MA1S11 (Calculus) - Sample Examination Questions 2017

5. (a) Determine the unique rational numbers p, q and r for which

$$\frac{\sqrt[3]{x^{\frac{1}{4}} - x^{\frac{2}{9}}}}{\sqrt[4]{x^{\frac{2}{5}} - x^{\frac{1}{6}}}} = \frac{x^p \sqrt[3]{1 - x^q}}{\sqrt[4]{1 - x^r}}$$

for all real numbers x satisfying $0 < x < 1$.

(10 marks)

(b) For each real number k let $p_k(x)$ be the polynomial in x defined so that

$$p_k(x) = x^2 - (1 + k)x + k.$$

For which values of k (if any) does the polynomial $p_k(x)$ have just one real root? For which values of k (if any) does this polynomial have two distinct real roots?

(4 marks)

(c) Let $f: D \to \mathbb{R}$ be a strictly increasing function defined over a subset D of the set $\mathbb{R}$ of real numbers. Prove that the function $f: D \to \mathbb{R}$ is injective.

(6 marks)

6. (a) Let $f\colon (0,+\infty) \to \mathbb{R}$ be the real-valued function on the set $(0,+\infty)$ of positive real numbers defined such that

$$f(x) = (2x^2 + 3x^3)\,\sin\frac{1}{x}.$$

Write down a formula, expressing δ as a function of ε, that determines, for each real number ε satisfying $0 < \varepsilon \leq 1$, a corresponding strictly positive real number δ with the property that $|f(x)| < \varepsilon$ for all real numbers x satisfying $0 < x < \delta$.

(10 marks)

(b) Let f, g and h be real-valued functions defined over a subset D of the set $\mathbb{R}$ of real numbers, let s be a limit point of D, and let L be a real number. Suppose that $f(x) \leq g(x) \leq h(x)$ for all real numbers x satisfying $x \neq s$ that belong to D. Suppose also that

$$\lim_{x\to s} f(x) = \lim_{x\to s} h(x) = L,$$

so that the real number L is the limit both of $f(x)$ and of $h(x)$ as x tends to s in D. Prove that

$$\lim_{x\to s} g(x) = L.$$

[You are thus asked to prove the validity of the *Squeeze Theorem*.]

(10 marks)

7. Let $f\colon [1,6] \to \mathbb{R}$ be the real-valued function on the interval $[1,6]$ defined so that

$$f(x) = \frac{6x^2 - 91x - 936}{\sqrt[7]{x}}.$$

(a) Determine all values of x in the interval $[1,6]$ (if any) at which the function f achieves local minima, and also all values (if any) at x in this interval at which the function f achieves local maxima.

(12 marks)

(b) Let $g\colon [3,6] \to \mathbb{R}$ be the function from $[3,6]$ to $\mathbb{R}$ defined so that $g(x) = f(x)$ for all $x \in [3,6]$. Is the function g injective? [Briefly justify your answer.]

(4 marks)

(c) What is the derivative with respect to x of

$$\sin^3 \frac{1}{1+2x^2}?$$

(4 marks)

8. Evaluate the following definite integrals:—

(i) $\displaystyle\int_2^3 (8x^3 - 6x^2 + 3)\,dx;$

(5 marks)

(ii) $\displaystyle\int_0^{\sqrt[3]{\frac{1}{2}\pi}} 6x^2 \cos x^3\,dx;$

(5 marks)

(iii) $\displaystyle\int_3^6 \frac{(x+3)^2}{\sqrt{x-2}}\,dx;$

(5 marks)

(iv) $\displaystyle\int_0^{\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx.$

(5 marks)