Trigonometry

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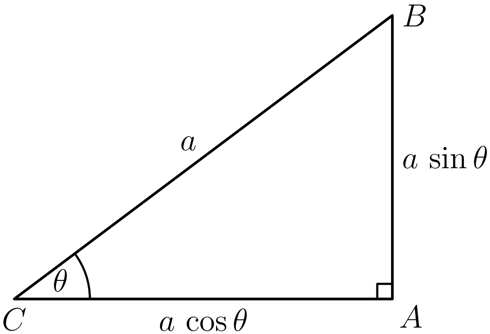
1. Trigonometry

1.1. Trigonometric Functions

There are six standard trigonometric functions. They are the *sine function* (sin), the *cosine function* (cos), the *tangent function* (tan), the *cotangent function* (cot), the *secant function* (sec) and the *cosecant function* (csc).

Angles will always be represented in the following discussion using *radian measure*. If one travels a distance s around a circle of radius r, then the angle subtended by the starting and finishing positions at the centre of the circle is s/r radians.

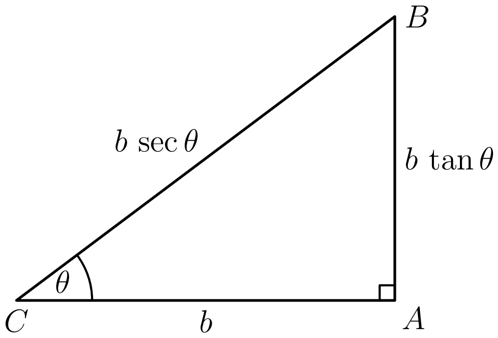
The standard trigonometrical functions represent ratios of sides of right-angled triangles, as indicated in the following diagrams.



In the above triangle ABC, in which the angle at the vertex B is a right angle, the lengths a, b and c satisfy the identities

$$b = a\cos\theta, \quad b = a\sin\theta,$$

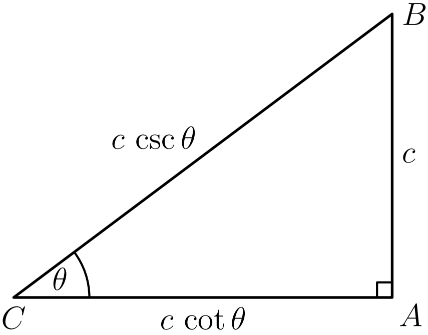
where θ denotes the angle of the triangle at the vertex A.



In the above triangle ABC, in which the angle at the vertex B is a right angle, the lengths a, b and c satisfy the identities

$$c = b\tan\theta, \quad a = b\sec\theta,$$

where θ denotes the angle of the triangle at the vertex A.



In the above triangle ABC, in which the angle at the vertex B is a right angle, the lengths a, b and c satisfy the identities

$$b = c \cot\theta, \quad a = c \csc\theta,$$

where θ denotes the angle of the triangle at the vertex A.

The identities described above that determine the ratios of the sides of a right angled triangle are summarized in the following proposition.

Proposition 1.1

Let ABC be a triangle in which the angle at A is a right angle, and let θ denote the angle at C. Then the lengths $|BC|$, $|AC|$ and $|AB|$ of the sides BC, AC and AB respectively satisfy the following identities:—

$$|AC| = |BC| \cos\theta, \quad |AB| = |BC| \sin\theta;$$

$$|AB| = |AC| \tan\theta, \quad |BC| = |AC| \sec\theta;$$

$$|AC| = |AB| \cot\theta, \quad |BC| = |AB| \csc\theta.$$

The following trigonometrical formulae follow directly from the results stated in Proposition 1.1.

Proposition 1.2

The tangent, cotangent, secant and cosecant functions are determined by the sine and cosine functions in accordance with the following identities:—

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\sec\theta = \frac{1}{\cos\theta}; \quad \csc\theta = \frac{1}{\sin\theta}.$$

Proposition 1.3

The sine and cosine functions are related by the following relationship, when angles are specified using radian measure:—

$$\sin\theta = \cos(\tfrac{1}{2}\pi - \theta); \quad \cos\theta = \sin(\tfrac{1}{2}\pi - \theta).$$

Proof

The trigonometrical functions are determined by ratios of edges of a right angled triangle ABC in which the angle B is a right angle and the angle A is θ radians. The angles of a triangle add up to two right angles, and two right angles are equal to π in radian measure. Thus if $\angle B$ denotes the angle of the right-angled triangle ABC then

$$\angle A + \angle B + \angle C = \pi,$$

and thus

$$\theta + \tfrac{1}{2}\pi + \angle C = \pi,$$

and therefore $C = \frac{1}{2}\pi - \theta$. The result then follows from the definitions of the sine and cosine functions. ∎

The nth powers of trigonometric functions are usually presented using the following traditional notation, in instances where n is a positive integer:—

$$\sin^n \theta = (\sin \theta)^n, \quad \cos^n \theta = (\cos \theta)^n, \quad \tan^n \theta = (\tan \theta)^n, \quad \text{etc.}$$

Proposition 1.4

The trigonometric functions satisfy the following identities:—

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1; \\ 1 + \tan^2 \theta &= \sec^2 \theta; \\ 1 + \cot^2 \theta &= \csc^2 \theta; \end{aligned}$$

Proof

These identities follow from the definitions of the trigonometric functions on applying Pythagoras' Theorem. ∎

1.2. Periodicity of the Trigonometrical Functions

Suppose that a particle moves with speed v around the circumference of a circle of radius r, where that circle is represented in Cartesian coordinates by the equation

$$x^2 + y^2 = r^2.$$

The centre of the circle is thus at the origin of the Cartesian coordinate system. We suppose that the particle travels in an anticlockwise direction and passes through the point $(r, 0)$ when $t = 0$. Then the particle will be at the point

$$\left(r \cos \frac{vt}{r}, r \sin \frac{vt}{r} \right).$$

at time t. The quantities $\sin \theta$ and $\cos \theta$ are defined for all real numbers θ so that the above formula for the position of the particle moving around the circumference of the circle at a constant speed remains valid for all times.

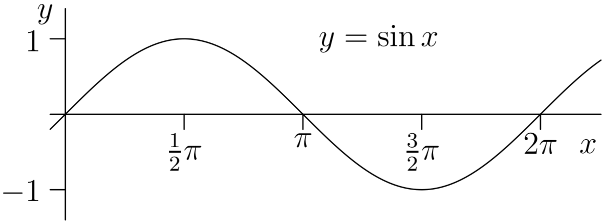
Now the particle moving round the circumference of the circle of radius r with speed v will complete each revolution in time $\dfrac{2\pi r}{v}$. Thus

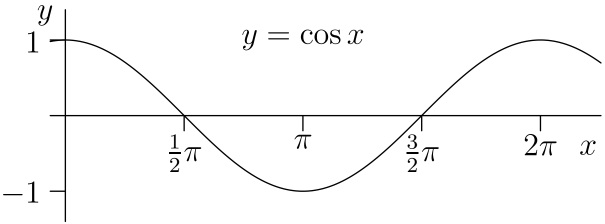
$$\cos(\theta + 2\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2\pi) = \sin\theta$$

for all real numbers θ. It follows that

$$\cos(\theta + 2n\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2n\pi) = \sin\theta$$

for all real numbers θ and for all integers n. These equations express the *periodicity* of the sine and cosine functions.


y
1
-1
$y = \sin x$
$\frac{1}{2}\pi$
π
$\frac{3}{2}\pi$
2π
x


y
1
-1
$y = \cos x$
$\frac{1}{2}\pi$
π
$\frac{3}{2}\pi$
2π
x

1.3. Values of Trigonometric Functions at Particular Angles

The following table sets out the values of $\sin\theta$ and $\cos\theta$ for some angles θ that are multiples of $\frac{1}{2}\pi$:—

θ	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π	$\frac{5}{2}\pi$
$\sin\theta$	0	-1	0	1	0	-1	0	1
$\cos\theta$	-1	0	1	0	-1	0	1	0

The following values of the sine and cosine functions can be derived using geometric arguments involving the use of Pythagoras' Theorem:—

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

1.4. The Cosine and Sine Rules

Proposition 1.5

Let $\triangle ABC$ be a triangle, let $a = |BC|$, $b = |CA|$ and $c = |AB|$, and let $|\angle A| = |\angle CAB|$. Then

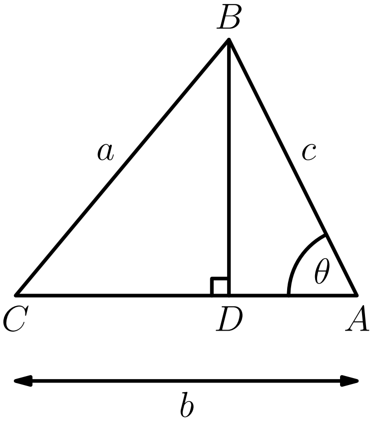
$$a^2 = b^2 + c^2 - 2bc\cos|\angle A|.$$

Proof

Let $\theta = |\angle A|$. Consider first the case depicted where the angle θ is acute, but the perpendicular dropped from B to the line CA meets that line at a point D lying between C and A. Let $|BC| = a$, $|AC| = b$ and $|AB| = c$.

Then $\triangle BCD$ is a right-angled triangle with its right angle at D, and

$$\begin{aligned} |CD| &= b - c\cos\theta, \\ |DB| &= c\sin\theta. \end{aligned}$$



Consideration of possibilities shows that the formulae

$$|CD| = |b - c\,\cos\theta| \quad \text{and} \quad |DB| = |c\,\sin\theta|$$

hold in all cases, including those cases where the angle θ is null, right, straight or obtuse, and those cases where the perpendicular dropped from the point B to the line through AB does not pass between A and D. It follows from Pythagoras's Theorem that

$$\begin{aligned} a^2 &= (b - c\,\cos\theta)^2 + c^2\sin^2\theta \\ &= b^2 - 2bc\,\cos\theta + c^2(\cos^2\theta + \sin^2\theta) \\ &= b^2 + c^2 - 2bc\,\cos\theta, \end{aligned}$$

as required. ∎

Proposition 1.6

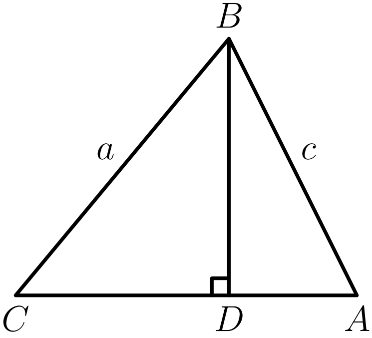
Let $\triangle ABC$ be a triangle, let $a = |BC|$, $b = |CA|$ and $c = |AB|$. Then

$$\frac{a}{\sin|\angle A|} = \frac{b}{\sin|\angle B|} = \frac{c}{\sin|\angle C|}.$$

Proof

We consider the case when the angles of the triangle are all acute. Let the triangle be as depicted, and let the perpendicular from the vertex B to the line AC intersect that line at D. Then AC and BD are perpendicular. Then

$$|BD| = a \sin|\angle C| = c \sin|\angle A|.$$



It follows that
$$\frac{a}{\sin|\angle A|} = \frac{c}{\sin|\angle C|}.$$
The proof in this case is completed by replacing A, B and C by B, C and A. The proof in other cases is analogous. ∎

1.5. Sines and Cosines of Angle Sums

Proposition 1.7

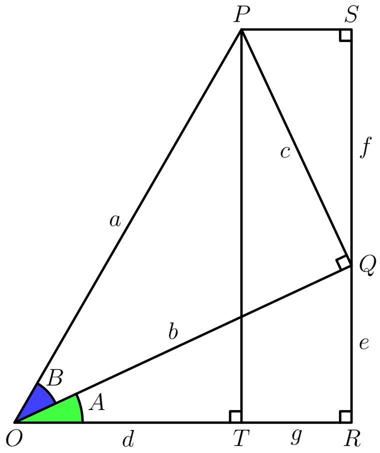
The sine and cosine functions satisfy the following identities for all real numbers A and B:—

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B,\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B.\end{aligned}$$

1st Proof of Proposition 1.7 (for acute angles only)

Let us consider the special case when angles A, B and $A+B$ all lie between 0° and 90°.

Consider the geometrical configuration depicted to the right, in which $A = |\angle ROQ|$ and $B = |\angle QOP|$, and thus $A-B = |\angle ROP|$. The lengths of the sides are denoted by letters from a to g, where $a = |OP|$, $b = |OQ|$, $c = |QP|$, $d = |OT|$, $e = |RQ|$, $f = |QS|$ and $g = |TR|$. In this configuration $|PS| = g$, $|PT| = e + f$ and $|OR| = d + g$.

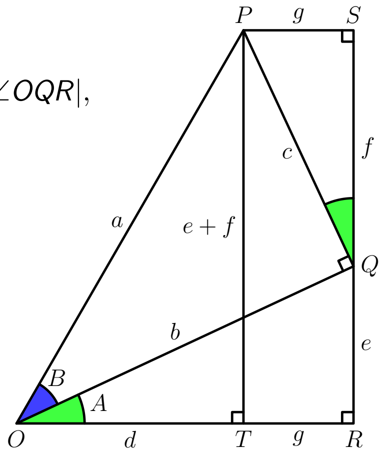


Then $|\angle ROP| = A + B$. Also

$|\angle SQP| + |\angle OQR| = 90^\circ = |\angle ROQ| + |\angle OQR|$,

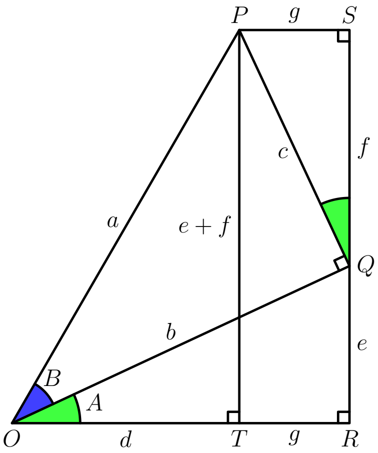
and therefore

$$|\angle SQP| = |\angle ROQ| = A.$$



It follows that

$$\begin{aligned} b &= a\cos B, \\ c &= a\sin B, \\ d &= a\cos(A+B), \\ e+f &= a\sin(A+B), \\ d+g &= b\cos A, \\ e &= b\sin A, \\ f &= c\cos A, \\ g &= c\sin A. \end{aligned}$$



1. Trigonometry (continued)

It follows that

$$\begin{aligned} a\cos(A+B) &= d = (d+g) - g = b\cos A - c\sin A \\ &= a\ \cos A\cos B - a\ \sin A\sin B, \\ a\sin(A+B) &= e + f = b\ \sin A + c\ \cos A \\ &= a\ \sin A\cos B + a\ \cos A\sin B. \end{aligned}$$

Thus

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B, \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B. \qquad \blacksquare \end{aligned}$$

We now consider another approach for establishing the addition formulae for the sine and cosine functions, using coordinate geometry, which does not limit the verification to angles between 0° and 90°.

2nd Proof of Proposition 1.7

Let (x, y) be a point of the plane, specified in Cartesian coordinates relative to some chosen origin, and let A be an angle. Using the identity $\cos^2 A + \sin^2 A = 1$, we see that

$$\begin{aligned} x &= (x \cos A + y \sin A) \cos A - (y \cos A - x \sin A) \sin A, \\ y &= (x \cos A + y \sin A) \sin A + (y \cos A - x \sin A) \cos A, \end{aligned}$$

Let points O, P, Q and R of the plane be determined so that $O = (0, 0)$, $P = (\cos A, \sin A)$, $Q = (b \cos A, b \sin A)$ and

$$R = (b \cos A - c \sin A, b \sin A + c \cos A),$$

where b and c are real numbers. Now the point Q lies on the line OP that passes through the points O and P, and, , for all real numbers t, the point $(b \cos A - t \sin A, b \sin A + t \cos A)$ lies on a line through the point Q that is perpendicular to the line OP. It follows that the points O, Q and R are the vertices of a right-angled triangle whose sides $|OQ|$ and $|QR|$ meeting at the right angle Q are of length $|b|$ and $|c|$ respectively. Moreover $b = 0$ if and only if $\angle POR$ is right, and $b < 0$ if and only if $\angle POR$ is obtuse. Also if $c > 0$ then R makes an anticlockwise angle less than 180° degrees with the ray $[OP$, and if $c < 0$ then R makes a clockwise angle less than 180° with the ray $[OP$.

1. Trigonometry (continued)

It follows that if the point R lies at a distance r from the origin and the ray $[OR$ makes an anticlockwise angle B with the ray $[OP$ then $b = r\cos B$ and $c = r\sin B$.

We can apply this result when $R = (\cos(A+B), \sin(A+B)$. where A and B are angles. In this case the ray $[OR$ makes an anticlockwise angle B with the ray $[OP$. In this case $b = \cos B$ and $c = \sin B$, and therefore It follows that

$$\begin{aligned}
\cos(A+B) &= b \cos A - c \sin A \\
&= \cos A \cos B - \sin A \sin B, \\
\sin(A+B) &= b \sin A + c \cos A \\
&= \sin A \cos B + \cos A \sin B,
\end{aligned}$$

as required. $\blacksquare$

The next proof of Proposition 1.7] develops the formulae for changes of coordinates resulting from rotations about the origin.

3rd Proof of Proposition 1.7 (using coordinate transformations)

An anticlockwise rotation about the origin through an angle of A radians sends a point (x, y) of the plane to the point (x', y'), where

$$\left\{ \begin{array}{lcl} x' & = & x\cos A - y\sin A \\ y' & = & x\sin A + y\cos A \end{array} \right.$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos A, \sin A)$ and takes the point $(0, 1)$ to the point $(-\sin A, \cos A)$.) An anticlockwise rotation about the origin through an angle of B radians then sends the point (x', y') of the plane to the point (x'', y''), where

$$\left\{ \begin{array}{lcl} x'' & = & x'\cos B - y'\sin B \\ y'' & = & x'\sin B + y'\cos B \end{array} \right.$$

Now an anticlockwise rotation about the origin through an angle of $A + B$ radians sends the point (x, y), of the plane to the point (x'', y''), and thus

$$\left\{ \begin{array}{rcl} x'' & = & x\cos(A+B) - y\sin(A+B) \\ y'' & = & x\sin(A+B) + y\cos(A+B) \end{array} \right.$$

But if we substitute the expressions for x' and y' in terms of x, y and A obtained previously into the above equation, we find that

$$\left\{ \begin{array}{rcl} x'' & = & x(\cos A \cos B - \sin A \sin B) - y(\sin A \cos B + \cos A \sin B) \\ y'' & = & x(\sin A \cos B + \cos A \sin B) + y(\cos A \cos B - \sin A \sin B) \end{array} \right.$$

On comparing equations, we see that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

and

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

as required. ∎

4th Proof of Proposition 1.7 (using matrix multiplication)
Let a vector (u, v) be rotated about the origin through an anticlockwise angle A, yielding a vector (u', v'). Then let the vector (u', v') in turn be rotated about the origin through an anticlockwise angle B, yielding a vector (u'', v''). Then

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$
$$\begin{pmatrix} u'' \\ u'' \end{pmatrix} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Moreover the combined effect of these two rotations has the effect of rotating the vector (u, v) through an anticlockwise angle of $A + B$ to yield the vector (u'', v''). Therefore

$$\begin{pmatrix} u'' \\ v'' \end{pmatrix} = \begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix},$$

and therefore

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B, \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B, \end{aligned}$$

as required. $\blacksquare$

On replacing B by $-B$, and noting that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, we find that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

and

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

We have therefore established the addition formulae for the sine and cosine functions stated in the following proposition.

Corollary 1.8

The sine and cosine functions satisfy the following identities for all real numbers A and B:—

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \cos A \sin B,\\ \cos(A - B) &= \cos A \cos B + \sin A \sin B.\end{aligned}$$

Proof

The result follows from the formulae for $\sin(A + B)$ and $\cos(A + B)$ on simply replacing B by $-B$. ∎

Remark

The equations describing how Cartesian coordinates of points of the plane transform under rotations about the origin may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (31) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{aligned}&\begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix} \\ &\qquad = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}.\end{aligned}$$

Therefore

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

This provides an alternative derivation of the addition formulae stated in Proposition 1.7.

Corollary 1.9

The sine and cosine functions satisfy the following identities for all real numbers A:—

$$\begin{array}{rcl} \sin(A + \frac{1}{2}\pi) & = & \cos A, \\ \cos(A + \frac{1}{2}\pi) & = & -\sin A, \\ \sin(A + \pi) & = & -\sin A, \\ \cos(A + \pi) & = & -\cos A, \end{array}$$

Proof

These results follow directly on applying Proposition 1.7 in view of the identities

$$\sin\tfrac{1}{2}\pi = 1, \quad \cos\tfrac{1}{2}\pi = 0, \quad \sin\pi = 0 \quad \text{and} \quad \cos\pi = -1. \qquad \blacksquare$$

The formulae stated in the following corollary follow directly from the addition formulae stated in Proposition 1.7 on adding and subtracting those addition formulae.

Corollary 1.10

The sine and cosine functions satisfy the following identities for all real numbers A and B:—

$$\begin{aligned}\sin A \sin B &= \tfrac{1}{2}(\cos(A-B) - \cos(A+B));\\ \cos A \cos B &= \tfrac{1}{2}(\cos(A+B) + \cos(A-B));\\ \sin A \cos B &= \tfrac{1}{2}(\sin(A+B) + \sin(A-B)).\end{aligned}$$

Corollary 1.11

The sine and cosine functions satisfy the following identities for all real numbers A:—

$$\begin{aligned}\sin 2A &= 2\sin A\cos A;\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A.\end{aligned}$$

Proof

The formula for $\sin 2A$ and the first formula for $\cos 2A$ follow from the identities stated in Proposition 1.7 on setting $B = A$ in the formulae for $\sin(A+B)$ and $\cos(A+B)$. The second and third formulae for $\cos 2A$ then follow on making use of the identity $\sin^2 A + cos^2 A = 1$. ∎

The following formulae then follow directly from those stated in Corollary 1.11.

Corollary 1.12

The sine and cosine functions satisfy the following identities for all real numbers A:—

$$\begin{aligned}\sin^2 A &= \tfrac{1}{2}(1 - \cos 2A);\\ \cos^2 A &= \tfrac{1}{2}(1 + \cos 2A).\end{aligned}$$

1.6. Limits of Functions of a Single Real Variable

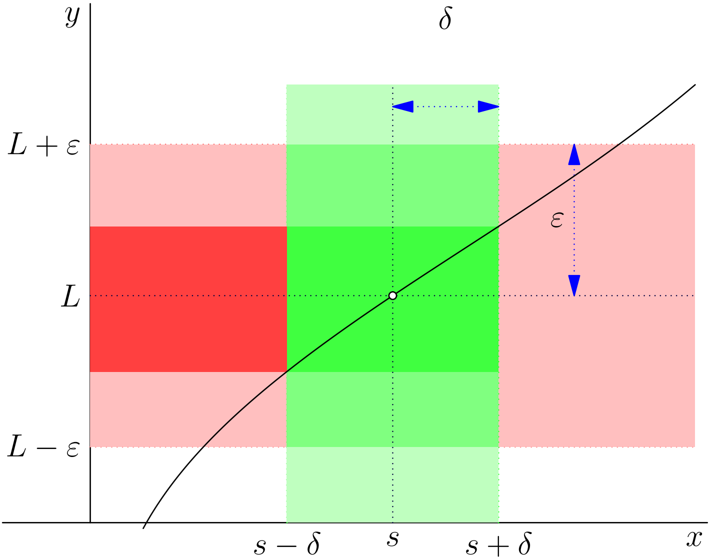
Definition

Let s and L be real numbers, and let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of $\mathbb{R}$ that, for some strictly positive real number δ_0, includes all real numbers x satisfying $0 < |x - s| < \delta_0$. We say that L is the *limit* of $f(x)$ as x tends to s, and write

$$\lim_{x \to s} f(x) = L,$$

if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $L - \varepsilon < f(x) < L + \varepsilon$ for all real numbers x in D that satisfy both

$$s - \delta < x < s + \delta \quad \text{and} \quad x \neq s.$$



y
δ
$L+\varepsilon$
ε
L
$L-\varepsilon$
$s-\delta$
s
$s+\delta$
x

1.7. Derivatives of Trigonometrical Functions

Lemma 1.13

Let ε *be a positive real number. Then there exists some positive real number* δ *satisfying* $0 < \delta < \frac{1}{2}\pi$ *with the property that* $1 - \varepsilon < \cos\theta < 1$ *whenever* $0 < \theta < \delta$.

Proof

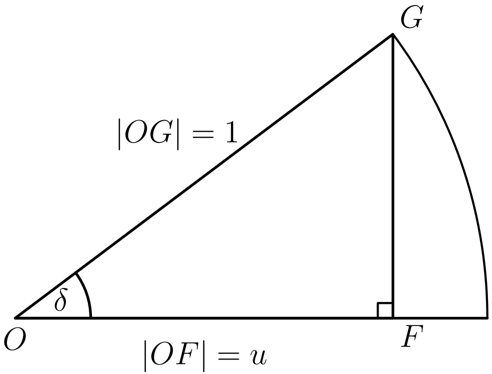
Choose a real number u satisfying $0 < u < 1$ for which $1 - \varepsilon \leq u$. Let a right-angled triangle $\triangle OFG$ be constructed so that the angle at F is a right angle, $|OF| = u$ and $|FG| = \sqrt{1 - u^2}$, and let δ be the angle of this triangle at the vertex O. Then

$$|OG|^2 = |OF|^2 + |FG|^2 = 1,$$

and therefore $u = \cos\delta$. It follows that if θ is a positive real number satisfying $0 < \theta < \delta$ then

$$1 - \varepsilon \leq \cos\delta < \cos\theta < 1.$$

The result follows. ∎



G
$|OG| = 1$
δ
O
F
$|OF| = u$

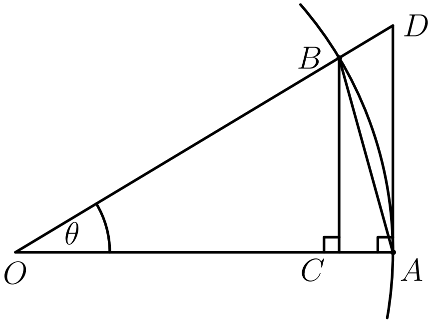
Proposition 1.14

Let $\sin\colon \mathbb{R} \to \mathbb{R}$ *be the sine function whose value* $\sin\theta$, *for a given real number* θ *is the sine of an angle of* θ *radians. Then*

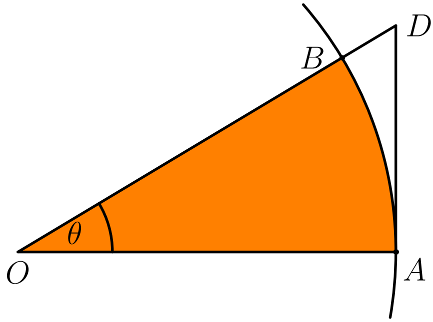
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Proof

Let a circle of unit radius pass through points A and B, so that the angle θ in radians between the line segements OA and OB at the centre O of the circle satisfies the inequalities $0 < \theta < \frac{1}{2}\pi$. Let C be the point on the line segment OA for which the angle OCB is a right angle, and let the line OB be produced to the point D determined so that the angle OAD is a right angle.



The sector OAB of the unit circle is by definition the region bounded by the arc AB of the circle and the radii OA and OB. Now the area of a sector of a circle subtending at the centre an angle of θ radians is equal to the area of the circle multiplied by $\dfrac{\theta}{2\pi}$. But the area of a circle of unit radius is π. It follows that a sector of the unit circle subtending at the centre an angle of θ radians has area $\frac{1}{2}\theta$.

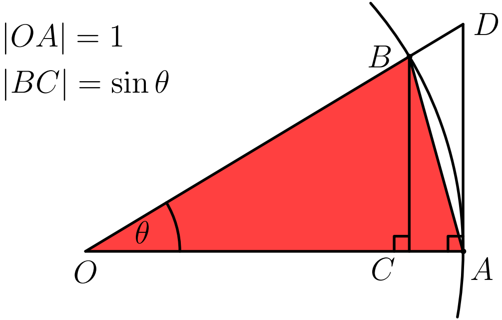


The the area of a triangle is half the base of the triangle multiplied by the height of the triangle. The base $|OA|$ and height $|BC|$ of the triangle AOB satisfy

$$|OA| = 1, \quad |BC| = \sin\theta.$$

It follows that

$$\text{area of triangle } OAB = \tfrac{1}{2} \times |OA| \times |BC| = \tfrac{1}{2}\sin\theta.$$



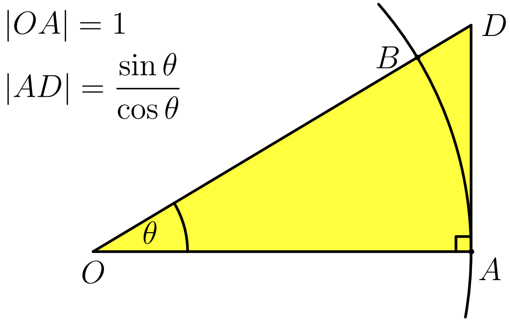
1. Trigonometry (continued)

Also the base $|OA|$ and height $|AD|$ of the triangle AOD satisfy

$$|OA| = 1, \quad |AD| = \frac{\sin\theta}{\cos\theta}.$$

It follows that

$$\text{area of triangle } OAD = \tfrac{1}{2} \times |OA| \times |AD| = \frac{\sin\theta}{2\cos\theta}.$$



The results concerning areas just obtained can be summarized as follows:—

$$\begin{aligned}
\text{area of triangle } OAB &= \tfrac{1}{2} \times |OA| \times |BC| \\
&= \tfrac{1}{2} \sin\theta, \\
\text{area of sector } OAB &= \frac{\theta}{2\pi} \times \pi = \tfrac{1}{2}\theta, \\
\text{area of triangle } OAD &= \tfrac{1}{2} \times |OA| \times |AD| \\
&= \tfrac{1}{2} \tan\theta = \frac{\sin\theta}{2\cos\theta}.
\end{aligned}$$

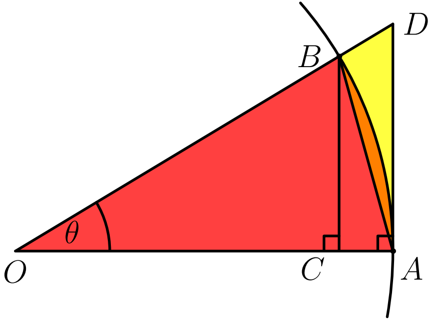
Moreover the triangle OAB is strictly contained in the sector OAB, which in turn is strictly contained in the triangle OAD. It follows that

$$\mathrm{area}(\triangle OAB) < \mathrm{area}(\mathrm{sector}\ OAB) < \mathrm{area}(\triangle OAD),$$

and thus

$$\tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \frac{\sin\theta}{2\cos\theta}$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$.



Multiplying by 2, and then taking reciprocals, we find that

$$\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$. If we then multiply by $\sin\theta$, we obtain the inequalities

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

for all real numbers θ satisfying $0 < \theta < \frac{1}{2}\pi$.

Now, given any positive real number ε, there exists some positive real number δ satisfying $0 < \delta < \frac{1}{2}\pi$ such that $1 - \varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$ (see Lemma 1.13). But then

$$1 - \varepsilon < \frac{\sin\theta}{\theta} < 1$$

whenever $0 < \theta < \delta$. These inequalities also hold when $-\delta < \theta < 0$, because the value of $\dfrac{\sin\theta}{\theta}$ is unchanged on replacing θ by $-\theta$. It follows that $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, as required. ∎

Corollary 1.15

Let $\cos\colon \mathbb{R} \to \mathbb{R}$ *be the cosine function whose value* $\cos\theta$*, for a given real number* θ *is the cosine of an angle of* θ *radians. Then*

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Proof

Basic trigonometrical identities ensure that

$$1 - \cos\theta = 2\sin^2 \tfrac{1}{2}\theta \quad \text{and} \quad \sin\theta = 2\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta$$

for all real numbers θ (see Corollary 1.11 and Corollary 1.12). Therefore

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

for all real numbers θ. It follows that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \tan\tfrac{1}{2}\theta = 0,$$

and therefore

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \times \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0 \times 1 = 0,$$

as required. ∎

Proposition 1.16

The derivatives of the sine and cosine functions satisfy

$$\frac{d}{dx}(\sin x) = \cos x, \quad \textit{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Proof

Limits of sums, differences and products of functions are the corresponding sums, differences and products of the limits of those functions, provided that those limits exist. Also

$$\sin(x+h) = \sin x \, \cos h + \cos x \, \sin h$$

and

$$\cos(x+h) = \cos x \, \cos h - \sin x \, \sin h$$

for all real numbers h (see Proposition 1.7). Applying these results, together with those of Proposition 1.14 and Corollary 1.15, we see that

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\, \cos h + \cos x\, \sin h - \sin x}{h} \\
&= \cos x\, \lim_{h\to 0} \frac{\sin h}{h} - \sin x\, \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= \cos x.
\end{aligned}
$$

Similarly

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\ \cos h - \sin x\ \sin h - \cos x}{h} \\
&= -\sin x\ \lim_{h\to 0} \frac{\sin h}{h} - \cos x\ \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= -\sin x,
\end{aligned}$$

as required. ∎

Corollary 1.17

The derivative of the tangent function satisfies

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Proof

Using the formulae for the derivatives of the sine and cosine functions (Proposition 1.16), together with the Quotient Rule of differential calculus we find that

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{1}{\cos^2 x}\left(\frac{d}{dx}(\sin x)\cos x - \frac{d}{dx}(\cos x)\sin x\right) \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

as required. $\blacksquare$