The conditions listed above ensure that $|f(u_i) - f(u_{i-1})| \leq K(u_i - u_{i-1})$ for $i = 1, 2, \ldots, N$.) Prove that

$$\mathbf{FI}(f; a, b) = \tfrac{1}{2}(L(P, f) + U(P, f)),$$

where $L(P, f)$ and $U(P, f)$ denote the Darboux lower sum and upper sum for the partition P and the function f.

Also let δ be a positive real number, and let the partition P be chosen such that $u_i - u_{i-1} < \delta$ for $i = 1, 2, \ldots, N$. Prove that

$$U(P, f) - L(P, f) \leq K(b - a)\delta$$

(On each subinterval $[u_{i-1}, u_i]$ of the partition, you should explicitly consider both the case where $f(x_{i-1}) \leq f(x_i)$ and also the case where $f(x_{i-1}) \geq f(x_i)$.)

3. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded real-valued function on the closed bounded interval $[a, b]$ and let $p\colon [a, b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on $[a, b]$ that satisfies $p(x) \leq f(x)$ for all $x \in [a, b]$. Using the result of the previous question, or otherwise, prove that, given any real number ε satisfying $\varepsilon > 0$, there exists a partition P of $[a, b]$ with the property that $L(P, f) > \mathbf{FI}(p; a, b) - \varepsilon$. Then use this result to prove that

$$\mathbf{FI}(p; a, b) \leq \mathcal{L} \int_a^b f(x)\, dx,$$

where $\mathcal{L} \int_a^b f(x)\, dx$ denotes the *lower Riemann integral* of the function f on $[a, b]$.

Remarks. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded real-valued function on the closed bounded interval $[a, b]$ and let $q\colon [a, b] \to \mathbb{R}$ be a continuous and piecewise linear real-valued function on $[a, b]$ that satisfies $q(x) \geq f(x)$ for all $x \in [a, b]$. Let $g\colon [a, b] \to \mathbb{R}$ and $p\colon [a, b] \to \mathbb{R}$ be defined such that $g(x) = -f(x)$ and $p(x) = -q(x)$ for all $x \in [a, b]$. It follows from the inequality proved in the previous question that

$$\mathbf{FI}(q; a, b) = -\mathbf{FI}(p; a, b) \geq -\mathcal{L} \int_a^b g(x)\, dx = \mathcal{U} \int_a^b f(x)\, dx.$$

Thus $\mathbf{FI}(q; a, b) \geq \mathcal{U} \int_a^b f(x)\, dx$ for all continuous and piecewise linear functions $q\colon [a, b] \to \mathbb{R}$ satisfying $q(x) \geq f(x)$ for all $x \in [a, b]$.

We now set up some notation for the next question.

Let $f\colon [a, b] \to \mathbb{R}$ be a bounded real-valued function on the closed bounded interval $[a, b]$ and let let m and M be real constants detetermined so that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Let P be a partition of the interval $[a, b]$, and let $P = \{u_0, u_1, \ldots u_N\}$, where

$$a = u_0 < u_1 < \cdots < u_N = b.$$

For each integer i between 1 and r let

$$m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\} \quad \text{and} \quad M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}.$$

Let z be a real number satisfying $0 < z < \frac{1}{2}(u_i - u_{i-1})$ for $i = 1, 2, \ldots, N$, and let $p_z\colon [a, b] \to \mathbb{R}$ and $q_z\colon [a, b] \to \mathbb{R}$ be the continuous and piecewise linear functions defined so that

$$p_z(x) = \frac{(u_{i-1} + z - x)m + (x - u_{i-1})m_i}{z} \quad \text{(whenever } u_{i-1} \leq x \leq u_{i-1} + z)),$$

$$q_z(x) = \frac{(u_{i-1} + z - x)M + (x - u_{i-1})M_i}{z} \quad \text{(whenever } u_{i-1} \leq x \leq u_{i-1} + z)),$$

whenever $u_{i-1} \leq x \leq u_{i-1} + z$,

$$p_z(x) = m_i \quad \text{(whenever } u_{i-1} + z \leq x \leq u_i - z)),$$

$$q_z(x) = M_i \quad \text{(whenever } u_{i-1} + z \leq x \leq u_i - z)),$$

$$p_z(x) = \frac{(u_i - x)m_i + (x - u_i + z)m}{z} \quad \text{(whenever } u_i - z \leq x \leq u_i),$$

$$q_z(x) = \frac{(u_i - x)M_i + (x - u_i + z)M}{z} \quad \text{(whenever } u_i - z \leq x \leq u_i).$$

Note that these functions p_z, q_z are indeed continuous and piecewise linear and satisfy $p_z(u_i) = m$ and $q_z(u_i) = M$ for $i = 0, 1, 2, \ldots, N$. Also $p_z(u_{i-1} + z) = p_z(u_i - z) = m_i$ and $q_z(u_{i-1} + z) = q_z(u_i - z) = M_i$ for $i = 1, 2, \ldots, N$. Moreover $p_z(x) \leq f(x) \leq q_z(x)$ for all $x \in [a, b]$.

4. In the context just described, calculate the values of the formal integrals $\mathbf{FI}(p_z; a, b)$ and $\mathbf{FI}(q_z; a, b)$ for the functions p_z and q_z on the interval $[a, b]$, and prove that

$$\lim_{z \to 0^+} \mathbf{FI}(p_z; a, b) = L(P, f) \quad \text{and} \quad \lim_{z \to 0^+} \mathbf{FI}(q_z; a, b) = U(P, f).$$

5. Using the results proved in previous questions, prove that the lower Riemann integral $\mathcal{L}\int_a^b f(x)\,dx$ is the least upper bound of the formal integrals $\mathbf{FI}(p; a, b)$ as p ranges over all continuous and piecewise-linear functions on the interval $[a, b]$ that satisfy $p(x) \leq f(x)$ for all $x \in [a, b]$. Similarly prove that the upper Riemann integral $\mathcal{U}\int_a^b f(x)\,dx$ is the greatest lower bound of the formal integrals $\mathbf{FI}(q; a, b)$ as q ranges over all continuous and piecewise-linear functions on the interval $[a, b]$ that satisfy $q(x) \geq f(x)$ for all $x \in [a, b]$.

Module MA2224—Lebesgue Integral,
Hilary Term 2019.
Assignment I.

Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

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I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at

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Signed:

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