Module MA1S11 (Calculus)
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Section 7: Integration

D. R. Wilkins

Contents

7 Integration

7.1 Darboux Sums of a Bounded Function

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral.*

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on a closed interval $[a,b]$ that is bounded above and below on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Then there exist real numbers m and M such that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. We seek to define a quantity $\int_a^b f(x)\,dx$, the *definite integral* of the function f on the interval $[a,b]$, where the value of this quantity represents the area "below" the graph of the function where the function is positive, minus the area "above" the graph of the function where the function is negative.

We now introduce the definition of a *partition* of the interval $[a,b]$.

Definition A *partition* P of an interval $[a,b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

A partition P of the closed interval $[a,b]$ provides a decomposition of that interval as a union of the subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Successive subintervals of the partition intersect only at their endpoints.

Let P be a partition of the interval $[a,b]$. Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $x_0, x_1, \ldots, x_n$ are real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The values of the bounded function $f\colon [a,b] \to \mathbb{R}$ satisfy $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. It follows that, for each integer i between i and n, the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

is a set of real numbers that is bounded below by m and bounded above by M. The Least Upper Bound Principle then ensures that the set $\{f(x) \mid x_{i-1} \leq x \leq x_i\}$ has a well-defined greatest lower bound and a well-defined least upper bound (see the discussion of least upper bounds and greatest lower bounds in Subsections 1.1.15 to 1.1.19).

For each integer i between 1 and n, let us denote by m_i the greatest lower bound on the values of the function f on the interval $[x_{i-1}, x_i]$, and let us denote by M_i the least upper bound on the values of the function f on the interval $[x_{i-1}, x_i]$, so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Then the interval $[m_i, M_i]$ can be characterized as the smallest closed interval in $\mathbb{R}$ that contains the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

We now consider what the values of the greatest lower bound and least upper bound on the values of the function are determined in particular cases where the function has some special behaviour.

First suppose that the function f is non-decreasing on the interval $[x_{i-1}, x_i]$. Then $m_i = f(x_{i-1})$ and $M_i = f(x_i)$, because in this case the values of the function f satisfy $f(x_{i-1}) \leq f(x) \leq f(x_i)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$.

Next suppose that the function f is non-increasing on the interval $[x_{i-1}, x_i]$. Then $m_i = f(x_i)$ and $M_i = f(x_{i-1})$, because in this case the values of the function f satisfy $f(x_{i-1}) \geq f(x) \geq f(x_i)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$.

Next suppose that that the function f is continuous on the interval $[x_{i-1}, x_i]$. The Extreme Value Theorem (Theorem 4.29) then ensures the existence of real numbers u_i and v_i, where $x_{i-1} \leq u_i \leq x_i$ and $x_{i-1} \leq v_i \leq x_i$ with the property that

$$f(u_i) \leq f(x) \leq f(v_i)$$

for all real numbers x satisfying $x_{i-1} \leq u_i \leq x_i$. Then $m_i = f(u_i)$ and $M_i = f(v_i)$.

Finally consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined such that $f(x) = x - \lfloor x \rfloor$ for all real numbers x, where $\lfloor x \rfloor$ is the greatest integer satisfying the inequality $\lfloor x \rfloor \leq x$. Then $0 \leq f(x) < 1$ for all real numbers x. If the interval $[x_{i-1}, x_i]$ includes an integer in its interior then

$$\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} = 1,$$

and thus $M_i = 1$, even though there is no real number x for which $f(x) = 1$.

We now summarize the essentials of the discussion so far.

The function $f\colon [a, b] \to \mathbb{R}$ is a bounded function on the closed interval $[a, b]$, where a and b are real numbers satisfying $a < b$. There then exist real numbers m and M such that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$. We are given also a partition P of the interval $[a, b]$. This partition P is representable as a finite set of real numbers in the interval $[a, b]$ that includes the endpoints of the interval. Thus

$$P = \{x_0, x_1, \ldots, x_n\}$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The quantities m_i and M_i are then defined so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

for $i = 1, 2, \ldots, n$. Then $m_i \leq f(x) \leq M_i$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. Moreover $[m_i, M_i]$ is the smallest closed interval that contains all the values of the function f on the interval $[x_{i-1}, x_i]$.

Definition Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function defined on a closed bounded interval $[a, b]$, where $a < b$, and let the partition P be a partition of $[a, b]$ given by $P = \{x_0, x_1, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then the *lower sum* (or *lower Darboux sum*) $L(P, f)$ and the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ are defined so that

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}$.

Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{n} (x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Remark Let us consider how the lower and upper sum of a bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ are related to the notion of the area "under the graph of the function f" on the interval a, in the case where the function f is non-negative on the interval $[a,b]$. Thus suppose that $f(x) \geq 0$ for all $x \in [a,b]$, and let X denote the region of the plane bounded by the graph of the function f from $x = a$ to $x = b$ and the lines $x = a$, $x = b$ and $y = 0$. Then

$$X = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\},$$

where $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. (The elements of $\mathbb{R}^2$ are then regarded as Cartesian coordinates of points of the plane.)

For each integer i let

$$\begin{aligned} X_i &= \{(x,y) \in X \mid x_{i-1} \leq x \leq x_i\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x_{i-1} \leq x \leq x_i \text{ and } 0 \leq y \leq f(x)\}. \end{aligned}$$

If the regions X and X_i have well-defined areas for $i = 1, 2, \ldots, n$ satisfying the properties that areas of planar regions are expected to satisfy, then

$$\operatorname{area}(X) = \sum_{i=1}^{n} \operatorname{area}(X_i),$$

because, where subregions X_i for different values of i intersect one another, they intersect only along their bounding edges.

Let i be an integer between 1 and n. Then $0 \leq m_i \leq f(x)$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. It follows that the rectangle with vertices $(x_{i-1}, 0)$, $(x_i, 0)$, (x_i, m_i) and (x_{i-1}, m_i) is contained in the region X_i. This rectangle has width $x_i - x_{i-1}$ and height m_i, and thus has area $m_i(x_i - x_{i-1})$. It follows that

$$m_i(x_i - x_{i-1}) \leq \operatorname{area}(X_i)$$

for all integers i between 1 and n. Summing these inequalities over i, we find that

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \operatorname{area}(X_i) = \operatorname{area}(X).$$

An analogous inequality holds for upper sums. For each integer i between x_{i-1} and x_i the region X_i of the plane $\mathbb{R}^2$ is contained within the rectangle with vertices $(x_{i-1}, 0)$, $(x_i, 0)$, (x_i, M_i) and (x_{i-1}, M_i). This rectangle has width $x_i - x_{i-1}$ and height M_i, and thus has area $M_i(x_i - x_{i-1})$. It follows that

$$M_i(x_i - x_{i-1}) \geq \operatorname{area}(X_i)$$

for all integers i between 1 and n. Summing these inequalities over i, we find that

$$U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} \text{area}(X_i) = \text{area}(X).$$

We conclude therefore that if the function f is non-negative on the interval $a, b]$, and if the region X "under the graph of the function" on the interval $[a, b]$ has a well-defined area, then

$$L(P, f) \leq \text{area}(X) \leq U(P, f).$$

7.2 Upper and Lower Integrals and Integrability

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\{U(P, f) \mid P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\{L(P, f) \mid P \text{ is a partition of } [a, b]\}. \end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P, f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$.

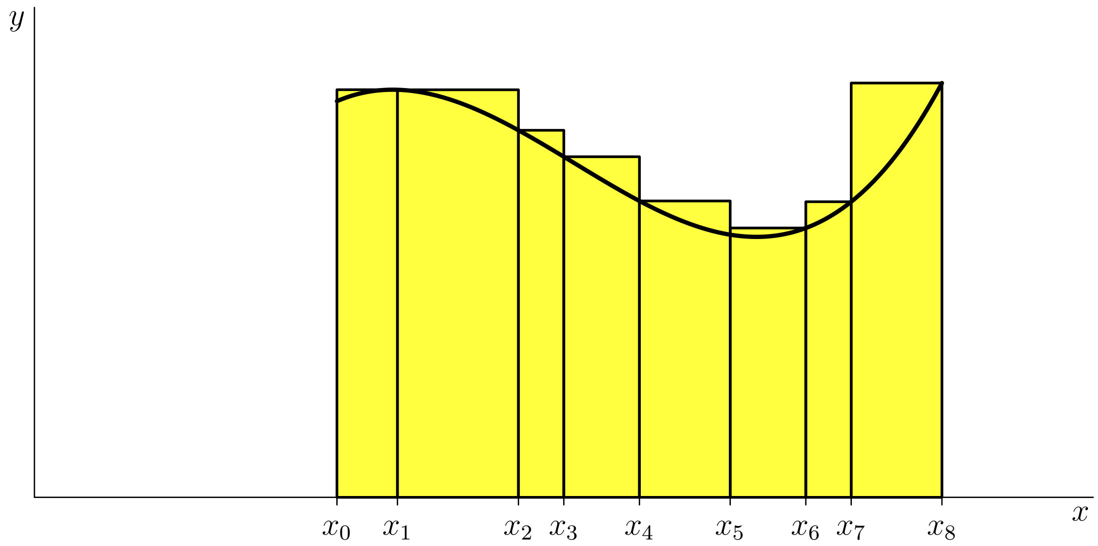
Remark Let us consider how the lower and upper Riemann integrals of a bounded function $f\colon [a, b] \to \mathbb{R}$ on a closed bounded interval $[a, b]$ are related to the notion of the area "under the graph of the function f" on the interval a, in the case where the function f is non-negative on the interval $[a, b]$. Thus suppose that the region X has a well-defined area area(X), where

$$X = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

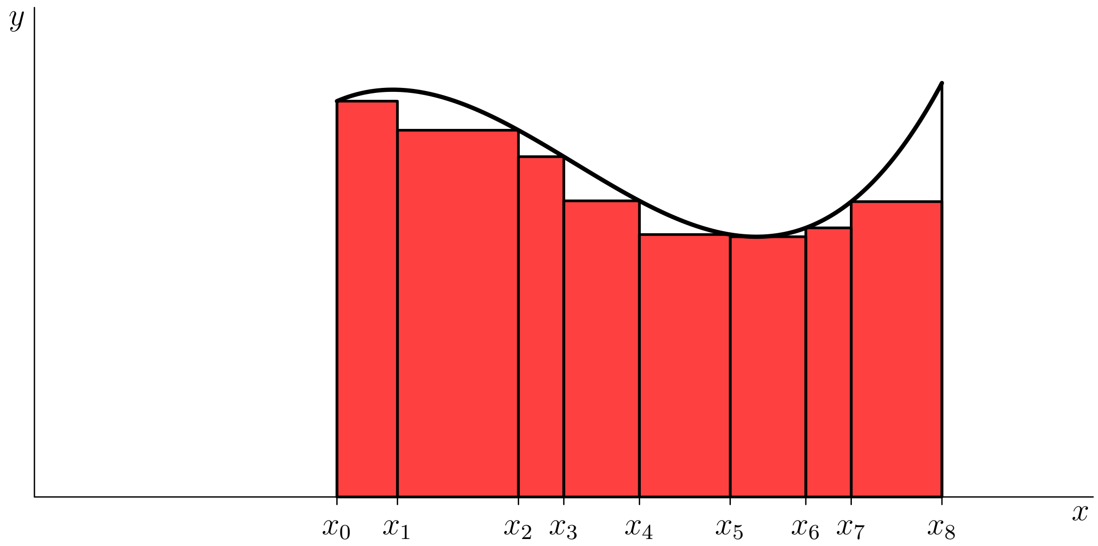
We have already shown that

$$L(P, f) \leq \text{area}(X) \leq U(P, f)$$

for all partitions P of the interval $[a, b]$. It follows that area(X) is an upper bound on all the lower sums determined by all the partitions P of $[a, b]$. It



The upper sum $U(P, f)$



The lower sum $L(P, f)$

is therefore not less than the least upper bound on all these lower sums. Therefore

$$\mathcal{L}\int_a^b f(x)\,dx \leq \text{area}(X),$$

An analogous argument shows that

$$\mathcal{U}\int_a^b f(x)\,dx \geq \text{area}(X).$$

Thus if the region X has a well-defined area, then that area must satisfy the inequalities

$$\mathcal{L}\int_a^b f(x)\,dx \leq \text{area}(X) \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P,f) \leq L(P,g)$ and $U(P,f) \leq U(P,g)$ for all partitions P of $[a,b]$.

We recall the basic definitions associated with the definition of the Riemann integral (or Riemann-Darboux) integral of a bounded real-valued function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a < b$. The function f is required to be bounded, and therefore there exist real numbers m and M with the property that $m \leq f(x) \leq M$ for all real numbers x satisfying $a \leq x \leq b$.

A *partition* P of the interval $[a,b]$, may be specified in the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $x_0, x_1, \ldots, x_n$ are real numbers satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

The quantities m_i and M_i are defined for $i = 1, 2, \ldots, n$ so that

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

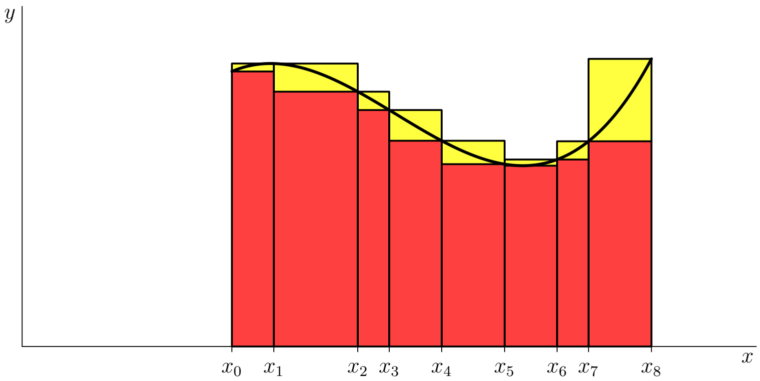
$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Then the interval $[m_i, M_i]$ can be characterized as the smallest closed interval in $\mathbb{R}$ that contains the set

$$\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

The *Darboux lower sum* $L(P, f)$ and *Darboux upper sum* $U(P, f)$ determined by the function f and the partition P of the interval $[a, b]$ are then defined by the identities

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$



The *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on the interval $[a, b]$ is defined to be the least upper bound of the Darboux lower sums $L(P, f)$ as P ranges over all partitions of the interval $[a, b]$.

Similarly the *upper Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on the interval $[a, b]$ is defined to be the greatest lower bound of the Darboux upper sums $U(P, f)$ as P ranges over all partitions of the interval $[a, b]$.

The lower and upper Riemann integrals of the function f on the interval $[a, b]$ are therefore characterized by the properties presented in the following lemmas.

Lemma 7.1 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then the lower Riemann integral is the unique real number characterized by the following two properties:—*

(i)

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx$$

for all partitions P of the interval $[a,b]$.

(ii) given any positive real number ε, there exists a partition P of the interval $[a,b]$ for which

$$L(P,f) > \mathcal{L}\int_a^b f(x)\,dx - \varepsilon.$$

Lemma 7.2 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Then the upper Riemann integral is the unique real number characterized by the following two properties:—*

(i)

$$U(P,f) \geq \mathcal{U}\int_a^b f(x)\,dx$$

for all partitions P of the interval $[a,b]$.

(ii) given any positive real number ε, there exists a partition P of the interval $[a,b]$ for which

$$U(P,f) < \mathcal{U}\int_a^b f(x)\,dx + \varepsilon.$$

A bounded function f on the interval $[a,b]$ is then *Riemann-integrable* if and only if

$$\mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx.$$

The *integral* $\int_a^b f(x)\,dx$ of a Riemann-integrable function f on the interval $[a,b]$ is then the common value of the upper and lower Riemann integrals.

In order to develop further the theory of integration, we introduce the notion of a *refinement* of a partition, and prove that if we replace a partition P by a refinement R of that partition, then the Darboux upper and lower sums satisfy the inequalities

$$L(R,f) \geq L(P,f) \quad \text{and} \quad U(R,f) \leq U(P,f).$$

for all bounded functions f on $[a,b]$. This result is an essential tool in developing the theory of the Riemann integral.

Definition Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 7.3 *Let R be a refinement of some partition P of $[a, b]$. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f\colon [a, b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

and

$$U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

Suppose that we add an extra division point to P to obtain a partition Q. We suppose that the extra division point z is added between x_{k-1} and x_k, where k is some integer between 1 and n, so that $x_{k-1} < z < x_k$. Let

$$\begin{aligned}
m'_k &= \inf\{f(x) \mid x_{k-1} \leq x \leq z\}, \\
M'_k &= \sup\{f(x) \mid x_{k-1} \leq x \leq z\}, \\
m''_k &= \inf\{f(x) \mid z \leq x \leq x_k\}, \\
M''_k &= \sup\{f(x) \mid z \leq x \leq x_k\}.
\end{aligned}$$

Then $m_k \leq m'_k$, $m_k \leq m''_k$, $M_k \geq M'_k$ and $M_k \geq M''_k$.

It follows that

$$\begin{aligned}
m_k(x_k - x_{k-1}) &= m_k(z - x_{k-1}) + m_k(x_k - z) \\
&\leq m'_k(z - x_{k-1}) + m''_k(x_k - z)
\end{aligned}$$

and

$$\begin{aligned} M_k(x_k - x_{k-1}) &= M_k(z - x_{k-1}) + M_k(x_k - z) \\ &\geq M'_k(z - x_{k-1}) + M''_k(x_k - z) \end{aligned}$$

But the lower sum $L(P, f)$ is the sum of the quantities $m_i(x_i - x_{i-1})$ as i ranges from 1 to n, and the lower sum $L(Q, f)$ is the analogous sum for the partition Q, obtained on replacing the summand $m_k(x_k - x_{k_1})$ by the quantity

$$m'_k(z - x_{k-1}) + m''_k(x_k - z),$$

which is no smaller than $m_k(x_k - x_{k_1})$. It follows that $L(P, f) \leq L(Q, f)$.

Similarly $U(P, f)$ is the sum of the quantities $M_i(x_i - x_{i-1})$ as i ranges from 1 to n, and the upper sum $U(Q, f)$ is the analogous sum for the partition Q, obtained on replacing the summand $M_k(x_k - x_{k-1})$ by the quantity

$$M'_k(z - x_{k-1}) + M''_k(x_k - z),$$

which is no larger than $M_k(x_k - x_{k_1})$. It follows that $U(P, f) \geq U(Q, f)$.

If the partition R of the interval $[a, b]$ is a refinement of the partition P, then one can obtain R from P by successively adding extra division points, one at a time. We have shown that the lower sums do not decrease, and the upper sums do not increase, each time a new division point is added. It follows that

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f),$$

as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. Indeed we can take R to be the partition of $[a, b]$ obtained in taking as division points all the division points belonging to the partitions P and Q. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 7.4 *Let f be a bounded real-valued function on the interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 7.3 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq$

$U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Proposition 7.5 *Let f be a bounded real-valued function on the interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then the function f is Riemann-integrable on f, with Riemann integral $\int_a^b f(x)\,dx$ if and only if the following two properties are satisfied:*

(i)

$$L(P, f) \leq \int_a^b f(x)\,dx \leq U(P, f)$$

for all partitions P of the interval $[a, b]$;

(ii) given any positive real number ε, there exists a partition P of the interval $[a, b]$ for which

$$\int_a^b f(x)\,dx - \varepsilon < L(P, f) \leq U(P, f) < \int_a^b f(x) + \varepsilon.$$

Proof Let A be a real number. Suppose that $L(P, f) \leq A \leq U(P, f)$ for all partitions P of $[a, b]$, and that, given any positive real number ε, there exists a partition P of $[a, b]$ for which $A - \varepsilon < L(P, f) \leq U(P, f) < A + \varepsilon$. It then follows from Lemma 7.1 and Lemma 7.2 that $A = \mathcal{L}\int_a^b f(x)\,dx$ and $A = \mathcal{U}\int_a^b f(x)\,dx$. Therefore the function f is Riemann-integrable on $[a, b]$, and $\int_a^b f(x)\,dx = A$.

Conversely, suppose that the function f is Riemann-integrable on $[a, b]$, with Riemann integral equal to the real number A. Then $A = \mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx$, and therefore $L(P, f) \leq A \leq U(P, f)$ for all partitions P of $[a, b]$. Moreover it follows from Lemma 7.1 and Lemma 7.2 that there exist partitions P_1 and P_2 of $[a, b]$ for which $L(P_1, f) > A - \varepsilon$ and $U(P_2, f) < A + \varepsilon$. Let P be a common refinement of the partitions P_1 and P_2. It follows from Lemma 7.3 that

$$A - \varepsilon < L(P_1, f) \leq L(P, f) \leq U(P, f) < U(P_2, f) < A + \varepsilon.$$

The result follows. ∎

Corollary 7.6 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfing $a \leq b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof Suppose that the bounded function f is Riemann-integrable on $[a,b]$. Let $A = \int_a^b f(x)\,dx$. It follows from Proposition 7.5 that, given any positive real number ε, there exists a partition P of $[a,b]$ for which

$$A - \tfrac{1}{2}\varepsilon < L(P,f) \leq U(P,f) < A + \tfrac{1}{2}\varepsilon.$$

Then $U(P,f) - L(P,f) < \varepsilon$.

for which there exists a partition P with $U(P,f) - L(P,f) < \varepsilon$. Then

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P,f)$$

(see Lemma 7.4). Therefore

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all positive real numbers ε. But the difference of the upper and lower Riemann integrals is independent of ε. It follows that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx = 0,$$

and thus the function f is Riemann-integrable on $[a,b]$, as required. ∎

Corollary 7.7 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed bounded interval $[a,b]$, where $a < b$, and let u and v be real numbers belonging to $[a,b]$. Then the function f is Riemann-integrable on the interval with endpoints u and v, and*

$$\int_u^v f(x)\,dx = -\int_v^u f(x)\,dx.$$

Proof First suppose that $a \leq u < v \leq b$. Let some positive real number ε be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let Q be the partition of $[u,v]$ consisting of the endpoints u and v of the closed interval $[u,v]$ together with those division points of P that lie in the interior of this interval. An examination of the relevant definitions shows that

$$U(Q,f) - L(Q,f) \leq U(P,f) - L(P,f) < \varepsilon.$$

It follows that if $a \leq u < v \leq b$ then the function f is Riemann-integrable on $[u,v]$. The definition of the relevant integrals then ensures that

$$\int_v^u f(x)\,dx = -\int_u^v f(x)\,dx.$$

(see subsection 7.7.2).

If $a \leq v < u \leq b$ then the required result follows from the case already proved on interchanging u and v. If $a \leq u = v \leq b$ then the integrals $\displaystyle\int_u^v f(x)\,dx$ and $\displaystyle\int_v^u f(x)\,dx$ are equal to zero, and therefore the result follows in this case also. This completes the proof. ∎

Lemma 7.8 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed interval $[a,b]$, where $a < b$. Suppose that $f(x) \leq g(x)$ for all real numbers x satisfying $a \leq x \leq b$. Then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof The Darboux lower and uppse sums of the functions f and g satisfy $L(P,f) \leq L(P,g)$ and $U(P,f) \leq U(P,g)$. It follows that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{L}\int_a^b g(x)\,dx \quad \text{and} \quad \mathcal{U}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b g(x)\,dx.$$

The result follows. ∎

Proposition 7.9 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof Let $\int_a^b f(x)\,dx = A'$ and $\int_a^b g(x)\,dx = A''$, and let $A = A' + A''$. Let some positive number ε be given. It follows from Proposition 7.5 that there exist partitions P' and P'' of $[a,b]$ that satisfy

$$A' - \tfrac{1}{2}\varepsilon < L(P',f) < U(P',f) < A' + \tfrac{1}{2}\varepsilon$$

and

$$A'' - \tfrac{1}{2}\varepsilon < L(P'', g) < U(P'', g) < A'' + \tfrac{1}{2}\varepsilon$$

Let P be a common refinement of the partitions P' and P''. Then $L(P', f) \leq L(P, f)$, $L(P'', g) \leq L(P, g)$, $U(P', f) \geq U(P, f)$ and $U(P'', g) \geq L(P, g)$, and therefore

$$A - \varepsilon < L(P, f) + L(P, g) \leq U(P, f) + U(P, g) < A + \varepsilon.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned}
M_i' &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\},\\
M_i'' &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\},\\
M_i &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\},\\
m_i' &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\},\\
m_i'' &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\},\\
m_i &= \inf\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.
\end{aligned}$$

Let i be an integer between 1 and n. Then $m_i' \leq f(x) \leq M_i'$ and $m_i'' \leq g(x) \leq M_i''$ for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$, and therefore

$$m_i' + m_i'' \leq f(x) + g(x) \leq M_i' + M_i''$$

for all real numbers x satisfying $x_{i-1} \leq x \leq x_i$. It follows that

$$m_i' + m'' \leq m_i \leq M_i \leq M_i' + M''$$

for $i = 1, 2, \ldots, n$. Multiplying by $x_i - x_{i-1}$ and summing over i, we find that

$$\begin{aligned}
\sum_{i=1}^{n} m_i'(x_i - x_{i-1}) &+ \sum_{i=1}^{n} m_i''(x_i - x_{i-1})\\
&\leq \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1})\\
&\leq \sum_{i=1}^{n} M_i'(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i''(x_i - x_{i-1}).
\end{aligned}$$

Thus

$$\begin{aligned} L(P,f) + L(P,g) &\leq L(P, f+g) \\ &\leq U(P, f+g) \leq U(P,f) + U(P,g). \end{aligned}$$

It then follows from inequalities obtained earlier in the proof that

$$A - \varepsilon < L(P, f+g) \leq L(P, f+g) < A + \varepsilon.$$

The result therefore follows on applying Proposition 7.5 to the function $f+g$ on $[a,b]$. ∎

Lemma 7.10 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed bounded interval $[a,b]$, where $a < b$, and let c be a real number. Then cf is Riemann-integrable on $[a,b]$, and*

$$\int_a^b (cf(x))\,dx = c\int_a^b f(x)\,dx.$$

Proof Let $A = \int_a^b f(x)\,dx$. The result is immediate if $c = 0$. Suppose that $c > 0$. Then $L(P, cf) = cL(P,f)$ and $U(P, cf) = cU(P,f)$ for all partitions P of $[a,b]$. It follows that $L(P, cf) \leq cA \leq U(P, cf)$ for all partitions P of $[a,b]$. Also, given any positive real number ε, there exists a partition P of $[a,b]$ for which

$$A - \varepsilon/c < L(P,f) \leq U(P,f) < A + \varepsilon/c$$

(see Proposition 7.5). But then

$$cA - \varepsilon < L(P, cf) \leq U(P, cf) < cA + \varepsilon.$$

The result therefore follows in the case when $c > 0$.

The result is also true in the case where $c = -1$, because $L(P, -f) = -U(P,f)$ and $U(P,-f) = -L(P,f)$ for all partitions P of the interval $[a,b]$. Combining these results, we see that the result is true for all real numbers c, as required. ∎

Proposition 7.11 *Let f be a bounded real-valued function on the interval $[a,c]$. Suppose that f is Riemann-integrable on the intervals $[a,b]$ and $[b,c]$, where $a < b < c$. Then f is Riemann-integrable on $[a,c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let some positive real number ε be given. There exist partitions P_1 and P_2 of $[a,b]$ and $[b,c]$ respectively for which

$$\int_a^b f(x)\,dx - \tfrac{1}{4}\varepsilon < L(P_1,f) \le U(P_1,f) < \int_a^b f(x)\,dx + \tfrac{1}{4}\varepsilon$$

$$\int_b^c f(x)\,dx - \tfrac{1}{4}\varepsilon < L(P_2,f) \le U(P_2,f) < \int_b^c f(x)\,dx + \tfrac{1}{4}\varepsilon$$

(see Proposition 7.5). The partitions P_1 and P_2 combine to give a partition P of $[a,c]$, where $P = P_1 \cup P_2$. Moreover

$$L(P,f) = L(P_1,f) + L(P_2,f) \quad \text{and} \quad U(P,f) = U(P_1,f) + U(P_2,f).$$

It follows that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \tfrac{1}{2}\varepsilon & \\ &< L(P,f) \le U(P,f) \\ &< \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \tfrac{1}{2}\varepsilon, \end{aligned}$$

and therefore $U(P,f) - L(P,f) < \varepsilon$. It now follows from Corollary 7.6 that the function f is Riemann-integrable in $[a,b]$, and then follows from Proposition 7.5 that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,$$

as required. ∎

Corollary 7.12 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a,b]$, where $a < b$. Then*

$$\int_u^w f(x)\,dx = \int_u^v f(x)\,dx + \int_v^w f(x)\,dx$$

for all real numbers u, v and w belonging to $[a,b]$.

Proof In the case where $u = w$, the result follows from the identity

$$\int_v^u f(x)\,dx = -\int_u^v f(x)\,dx$$

(see Lemma 7.7). In the case where $u = v$ and when $v = w$ the result follows from the definition of the integral, which requires that $\int_u^u f(x)\,dx = 0$ and $\int_w^w f(x)\,dx = 0$.

In the case when $u < v < w$, the result follows directly from Proposition 7.11. In the case when $u < w < v$, it follows from Proposition 7.11 that

$$\int_u^v f(x)\,dx = \int_u^w f(x)\,dx + \int_w^v f(x)\,dx$$

It then follows that

$$\begin{aligned}
\int_u^w f(x)\,dx &= \int_u^v f(x)\,dx - \int_w^v f(x)\,dx \\
&= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.
\end{aligned}$$

It then follows that if either $v < w < u$ or $v < u < w$ then

$$\int_v^u f(x)\,dx = \int_v^w f(x)\,dx + \int_w^u f(x)\,dx,$$

and therefore

$$\begin{aligned}
\int_u^w f(x)\,dx &= -\int_w^u f(x)\,dx \\
&= -\int_v^u f(x)\,dx + \int_v^w f(x)\,dx \\
&= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.
\end{aligned}$$

Finally if $w < u < v$ or $w < v < u$ then

$$\int_w^v f(x)\,dx = \int_w^u f(x)\,dx + \int_u^v f(x)\,dx,$$

and therefore

$$\begin{aligned}
\int_u^w f(x)\,dx &= -\int_w^u f(x)\,dx \\
&= \int_u^v f(x)\,dx - \int_w^v f(x)\,dx \\
&= \int_u^v f(x)\,dx + \int_v^w f(x)\,dx.
\end{aligned}$$

This completes the proof. ∎

Proposition 7.13 *Let $f: [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable real-valued function on a closed bounded interval $[a, b]$, where $a < b$, and let k be a positive real number. Then*

$$\int_a^b f(x)\, dx = k \int_{a/k}^{b/k} f(ku)\, du.$$

Proof Let $g: [a/k, b/k] \to \mathbb{R}$ be defined so that $g(u) = f(ku)$ for all real numbers u satisfying $a/k \leq u \leq b/k$. Each partition P of $[a, b]$ determines a corresponding partition Q of $[a/k, b/k]$ so that if $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

then $Q = \{u_0, u_1, \ldots, u_n\}$, where $u_i = x_i/k$ for $i = 1, 2, \ldots, n$. Then $kL(Q, g) = L(P, f)$ and $kU(Q, g) = U(P, f)$. This ensures that

$$k \int_{a/k}^{b/k} f(ku)\, du = k \int_{a/k}^{b/k} g(u)\, du = \int_a^b f(x)\, dx,$$

as required. ∎

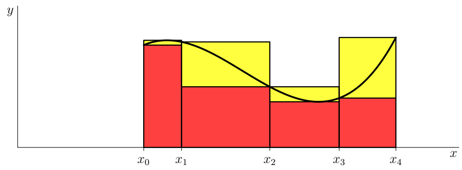
7.3 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f: [a, b] \to \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f: [a, b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f: [a, b] \to \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

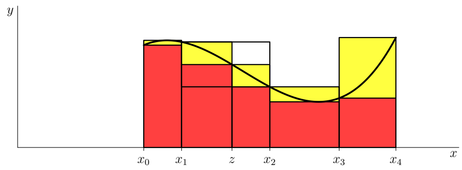
Proposition 7.14 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.*

Proof Let $f: [a, b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a, b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$, and therefore the function f is bounded on $[a, b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a, b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

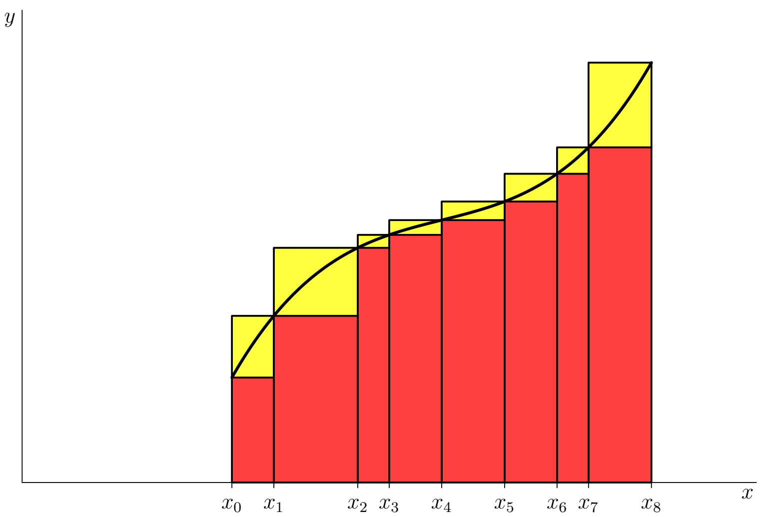
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$



Darboux sums before refinement



Darboux sums with new division point z between x_1 and x_2



and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$.

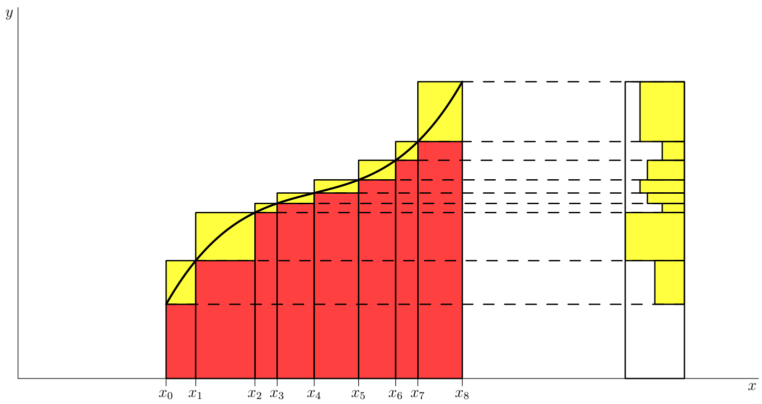
The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that



$$\begin{aligned} & U(P, f) - L(P, f) \\ & \quad = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ & \quad < \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the interval $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. It then follows from Corollary 7.6 that the function f is Riemann-integrable on $[a, b]$, as required. ∎

Corollary 7.15 *Let a and b be real numbers satisfing $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof The result follows immediately on applying the results of Proposition 7.11 and Proposition 7.14. ∎

Remark The result and proof of Proposition 7.14 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

7.4 Integrability of Continuous functions

The following theorem is stated *without proof*.

Theorem 7.16 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

7.5 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 7.16). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 7.17 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{ds}\left(\int_a^s f(x)\,dx\right) = f(s)$$

for all real numbers s satisfying $a < s < b$.

Proof Let

$$F(t) = \int_a^t f(x)\,dx$$

for all real numbers t satisfying $a \le t \le b$. If s is a real number satisfying $a < s < b$, and if h is a real number close enough to zero to ensure that $a \le h \le b$ then

$$\begin{aligned} F(s+h) &= \int_a^{s+h} f(x)\,dx = \int_a^s f(x)\,dx + \int_s^{s+h} f(x)\,dx \\ &= F(s) + \int_s^{s+h} f(x)\,dx \end{aligned}$$

(see Corollary 7.12). Also $\displaystyle\int_s^{s+h} c\,dx = hc$ for all real constants c. It follows that

$$F(s+h) - F(s) - hf(s) = \int_s^{s+h} (f(x) - f(s))\,dx$$

for all real numbers s satisfying $a < s < b$ and for all real numbers h close enough to zero to ensure that $a \leq s + h \leq b$.

Let s be a real number satisfying $a < s < b$, and let some strictly positive real number ε be given. Let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at s, where $a < s < b$. It follows that there exists some strictly positive real number δ such that $a \leq x \leq b$ and

$$f(s) - \varepsilon_0 \leq f(x) \leq f(s) + \varepsilon_0$$

for all real numbers x satisfying $s - \delta < x < s + \delta$. Now

$$-\varepsilon_0 \leq f(x) - f(s) \leq \varepsilon_0$$

for all real numbers x that lie between s and $s + h$. It follows that

$$-\varepsilon_0 |h| \leq \int_s^{s+h} (f(x) - f(s))\,dx \leq \varepsilon_0 |h|$$

for all real numbers h satisfying $0 < |h| < \delta$, Also

$$\frac{F(s+h) - F(s)}{h} - f(s) = \frac{1}{h} \int_s^{s+h} (f(x) - f(s))\,dx$$

for all real numbers h satisfying $0 < |h| < \delta$. It follows that

$$-\varepsilon < -\varepsilon_0 \leq \frac{F(s+h) - F(s)}{h} - f(s) \leq \varepsilon_0 < \varepsilon$$

for all real numbers h satisfying $0 < |h| < \delta$. We conclude from this that

$$F'(s) = \left.\frac{dF(x)}{dx}\right|_{x=s} = \lim_{h \to 0} \frac{F(s+h) - F(s)}{h} = f(s),$$

as required. ∎

Corollary 7.18 *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{ds}\left(\int_s^b f(x)\,dx\right) = -f(s)$$

for all real numbers s satisfying $a < s < b$.

Proof The integral satisfies

$$\int_a^b f(x)\,dx = \int_a^s f(x)\,dx + \int_s^b f(x)\,dx.$$

(see Proposition 7.11). Differentiating this identity, and applying the Fundamental Theorem of Calculus (Theorem 7.17), we find that

$$\begin{aligned} 0 &= \frac{d}{ds}\left(\int_a^b f(x)\,dx\right) \\ &= \frac{d}{ds}\left(\int_a^s f(x)\,dx\right) + \frac{d}{ds}\left(\int_s^b f(x)\,dx\right) \\ &= f(s) + \frac{d}{ds}\left(\int_s^b f(x)\,dx\right). \end{aligned}$$

The result follows. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}, \\ f'(b) &= \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \end{aligned}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(s) = \int_a^x f(x)\,dx$ for all $s \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h\to 0^+} \frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h\to 0^-} \frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 7.19 *Let f be a continuously-differentiable real-valued function on a closed interval with endpoints a and b. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).$$

Proof The result in the case when $b < a$ follows from that in the case when $a < b$ by interchanging the limits a and b of integration, since both sides of the identity change sign when a and b are interchanged. It therefore suffices to prove the result in the case when $a < b$.

Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfing $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 5.9) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

When evaluating definite integrals, it is customary to denote the difference in the values of a function between the endpoints of an interval by $[f(x)]_a^b$, where

$$[f(x)]_a^b = f(b) - f(a).$$

The result of Corollary 7.19 is therefore represented by the following identity: valid for all continuously-differentiable functions f on a closed interval with endpoints a and b:

$$\int_a^b \frac{df(x)}{dx}\,dx = [f(x)]_a^b = f(b) - f(a).$$

Corollary 7.20 *Let q be a rational number, where $q \neq -1$. Then*

$$\int_a^b x^q\,dx = \frac{1}{q+1}(b^{q+1} - a^{q+1})$$

for all positive real numbers a and b. Moreover this identity is valid for all real numbers a and b in the special case where q is a non-negative integer.

Proof Applying Corollary 7.19, we find that

$$\begin{aligned}\int_a^b x^q\,dx &= \frac{1}{q+1}\int_a^b \frac{d}{dx}\left(x^{q+1}\right)\,dx = \frac{1}{q+1}\left[x^{q+1}\right]_a^b \\ &= \frac{1}{q+1}(b^{q+1} - a^{q+1}),\end{aligned}$$

as required. ∎

Corollary 7.21 *Let $p(x)$ be a polynomial function of a real variable x, and let*

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n,$$

where $c_0, c_1, c_2, c_3, \ldots, c_n$ are real constants. Then

$$\int_a^b p(x)\,dx = \Big[P(x) \Big]_a^b,$$

for all real numbers a and b, where

$$P(x) = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \cdots + \frac{c_n}{n+1} x^{n+1}.$$

Proof Applying Corollary 7.19, we find that

$$\int_a^b p(x)\,dx = \int_a^b \frac{dP(x)}{dx}\,dx = [P(x)]_a^b = P(b) - P(a),$$

as required. ∎

Example We determine the value of

$$\int_1^3 (6x^2 - 4x + 3)\,dx$$

Applying Corollary 7.21 we find that

$$\begin{aligned} &\int_1^3 (6x^2 - 4x + 3)\,dx \\ &\quad = \Big[2x^3 - 2x^2 + 3x \Big]_1^3 \\ &\quad = (2 \times 3^3 - 2 \times 3^2 + 3 \times 3) - (2 \times 1^3 - 2 \times 1^2 + 3 \times 1) \\ &\quad = 42 \end{aligned}$$

Corollary 7.22 *Let k be a real number. Then*

$$\int_0^s \sin kx\,dx = \frac{1}{k}(1 - \cos ks),$$

and

$$\int_0^s \cos kx\,dx = \frac{1}{k} \sin ks.$$

Proof Applying Corollary 7.19, we find that

$$\begin{aligned}
\int_0^s \sin kx\,dx &= -\frac{1}{k}\int_0^s \frac{d}{dx}(\cos kx)\;dx = -\frac{1}{k}\left[\cos kx\right]_0^s \\
&= \frac{1}{k}(1-\cos ks)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^s \cos kx\,dx &= \frac{1}{k}\int_0^s \frac{d}{dx}(\sin kx)\;dx = \frac{1}{k}\left[\sin kx\right]_0^s \\
&= \frac{1}{k}\sin ks,
\end{aligned}$$

as required. ∎

Corollary 7.23 *Let a be a real number. Then*

$$\int_0^s \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{s}{a}\right).$$

Proof The derivative of the inverse tangent function satisfies

$$\frac{d}{du}(\arctan u) = \frac{1}{1+u^2}$$

(see Proposition 6.16). It follows from the Chain Rule (Proposition 5.5) that

$$\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{1}{a}\times\frac{1}{1+\dfrac{x^2}{a^2}} = \frac{a}{a^2+x^2}.$$

Applying Corollary 7.19, we now find that

$$\begin{aligned}
\int_0^s \frac{1}{a^2+x^2}\,dx &= \frac{1}{a}\int_0^s \frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right)\,dx \\
&= \frac{1}{a}\left[\arctan\left(\frac{x}{a}\right)\right]_0^s \\
&= \frac{1}{a}\arctan\left(\frac{s}{a}\right),
\end{aligned}$$

as required. ∎

Corollary 7.24 *Let a be a real number. Then*

$$\int_0^s \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{s}{a}\right).$$

Proof The derivative of the inverse sine function satisfies

$$\frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}$$

(see Proposition 6.18). It follows from the Chain Rule (Proposition 5.5) that

$$\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{a} \times \frac{1}{\sqrt{1-\dfrac{x^2}{a^2}}} = \frac{1}{\sqrt{a^2-x^2}}.$$

Applying Corollary 7.19, we now find that

$$\begin{aligned}\int_0^s \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int_0^s \frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right)\,dx \\ &= \left[\arcsin\left(\frac{x}{a}\right)\right]_0^s \\ &= \arcsin\left(\frac{s}{a}\right),\end{aligned}$$

as required. ∎

7.6 Integration by Parts

Proposition 7.25 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof This result follows from Corollary 7.19 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \quad ∎$$

Example We determine the value of

$$\int_0^s x\sin kx\,dx$$

where k is a non-zero real constant. Let

$$f(x) = x \quad \text{and} \quad g(x) = -\frac{1}{k}\cos kx$$

for all real numbers x. Then

$$\frac{dg(x)}{dx} = \sin kx.$$

It follows that

$$\begin{aligned}
\int_0^s x \sin kx \, dx &= \int_0^s f(x) \frac{dg(x)}{dx} \, dx \\
&= [f(x)g(x)]_0^s - \int_0^s \frac{df(x)}{dx} g(x) \, dx \\
&= -\frac{1}{k} [x \cos kx]_0^s + \frac{1}{k} \int_0^s \cos kx \, dx \\
&= -\frac{s}{k} \cos ks + \frac{1}{k^2} [\sin kx]_0^s \\
&= -\frac{s}{k} \cos ks + \frac{1}{k^2} \sin ks.
\end{aligned}$$

Thus

$$\int_0^s x \sin kx \, dx = \frac{1}{k^2} \sin ks - \frac{s}{k} \cos ks.$$

7.7 Integration by Substitution

Proposition 7.26 (Integration by Substitution) *Let $\varphi\colon [a,b] \to \mathbb{R}$ be a continuously-differentiable function on the interval $[a,b]$. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(u) \, du = \int_a^b f(\varphi(x)) \frac{d\varphi(x)}{dx} \, dx.$$

for all continuous real-valued functions f on the range $\varphi([a,b])$ of the function φ.

Proof Let $c = \varphi(a)$ and $d = \varphi(b)$, and let F and G be the functions on $[a,b]$ defined by

$$F(s) = \int_c^{\varphi(s)} f(u) du, \qquad G(s) = \int_a^s f(\varphi(x)) \frac{d\varphi(x)}{dx} dx.$$

Then $F(a) = 0 = G(a)$. Moreover $F(s) = H(\varphi(s))$, where

$$H(w) = \int_c^w f(u) \, du,$$

for all $w \in \varphi([a,b])$. Using the Chain Rule (Proposition 5.5) and the Fundamental Theorem of Calculus (Theorem 7.17), we find that

$$F'(s) = H'(\varphi(s))\varphi'(s) = f(\varphi(s))\varphi'(s) = G'(s)$$

for all $s \in (a,b)$. On applying the Mean Value Theorem (Theorem 5.9) to the function $F - G$ on the interval $[a,b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

Let x be a real variable taking values in a closed interval $[a, b]$, and let $u = \varphi(x)$ for all $x \in [a, b]$, where $\varphi\colon [a, b] \to \mathbb{R}$ be a continuously-differentiable function on the interval $[a, b]$. The rule for Integration by Substitution (Proposition 7.26) can then be stated as follows:

$$\int_{u(a)}^{u(b)} f(u)\, du = \int_a^b f(u(x)))\, \frac{du}{dx}\, dx.$$

for all continuous real-valued functions f whose domain includes $u(x)$ for all real numbers x satisfying $a \leq x \leq b$, where $u(a)$ and $u(b)$ denote the values of u when $x = a$ and $x = b$ respectively.

Example We determine the value of the integral

$$\int_0^s \frac{x^5}{\sqrt{1 - x^2}}\, dx,$$

where s is a real number satisfying $-1 < s < 1$. Let $u = \sqrt{1 - x^2}$. Then

$$\frac{du}{dx} = -2x \times \frac{1}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}.$$

Also $u^2 = 1 - x^2$ and therefore $x^2 = 1 - u^2$ and $x^4 = 1 - 2u^2 + u^4$. It follows that

$$\begin{aligned}
\int_0^s \frac{x^5}{\sqrt{1 - x^2}}\, dx &= -\int_0^s (1 - 2u^2 + u^4)\, \frac{du}{dx}\, dx \\
&= -\int_{u(0)}^{u(s)} (1 - 2u^2 + u^4)\, du \\
&= -\int_1^{\sqrt{1-s^2}} (1 - 2u^2 + u^4)\, du \\
&= -\left[u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5\right]_1^{\sqrt{1-s^2}} \\
&= \tfrac{8}{15} - \sqrt{1 - s^2}\left(1 - \tfrac{2}{3}(1 - s^2) + \tfrac{1}{5}(1 - s^2)^2\right) \\
&= \tfrac{8}{15} - \sqrt{1 - s^2}\left(\tfrac{8}{15} + \tfrac{4}{15}s^2 + \tfrac{1}{5}s^4\right).
\end{aligned}$$

Example We determine the value of the integral

$$\int_0^\pi \sin\theta\, \cos^4\theta\, d\theta$$

Let $u = \cos\theta$. Then $\dfrac{du}{d\theta} = -\sin\theta$. It follows that

$$\begin{aligned}
\int_0^\pi \sin\theta\,\cos^4\theta\,d\theta &= -\int_0^\pi u^4\,\frac{du}{d\theta}\,d\theta \\
&= -\int_{\cos(0)}^{\cos(\pi)} u^4\,du = -\int_1^{-1} u^4\,du \\
&= \int_{-1}^1 u^4\,du = \left[\frac{1}{5}u^5\right]_{-1}^1 = \frac{1}{5}(1^5 - (-1)^5) \\
&= \frac{2}{5}.
\end{aligned}$$

Example We determine the value of the integral

$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx.$$

Let $u = 1 + x^3$. Then $\dfrac{du}{dx} = 3x^2$. It follows that

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx &= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{u}}\,\frac{du}{dx}\,dx \\
&= \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int_1^9 u^{-\frac{1}{2}}\,du \\
&= \frac{1}{3}\left[2u^{\frac{1}{2}}\right]_1^9 = \frac{2}{3}(\sqrt{9} - \sqrt{1}) = \frac{4}{3}.
\end{aligned}$$

Example We determine the value of the integral

$$\int_0^s x^3 \sin^5(x^4)\cos(x^4)\,dx$$

for all real numbers s. Let $u = \sin(x^4)$. Then

$$\frac{du}{dx} = 4x^3\cos(x^4).$$

Applying the rule for Integration by Substitution, we see that

$$\begin{aligned}
\int_0^s x^3 \sin^5(x^4)\cos(x^4)\,dx &= \frac{1}{4}\int_0^s u^5\,\frac{du}{dx}\,dx = \frac{1}{4}\int_{u(0)}^{u(s)} u^5\,du \\
&= \frac{1}{24}\left[u^6\right]_{u(0)}^{u(s)} \\
&= \frac{1}{24}\sin^6(s^4).
\end{aligned}$$

Example We determine the value of the integral

$$\int_1^s \frac{1}{x^2}\sin\left(\frac{2\pi}{x}\right)\,dx$$

for all positive real numbers s. Let $u = \dfrac{2\pi}{x}$. Then

$$\frac{du}{dx} = -\frac{2\pi}{x^2}.$$

It follows that

$$\begin{aligned}\int_1^s \frac{1}{x^2}\sin\left(\frac{2\pi}{x}\right)\,dx &= -\frac{1}{2\pi}\int_1^s \sin u\,\frac{du}{dx}\,dx \\ &= -\frac{1}{2\pi}\int_{2\pi}^{\frac{2\pi}{s}} \sin u\,du \\ &= \frac{1}{2\pi}\Big[\cos u\Big]_{2\pi}^{\frac{2\pi}{s}} \\ &= \frac{1}{2\pi}\left(\cos\left(\frac{2\pi}{s}\right) - 1\right).\end{aligned}$$

In the examples we have considered above, we have been given an integral of the form $\int_a^b F(x)\,dx$, and we have evaluated the integral by finding a function u of x and a function $f(u)$ of u for which $F(x) = f(u(x))\dfrac{du}{dx}$. Some calculus texts refer to substitutions of this type as u-substitutions.

In some cases it may be possible to evaluate integrals using the method of Integration by Substitution, but expressing the variable x of integration as a function of some other real variable.

Example We evaluate

$$\int_0^1 \sqrt{1-x^2}\,dx.$$

Let $x = \sin\theta$. Then $0 = \sin 0$ and $1 = \sin\frac{1}{2}\pi$. It follows from the rule for Integration by Substitution (Proposition 7.26) that

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{1}{2}\pi} \sqrt{1-\sin^2\theta}\,\frac{d(\sin\theta)}{d\theta}\,d\theta.$$

But $\dfrac{d(\sin\theta)}{d\theta} = \cos\theta$ and $\sqrt{1-\sin^2\theta} = \cos\theta$ for all real numbers θ satisfying $0 \leq \theta = \frac{1}{2}\pi$. It follows that

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{1}{2}\pi} \cos^2\theta\,d\theta.$$

Now $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$. It follows that

$$\int_0^{\frac{1}{2}\pi} \cos^2\theta \, d\theta = \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{1}{2}\pi} = \tfrac{1}{4}(\pi + \sin\pi - \sin 0) = \tfrac{1}{4}\pi.$$

With the benefit of hindsight, this result should not seem too surprising! The curve $y = \sqrt{1-x^2}$ is an arc of a circle representing one quarter of the circle, and the definition of the integral as representing the area between this curve and the x-axis ensures that the integral measures the area of a sector of the unit circle subtending a right angle at the centre of the circle. The area of this sector is then a quarter of the area π of the unit circle.

Example We determine the value of the integral

$$\int_0^s \frac{1}{(a^2+x^2)^2}\, dx$$

for all positive real numbers s, where a is a positive real constant. We substitute $x = a\tan\theta$. Let $\beta = \arctan(s/a)$. Then $a\tan\beta = s$. The rule for Integration by Substitution (Proposition 7.26) then ensures that

$$\int_0^s \frac{1}{(a^2+x^2)^2}\, dx = \int_0^\beta \frac{1}{a^4(1+\tan^2\theta)^2} \frac{d(a\tan\theta)}{d\theta}\, d\theta.$$

Now

$$1 + \tan^2\theta = \sec^2\theta = \frac{1}{\cos^2\theta}.$$

Also

$$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta = \frac{1}{\cos^2\theta}$$

(Corollary 6.14). It follows that

$$\begin{aligned}
\int_0^s \frac{1}{(a^2+x^2)^2}\, dx &= \frac{1}{a^3}\int_0^\beta \cos^4\theta \times \frac{1}{\cos^2\theta}\, d\theta \\
&= \frac{1}{a^3}\int_0^\beta \cos^2\theta \, d\theta = \frac{1}{2a^3}\int_0^\beta (1+\cos 2\theta)\, d\theta \\
&= \frac{1}{2a^3}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^\beta = \frac{1}{2a^3}(\beta + \tfrac{1}{2}\sin 2\beta) \\
&= \frac{1}{2a^3}(\beta + \sin\beta\, \cos\beta)
\end{aligned}$$

Now

$$\begin{aligned}
\sin\beta\,\cos\beta &= \tan\beta\,\cos^2\beta = \frac{\tan\beta}{1+\tan^2\beta} \\
&= \frac{a^2\tan\beta}{a^2 + a^2\tan^2\beta} = \frac{as}{a^2+s^2}.
\end{aligned}$$

We conclude therefore that

$$\int_0^s \frac{1}{(a^2+x^2)^2}\,dx = \frac{1}{2a^3}\arctan\left(\frac{s}{a}\right) + \frac{s}{2a^2(a^2+s^2)}.$$

7.8 Indefinite Integrals

Let $f(x)$ be an integrable function of a real variable x. It is commonplace to use the notation $\int f(x)\,dx$ to denote some function $g(x)$ with the property that

$$\frac{d}{dx}(g(x)) = f(x).$$

This function $\int f(x)\,dx$ is said to be an *indefinite integral* of the function f.

It follows from the Fundamental Theorem of Calculus (Theorem 7.17), we find that that if $f(x)$ is an integrable function of x on an interval D, and if a is a real number of D then the function $g(x)$ is an indefinite integral of $f(x)$, where

$$g(x) = \int_a^x f(t)\,dt.$$

We can therefore write $g(x) = \int f(x)\,dx$.

Note that an indefinite integral is only defined up to addition of an arbitrary constant: if $\int f(x)\,dx$ is an indefinite integral of $f(x)$ then so is $\int f(x)\,dx + C$, where C is a real constant known as the *constant of integration.*

7.9 Riemann Sums

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where $a < b$, and let P be a partition of $[a,b]$. Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a_0 = x_1 < x_2 < x_2 < \cdots < x_n = b.$$

A *Riemann sum* for the function f on the interval $[a,b]$ is a sum of the form

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

where $x_{i-1} \le x_i^* \le x_i$ for $i = 1, 2, \ldots, n$.

The definition of the Darboux lower and upper sums ensures that

$$L(P,f) \le \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \le U(P,f)$$

for any Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$ associated with the partition P. Thus is the partition P is chosen fine enough to ensure that $U(P, f) - L(P, f) < \varepsilon$ then all Riemann sums associated with the partition P and its refinements will differ from one another by at most ε. Moreover if the function f is Riemann-integrable on $[a, b]$, then all Riemann sums associated with the partition P and its refinements will approximate to the value of the integral $\int_a^b f(x)\,dx$ to within an error of at most ε.

Some textbooks use definitions of integration that represent integrals as being, in an appropriate sense, limits of Riemann sums.