Selected Circle Theorems

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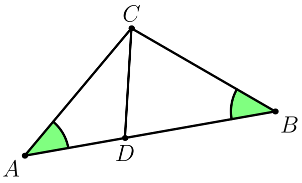
Lemma 4.1 (Elements, III-2)

Let A and B be distinct points on a circle. Then all points on the chord $[AB]$ between A and B lie inside the circle.

Proof

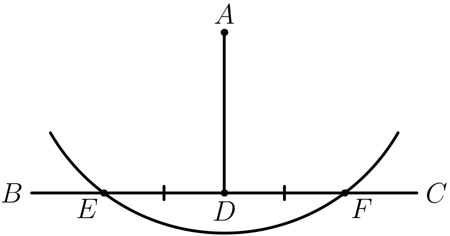
Let C be the centre of the circle. Then the sides $[CA]$ and $[CB]$ of the triangle $\triangle CAB$ are equal. It follows from the Isosceles Triangle Theorem (*Elements*, I, 5) (*PPG*, Theorem 2) that the angles $\angle CAB$ and $\angle CBA$ are equal.

Let D be a point on the chord between A and B. Then the exterior angle $\angle CDB$ of the triangle $\triangle CAD$ is greater than the opposite interior angle $\angle CAD$ (*Elements*, I, 16) (*PPG*, Theorem 6), and thus is greater than $\angle CBD$. If follows that, in the triangle $\triangle CDB$, the side $[CB]$ opposite the greater angle $\angle CDB$ is greater than the side $[CD]$ opposite the lesser angle $\angle CBD$. (*Elements*, I, 19). Thus the line segment $[CD]$ has length less than the radius of the circle. The result follows. ∎



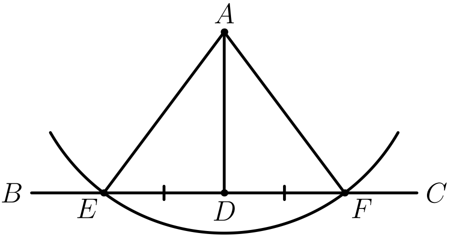
Proposition 4.2 (Elements, I-12 and III-3)

Let A be a point, let BC be a line that does not pass through the point A, let a circle with centre A cut the line BC at points E and F, and let the point D bisect the line segment $[EF]$. Then $\angle ADE$ is a right angle.



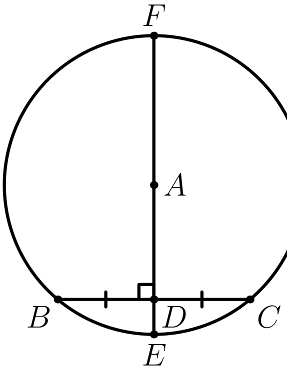
Proof

Join the centre A of the circle to points E and F. Then $|AE| = |AF|$ and $|ED| = |FD|$. It follows from the SSS Congruence Rule (*Elements*, I, 8) that the triangles $\triangle ADE$ and $\triangle ADF$ are congruent. It follows that the angles $\angle ADE$ and $\angle ADF$ are equal, and therefore $\angle ADE$ is a right angle. ∎

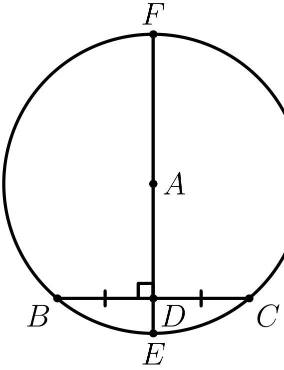


Corollary 4.3 (Elements, III-1)

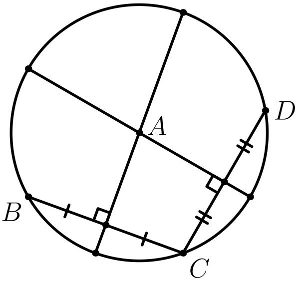
Given distinct points B and C on the circumference of a circle, the centre of the circle lies on the perpendicular bisector of the line segment $[BC]$.



The figure to the right represents the method of locating the centre A of a circle set out in Proposition 1 of Book III of Euclid's *Elements of Geometry*. Given two distinct points B and C on the circle, the perpendicular bisector is constructed. This perpendicular bisector intersects the circle in two points, and the point lying midway between those two points is the centre A of the circle.



The figure to the right represents the method of locating the centre A of a circle implicit in the proof of Proposition 10 of Book III of Euclid's *Elements of Geometry*. Given three distinct points B, C and D on the circumference of the circle, the perpendicular bisectors of the chords $[BC]$ and $[CD]$ are constructed. The point where those perpendicular bisectors intersect is the centre A of the circle.



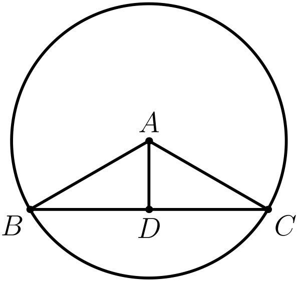
Lemma 4.4 (Elements, III-3)

Let A be the centre of a circle, and B and C be distinct points on the circumference of the circle, and let D be a point on the chord $[BC]$ joining the points B and C. Then D bisects $[BC]$ if and only if $[AD]$ is perpendicular to $[BC]$.

Proof

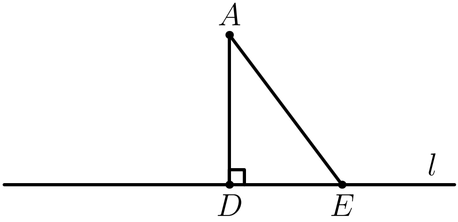
It follows from Proposition 4.2 that if D bisects $[BC]$ then $[AD]$ is perpendicular to $[BC]$. We must prove the converse.

Suppose therefore the $[AB]$ is perpendicular to $[BC]$. Now $\triangle BAC$ is an isosceles triangle. It follows from the Isosceles Triangle Theorem that the angles $\angle ABC$ and $\angle ACB$ are equal. Thus the angles of the triangle $\triangle ABD$ at A and B are equal to the angles of the triangle $\triangle ACD$ at A and C. The angles of these two triangles at A must therefore be equal, because the angles of a triangle add up to two right angles. It then follows from the ASA Congruence Rule that the triangles $\triangle ABD$ and $\triangle ACD$ are congruence, and therefore D bisects $[AC]$. This completes the proof. ∎



Proposition 4.5

Let A be a point, let l be a line that does not pass through the point A, and let D be the point on the line l for which $[AD]$ is perpendicular to l. Then the line segment joining the point A to any point of the line l distinct from D is greater than $[AD]$.



Proof

The sum of two angles of a triangle is less than two right angles. Therefore all angles of a right-angled triangle other than the right angle itself must be acute. Let E be a point of the line l distinct from D. Then the angle $\angle AED$ must be acute. It follows that the side $[AE]$ of the triangle $\triangle ADE$ opposite the right angle at D must be longer than the side $[AD]$ opposite the acute angle at E. The result follows. $\blacksquare$

A line passing through a point on the circumference of a circle is a *tangent line* of the circle, and is said to *touch* the circle, if and only if it does not pass through any point in the interior of the circle.

Proposition 4.6 (Elements, III-16)

Let A be a point on the circumference of a circle with centre B. Then a line through the point A is tangent to the circle if and only if the line is perpendicular to the line AB joining A to the centre of the circle.

Proof

It follows from Proposition 4.5 that if a line through A is perpendicular to AB then A is the closest point on that line to the centre of the circle, and therefore the line does not pass through any point in the interior of the circle.

Conversely suppose that a line l is a tangent line to the circle at A. A point D can be constructed on l for which $[BD]$ is perpendicular to l (*Elements*, I, 12) (see also Proposition 4.2). If D were distinct from A then $[BD]$ would be less than $[BA]$ (Proposition 4.5), and the point D would therefore lie in the interior of the circle. But then the line l would not be a tangent line to the circle. We conclude therefore that the tangent line must be perpendicular to AB at the point A. This completes the proof. ∎

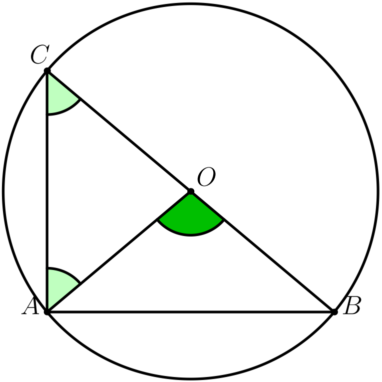
Proposition 4.7 (Elements, III-20)

Let $[AB]$ be a chord of a circle not passing through the centre of the circle. Then the angle $\angle AOB$ subtended by the chord $[AB]$ at the centre O of the circle is double the angle $\angle ACB$ subtended by that chord at a point C on the circumference that lies on the same side of the chord $[AB]$ as the centre of the circle.

Proof

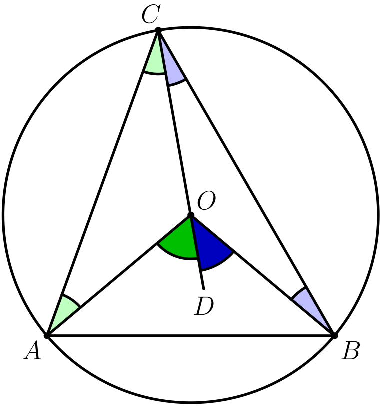
We divide up the proof into three cases. The first case is that in which the line through the point C on the circumference of the circle and the centre O of the circle passes through one of the endpoints of the chord $[AB]$.

In this case we may suppose, without loss of generality, that B, O and C are collinear. The triangle $\triangle OCA$ is then isosceles, with equal sides $[OC]$ and $[OA]$. The Isosceles Triangle Theorem (*Elements*, I, 5) (*PPG*, Theorem 2) ensures that the angles $\angle OCA$ and $\angle OAC$ are equal. The Exterior Angles Theorem (*Elements*, I, 32) (*PPG*, Theorem 6) then ensures that the sum of these two equal angles is equal to the exterior angle $\angle AOB$ of the triangle $\triangle OCA$ at the vertex O. Therefore the angle $\angle AOB$ is double the angle $\angle ACB$, as required in this case.

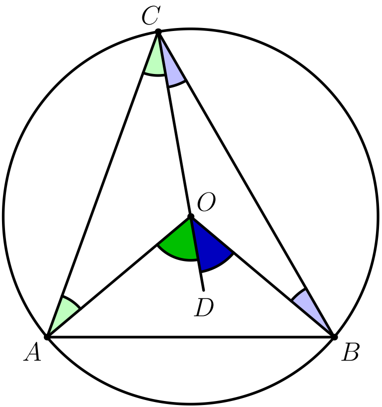


The next case is that in which the centre O of the circle lies inside the triangle $\triangle ABC$.

In this case join C to O be a line segment, and produce that line segment beyond O to a point D. Then $[OC]A$ is an isosceles triangle with equal sides $[OC]$ and $[OA]$. The Isosceles Triangle Theorem (*Elements*, I, 5) (*PPG*, Theorem 2) ensures that the angles $\angle OCA$ and $\angle OAC$ are equal.

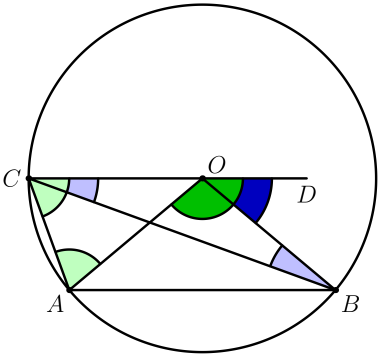


The Exterior Angles Theorem (*Elements*, I, 32) (*PPG*, Theorem 6) then ensures that the sum of these two equal angles is equal to the exterior angle $\angle AOD$ of the triangle $\triangle OCA$ at the vertex O. Therefore the angle $\angle AOD$ is double the angle $\angle ACB$. Similarly the angle $\angle BOD$ is double the angle $\angle BCA$. The angle $\angle AOB$ is the sum of the two angles $\angle AOD$ and $\angle BOD$. It is therefore equal to double the sum of the angles $\angle ACD$ and $\angle BCD$, and is therefore equal to double the angle $\angle ACB$, as required in this case.

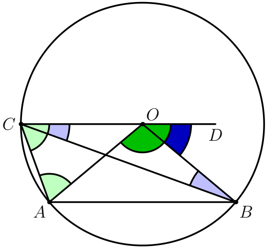


The final case is that in which the centre O of the circle lies outside the triangle $\triangle ABC$, but on the same side of $[AB]$ as the point C.

In this case join C to O be a line segment, and produce that line segment beyond O to a point D. As in the previous case, the Isosceles Triangle Theorem (*Elements*, I, 5) (*PPG*, Theorem 2) and the Exterior Angles Theorem (*Elements*, I, 32) (*PPG*, Theorem 6) then ensure that the angle $\angle AOD$ is double the angle $\angle ACB$. Similarly the angle $\angle BOD$ is double the angle $\angle BCA$.



Now the point O does not lie inside the triangle $\triangle ACD$, though both points O and C lie on the same side of $[AB]$. It follows that either O and A lie on opposite sides of $[BC]$ or else O and B lie on opposite sides of $[AC]$.



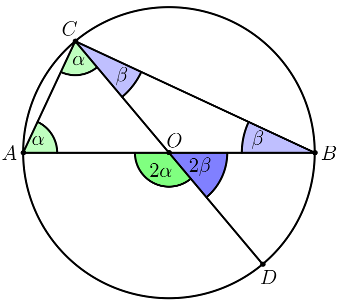
We may assume, without loss of generality, that (as depicted in the accompanying figure) the points O and A lie on opposite sides of $[CB]$, in which case the angle $\angle AOB$ is the difference obtained on subtracting the angle $\angle BOD$ from the angle $\angle AOD$, and the angle $\angle ACB$ is the difference obtained on subtracting the angle $\angle BCD$ from the angle $\angle ACD$. It follows therefore that, in this case also, the angle $\angle AOB$ is equal to double the angle $\angle ACB$, as required. This completes the proof. ∎

Proposition 4.8 (included in Elements, III-31)

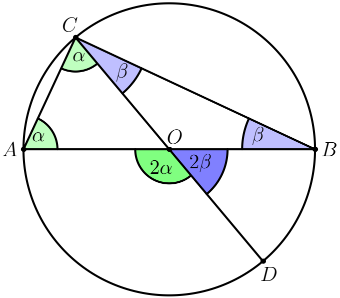
Let $[AB]$ be a diameter of a circle, and let C be a point on the circumference of the circle distinct from the endpoints A and B of the diameter. Then the angle $\angle ACB$ is a right angle.

Proof

Let O be the the centre of the circle, and let the diameter $[CD]$ having one endpoint at C be drawn. Let $\alpha = |\angle OCA|$ and $\beta = |\angle OCB|$. The triangle $\triangle OAC$ is isosceles. The Isosceles Triangle Theorem (*Elements*, I, 5) (*PPG*, Theorem 2) and therefore $|\angle OAC| = |\angle OCA| = \alpha$. Similarly $|\angle OBC| = |\angle OCB| = \beta$.



Now $\angle ACB$ is the sum of the angles $\angle ACO$ and $\angle BCO$. Moreover the Exterior Angles Theorem (*Elements*, I, 32) (*PPG*, Theorem 6) then ensures that the angles $\angle AOD$ and $\angle BOD$ are the doubles of the angles $\angle ACO$ and $\angle BCO$. It follows that the double of the angle $\angle ACB$ is equal to two right angles. Therefore $\angle ACB$ is a right angle, as required. ∎

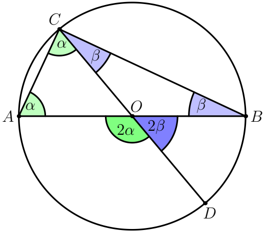


Remark

The proof may be presented symbolically, with angles measured in degrees, as follows.
$|\angle ACB| = |\angle ACO| + |\angle BCO| = \alpha + \beta$.
Moreover $|\angle AOD| = 2 \times \angle ACO = 2\alpha$ and $|\angle BOD| = 2 \times \angle BCO = 2\beta$, and $\angle AOD + \angle BOD = 180^\circ$.
Therefore

$$\begin{aligned} 2 \times |\angle ACB| \\ &= 2\alpha + 2\beta \\ &= |\angle AOD| + |\angle BOD| = 180^\circ, \end{aligned}$$

and therefore $|\angle ACB| = 90^\circ$,
as required.

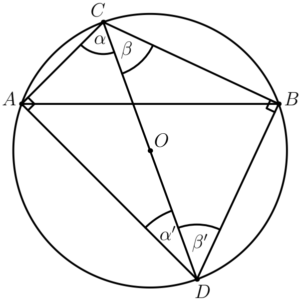


Corollary 4.9

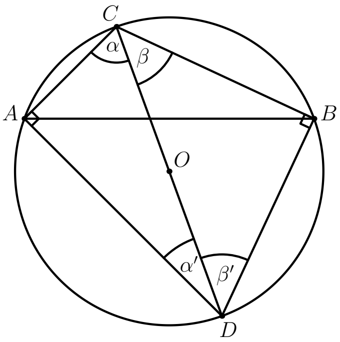
Let ABCD be a quadrilateral inscribed in a circle. Suppose that the line segment [CD] with endpoints C and D is a diameter of the circle. Then the sum of the angles $\angle ACB$ and $\angle ADB$ is equal to two right angles.

Proof

The sum of the two angles $\angle ACB$ and $\angle ACB$ is equal to the sum of the four angles $\angle ACD$, $\angle BCD$, $\angle ADC$ and $\angle BDC$.



Now it follows from Proposition 4.8 that the angles $\angle CAD$ and $\angle BCD$ are right angles. Also the sum of the three angles of any triangle is equal to two right angles (*Elements*, I, 32) (*PPG*, Theorem 4). It follows that the sum of the angles $\angle ACD$ and $\angle ADC$ is equal to one right angle. Similarly the sum of the angles $\angle BCD$ and $\angle BDC$ is equal to one right angle. Therefore the sum of the four angles $\angle ACD$, $\angle BCD$, $\angle ADC$ and $\angle BDC$ is equal to two right angles. The result follows. ∎

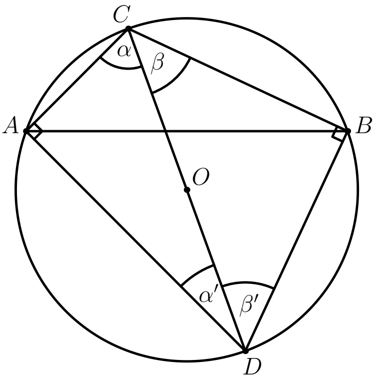


The proof may be presented symbolically, with angles measured in degrees, as follows.

Let $|\angle ACD| = \alpha$, $|\angle BCD| = \beta$, $|\angle ADC| = \alpha'$, $|\angle BDC| = \beta'$, The sum of the two acute angles in any right-angled triangle is equal to one right angle, Thus $\alpha + \alpha' = 90^\circ$ and $\beta + \beta' = 90^\circ$. It follows that

$$\begin{aligned} |\angle ACB| + |\angle ADB| &= \alpha + \alpha' + \beta + \beta' \\ &= 180^\circ, \end{aligned}$$

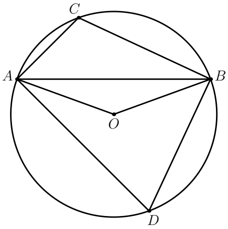
as required.



Corollary 4.10

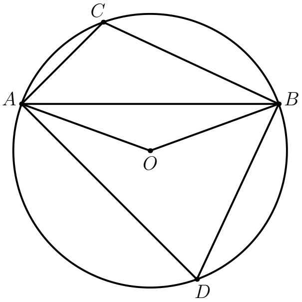
Let $[AB]$ be a chord of a circle that does not pass through the centre of the circle, and let C be a point on the circumference of that circle that lies on the opposite side of the line AB to the centre O of the circle. Then the sum of double the angle $\angle ACB$ and the angle $\angle AOB$ is equal to four right angles. Thus, if angles are measured in degrees,

$$2 \times |\angle ACB| + |\angle AOB| = 360^\circ.$$



Proof

Let the ray from C passing through the centre O of the circle intersect the circle again at D. Corollary 4.9 ensures that the sum of the angles $\angle ACB$ and $\angle ADB$ is equal to two right angles. Also it follows from Proposition 4.7 that the angle $\angle AOB$ is double the angle $\angle ADB$. Combining these results we see that the sum of double the angle $\angle ACB$ and the angle $\angle AOD$ is equal to four right angles, as required. ∎



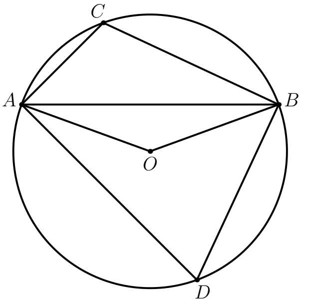
Remark

The proof may be presented symbolically, with angles measured in degrees, as follows.

The point D is determined so that the line segment $[CD]$ is a diameter of the circle. Corollary 4.9 ensures that $|\angle ACB| + |\angle ADB| = 180^\circ$. Moreover Proposition 4.7 ensures that $|\angle AOB| = 2 \times |\angle ADB|$. Therefore

$$\begin{aligned} 2 \times |\angle ACB| + |\angle AOB| & \\ = \quad & 2 \times (|\angle ACB| + |\angle ADB|) \\ = \quad & 2 \times 180^\circ = 360^\circ, \end{aligned}$$

as required.

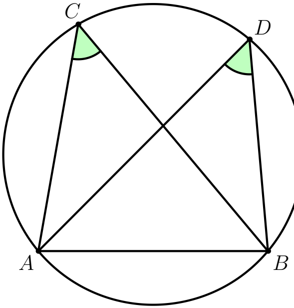


Proposition 4.11 (Elements, III-21)

Let $[AB]$ be a chord or diameter of a circle, and let C and D be points on the circumference of the circle that lie on the same side of the line AB. Then $|\angle ACB| = |\angle ADB|$.

Proof

The result follows directly on applying Proposition 4.7 in the case where the points C and D lie on the same side of AB as the centre of the circle, Proposition 4.8 in the case where AB is a diameter of the circle, and Corollary 4.10 in the case where the centre of the circle lies on the opposite side of AB to the points C and D. ■



Proposition 4.12 (Elements, III-22)

The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.

Proof

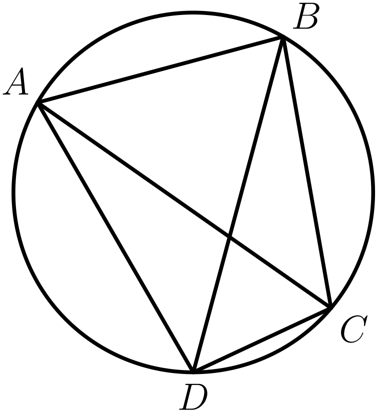
We denote the magnitude of the angle $\angle ABC$ by $|\angle ABC|$, and adopt analogous notation for other angles in the figure.

4. Selected Circle Theorems (continued)

It follows from Proposition 4.11 that $|\angle ABD| = |\angle ACD|$ and $|\angle DBC| = |\angle DAC|$. Therefore

$$\begin{aligned} |\angle ABC| &= |\angle ABD| + |\angle DBC| \\ &= |\angle ACD| + |\angle DAC| \end{aligned}$$

Therefore $|\angle ABC| + |\angle ADC|$ is the sum of the three angles of the triangle $\triangle ADC$ and is therefore equal to two right angles. $\blacksquare$

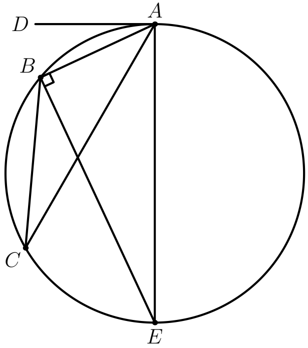


Proposition 4.13 (Elements, III-32)

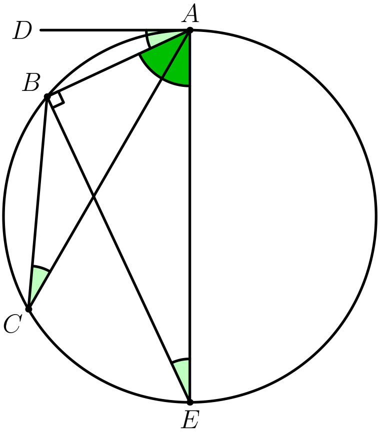
Let A, B and C be distinct points on the circumference of a circle, and let $[AD]$ be a line segment tangent to the circle at A, where D lies on the opposite side of the line AB to the point C. Then the angles $\angle DAB$ and $\angle BCA$ are equal to one another.

Proof

Let $[AE]$ be the diameter of the circle that meets the circumference at the point A. Then $\angle DAE$ is a right angle (*Elements*, III, 16) (*PPG*, Theorem 20). Moreover $\angle EBA$ is a right angle (Proposition 4.8 above) (*Elements*, III, 31) (*PPG*, Corollary 3 following Theorem 19).

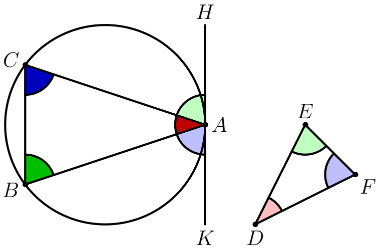


Thus if the angle $\angle BAE$ is added to $\angle DAB$, and to $\angle AEB$, the sum in both cases is equal to a right angle. It follows that the angles $\angle DAB$ and $\angle AEB$ are equal to one another. Now the angles $\angle AEB$ and $\angle ACB$ are also equal to one another (Proposition 4.11 above) (*Elements*, III, 21) (*PPG*, Corollary 2 following Theorem 19). Therefore the angles $\angle DAB$ and $\angle BCA$ are equal to one another, as required. $\blacksquare$



Proposition 4.14 (Elements, IV-2)

Let a circle given, and let HK be a line tangent to the circle at a point A on the circle. Let $\triangle DEF$ be a triangle. Let points B and C be found on the circle for which $|\angle HAC| = |\angle DEF|$ and $|\angle BAK| = |\angle EFD|$. Then the angles of the triangle $\triangle ABC$ at A, B and C are equal to the angles of the triangle $\triangle DEF$ at D, E and F respectively.



Proof

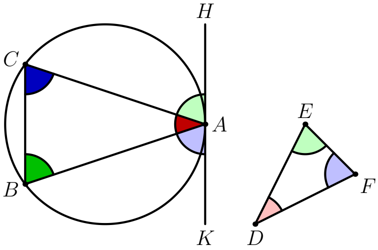
The angles of a triangle add up to two right angles. It follows that $|\angle CAB| = |\angle FDE|$. It follows directly on applying Proposition 4.13 that

$$|\angle ABC| = |\angle HAC| = |\angle DEF|$$

and

$$|\angle BCA| = |\angle BAK| = |\angle EFD|.$$

The result follows. ∎



In the discussion of the results that follow let us introduce some convenient notation for magnitudes representing areas of squares and rectangles. Given line segments $[AB]$, $[CD]$, $[EF]$ and $[GH]$, let us write

$$|AB| \times |AD| = |EF| \times |EH|$$

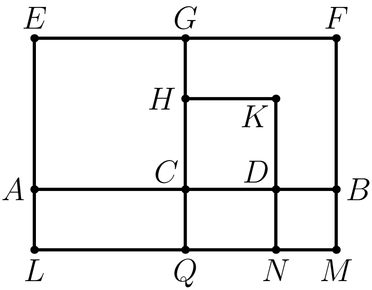
when we wish to specify that a first rectangle is equal in area to a second rectangle, in cases where the first rectangle has sides meeting at a corner that are equal in length to $[AB]$ and $[CD]$ and the second rectangle has sides meeting at a corner that are equal in length to $[EF]$ and $[GH]$ respectively. Moreover in cases where the line segments $[AB]$ and $[CD]$ meeting at a corner are equal in length, let us write $|AB|^2$ in place of $|AB| \times |CD|$.

We generalize such notation in an obvious fashion to indicate when some rectangle or a sum of some finite collection of rectangles is equal in area to some (often different) rectangle or finite collection of rectangles. The results stated in propositions included in Book II of Euclid's *Elements* can often be represented using such notation, which can prove useful in describing the results of the final three propositions included in Book III of Euclid's *Elements*.

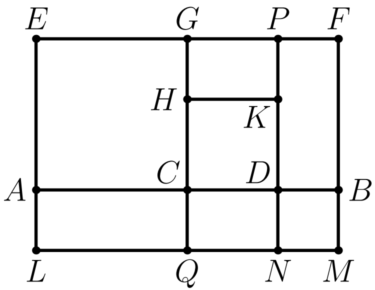
Lemma 4.15 (Elements, II-5)

Let [AB] be a line segment with midpoint C and let D be a point on that line segment distinct from C. Then

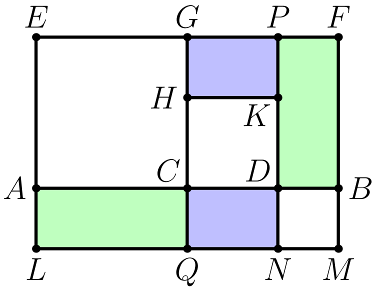
$$|AD| \times |DB| + |CD|^2 = |AC|^2 = |CB|^2.$$



Construct rectangles $ABFE$ and $ABML$ with $|AE| = |AC|$ and $|BM| = |DB|$ (as shown). Also construct a square $CDKH$ on side $[CB]$, and on the same side of $[CB]$ as the rectangle $ABFE$, produce $[CH]$ to G and Q, and produce $[DK]$ to N and P, where G and P lie on $[EF]$ and N and Q lie on $[LM]$. We must show that the the geometric figure $LNKHCA$ representing the sum of the rectangle $ADNL$ and the square $CDKH$ is equal in area to each of the squares $ACGE$ and $CBGF$.



Now the rectangles $ACQL$ and $DBFP$ are equal in area, because $|AC| = |BF|$ and $|AL| = |DB|$. Also the rectangles $CDNQ$ and $HKPG$ are equal in area, because $|CQ| = |GH|$. It follows that the figure $LNKHCA$ is equal in area to the square $CBFG$ on $[CB]$, and as thus equal in area to the square $ACGE$ on $[AC]$, as required. ∎



Remark

If we use modern notation, representing lengths and areas using real numbers relative to chosen units, then the result stated in Lemma 4.15 corresponds to the algebraic identity

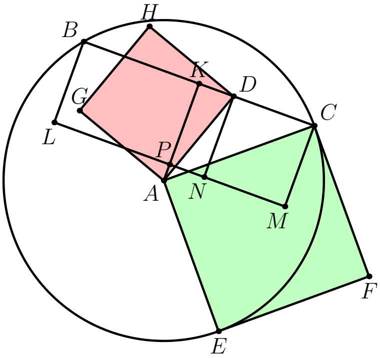
$$(u+v)(u-v)+v^2=u^2.$$

Here u represents $|AC|$ (or $|CB|$) and v represents $|CD|$, and therefore, in cases where D lies between C and B, $u+v$ represents $|AD|$ and $u-v$ represents $|DB|$.

Proposition 4.16

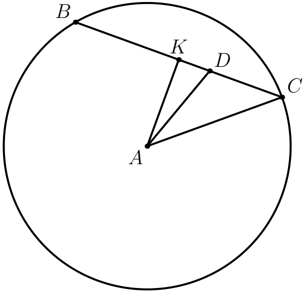
Let a circle with centre A be given, let B and C be distinct points on the circumference of that circle, and let D be a point on the chord $[BC]$. Then

$$|BD| \times |DC| + |AD|^2 = |AC|^2.$$



Proof

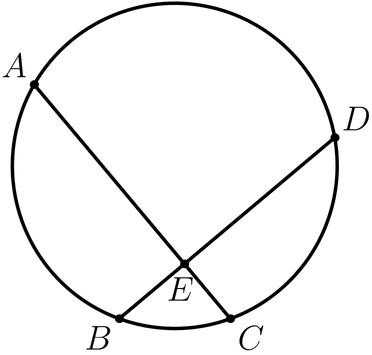
Let K denote the midpoint of the chord $[BC]$. It follows from Lemma 4.15 that $|BD| \times |DC| + |KD|^2 = |KC|^2$. Pythagoras's Theorem ensures that $|AK|^2 + |KC|^2 = |AC|^2$ and $|AK|^2 + |KD|^2 = |AD|^2$. It follows that $|BD| \times |DC| + |AD|^2 = |AC|^2$, as required. ∎



Corollary 4.17 (see Elements, III-35)

Let A, B, C and D be distinct points on the circumference of a circle, and let the chords $[AC]$ and $[BD]$ intersect at a point E in the interior of the circle. Then

$$|AE| \times |EC| = |BE| \times |ED|.$$

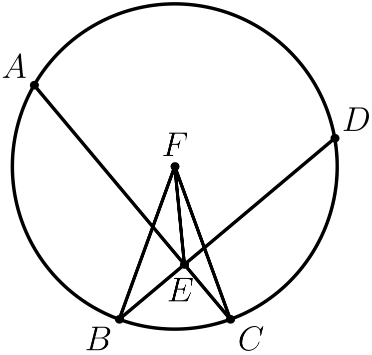


Proof

Let F denote the centre of the circle. It follows from Proposition 4.16 that

$$|AE| \times |EC| = |FC|^2 - |FE|^2 = |FD|^2 - |FE|^2 = |BE| \times |ED|,$$

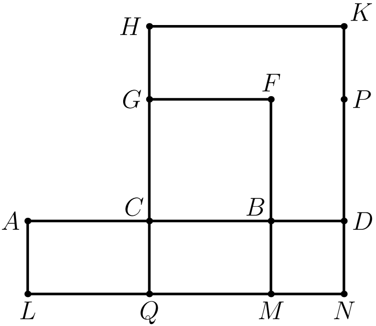
as required. ∎



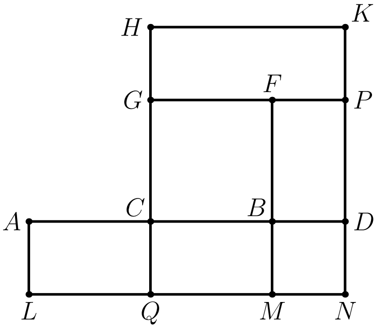
Lemma 4.18 (Elements, II-6)

Let $[AB]$ be a line segment with midpoint C and let D be a point on that line segment distinct from C. Then

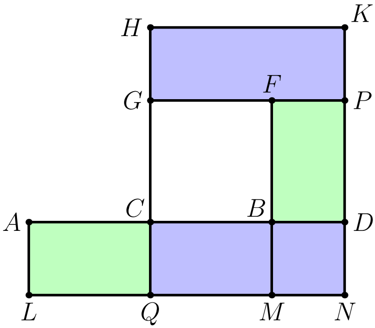
$$|AD| \times |DB| + |CB|^2 = |CD|^2.$$



Construct a rectangle $ADNL$ with $|DN| = |DB|$ (as shown). Also construct squares $CBFG$ and $CDKH$ on sides $[CB]$ and $[CD]$ respectively, and on the same side of $[CD]$, produce $[FB]$ and $[GC]$ to M and Q respectively, where M and Q lie on $[LM]$, and produce $[GF]$ to P, where P lies on $[DK]$. We must show that the geometric figure $LNDBFGCA$ representing the sum of the rectangle $ADNL$ and the square $CBFG$ is equal in area to the square $CDKH$.



Now the rectangles $ACQL$ and $BDPF$ are equal in area, because $|AC| = |BF|$ and $|AL| = |BD|$. Also the rectangles $CDNQ$ and $HKPG$ are equal in area, because $|CQ| = |GH|$. It follows that the figure $LNKHCA$ is equal in area to the square $CBFG$ on $[CB]$, as required. ∎



Remark

If we use modern notation, representing lengths and areas using real numbers relative to chosen units, then the result stated in Lemma 4.15 corresponds to the algebraic identity

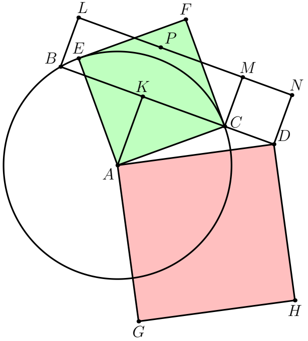
$$(2u + v)v + u^2 = (u + v)^2.$$

Here u represents $|CB|$ and v represents $|DB|$, and therefore $2u + v$ represents $|AD|$ and $u + v$ represents $|CD|$.

Proposition 4.19

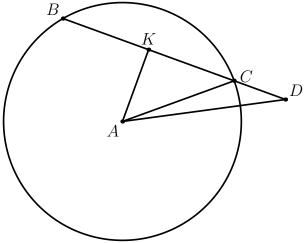
Let a circle with centre A be given, let B and C be distinct points on the circumference of that circle, and let the chord $[BC]$ be produced beyond C to a point D outside the circle. Then

$$|BD| \times |DC| + |AC|^2 = |AD|^2.$$



Proof

Let K denote the midpoint of the chord $[BC]$. It follows from Lemma 4.18 that $|BD| \times |DC| + |KC|^2 = |KD|^2$. Pythagoras's Theorem ensures that $|AK|^2 + |KC|^2 = |AC|^2$ and $|AK|^2 + |KD|^2 = |AD|^2$. It follows that $|BD| \times |DC| + |AC|^2 = |AD|^2$, as required. ∎

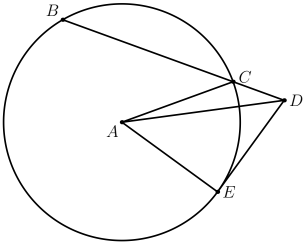


Proposition 4.20 (Elements, III-36 and III-37)

Let a circle with centre A be given, let B, D and E be distinct points on the circumference of that circle, and let the chord $[BC]$ be produced beyond C to a point D outside the circle. Then

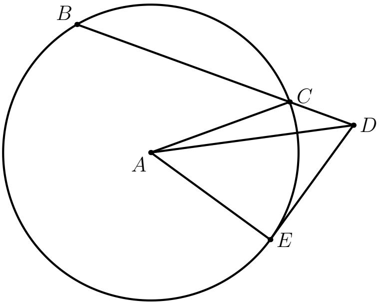
$$|BD| \times |DC| = |DE|^2$$

if and only if the line DE is tangent to the circle at the point E.

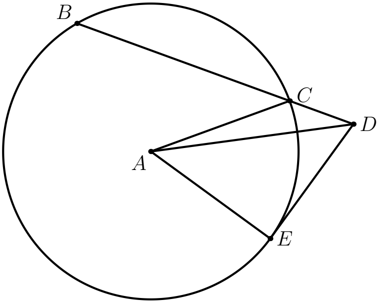


Proof

Let A denote the centre of the circle. It follows from Proposition 4.19 that $|BD| \times |DC| + |AC|^2. = |AD|^2$. Moreover $|AC| = |AE|$. It follows that $|BD| \times |DC| = |DE|^2$ if and only if $|AE|^2 + |DE|^2 = |AD|^2$.

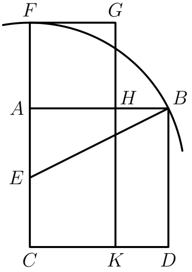


Pythagoras's Theorem (*Elements*, I, 47) (*PPG*, Theorem 14) and its converse (*Elements*, I, 48) (*PPG*, Theorem 15) together ensure that $|AE|^2 + |DE|^2 = |AD|^2$.
if and only if $|AE|D$ is a right angle. Moreover $\angle AED$ is a right angle if and only if the line $[DE]$ is tangent to the circle at E (Proposition 4.6). The result follows. ∎



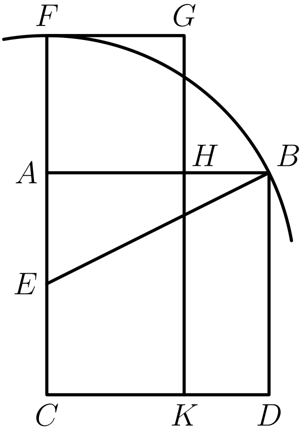
Proposition 4.21 (Elements, II-11)

Let $ABDC$ be a square as depicted in the accompanying figure, let E be the midpoint of the side $[AC]$ of that square, let F be a point on the line EA produced beyond A for which $|EF| = |EB|$, and let the square $AFGH$ be completed so that H lies between A and B. Let the side $[GH]$ of this square be produced beyond H to a point K on the side $[CD]$ of the square $ABDC$. Then $|AH|^2 = |AB| \times |HB|$.



Proof

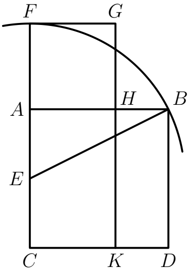
A direct application of Lemma 4.18 (*Elements*, II, 6) shows that the sum of the rectangle *CFGH* and a square with side [*EA*] is equal in area to a square with side [*EF*], and thus is equal to a square with side [*EF*].



In symbols, applying Pythagoras's Theorem,

$$|CF| \times |AH| + |EA|^2 = |EB|^2 = |EA|^2 + |AB|^2.$$

Thus $|CF| \times |AH| = |AB|^2$. But
$|CF| \times |AH| = |AC| \times |AH| + |AH|^2$ and
$|AB|^2 = |AC| \times |AH| + |AC| \times |HB|$.



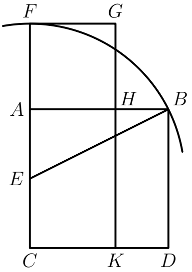
Thus

$$|AC| \times |AH| + |AH|^2 = |AC| \times |AH| + |AC| \times |HB|,$$

and therefore

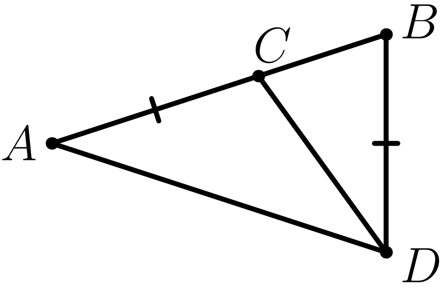
$$|AH|^2 = |AC| \times |HB| = |AB| \times |HB|,$$

as required. $\blacksquare$

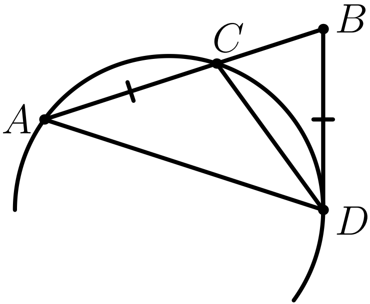


Proposition 4.22 (Elements, IV-10)

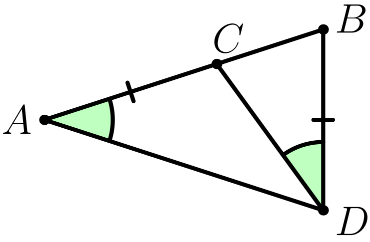
Let A and B be distinct points in the plane, let C be a point on the line segment $[AB]$ for which $|AC|^2 = |AB| \times |BC|$, and let D be a point of the plane for which $|AD| = |AB|$ and $|BD| = |AC|$. Then the angles $\angle DAB$, $\angle CDA$ and $\angle CDB$ are equal, the angles $\angle ABD$ and $\angle ADB$ are equal to one another, and are equal to twice the angle $\angle DAB$, and $|AC| = |CD| = |BD|$.



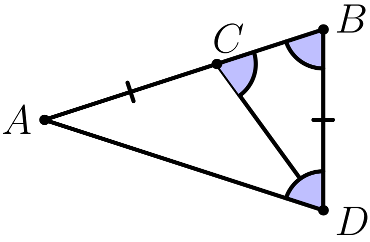
Let a circle be drawn through the points A, C and D. Now $|AB| \times |BC| = |BD|^2$. It follows from Proposition 4.20 that the line BD is tangent to the circle through A, C and D. It then follows from Proposition 4.13 that the angles $\angle BDC$ and $\angle DAC$ are equal.



Now $\angle BDA$ is the sum of the angles $\angle BDC$ and $\angle CDA$, and we have just shown that the angle $\angle BDC$ is equal to $\angle CAD$. It follows that the angle $\angle BDA$ is equal to the sum of the interior angles $\angle CAD$ and $\angle CDA$ of the triangle $\triangle CAD$. The sum of these angles is equal to the exterior angle $\angle BCD$ of this triangle at C (*Elements*, I, 32) (*PPG*, Theorem 6).



Now the conditions of the proposition require that $|AB|$ and $|AD|$ be equal. The Isosceles Triangle Theorem then (*Elements*, I, 5) (*PPG*, Theorem 2) ensures that the angles $\angle ADB$ and $\angle ABD$ are equal. Thus the three angles $\angle ADB$, $\angle CBD$ and $\angle DCB$ are equal. It then follows that $[CD]$ and $[BD]$ are equal (*Elements*, I, 6) (*PPG*, Theorem 2).



But $[CA]$ and $[BD]$ are also equal. Thus $[CA]$ and $[CB]$ are equal, and thus $\angle CAD$ is an isosceles triangle. It then follows that the angles $\angle CAD$ and $\angle CDA$ are equal. (*Elements*, I, 5) (*PPG*, Theorem 2) The angle $\angle BDA$ is thus the sum of two equal angles $\angle BDC$ and $\angle CDA$ which are both equal to $\angle DAB$. It follows that the equal angles $\angle ABD$ and $\angle ADB$ of the isosceles triangle $\triangle ABD$ are both equal to twice the remaining angle $\angle DAB$ of that triangle. This completes the proof. ∎