# Course 421: Academic Year 2002–3
# Problems II

## Problems

1. Prove that points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in $\mathbb{R}^k$ are geometrically independent if and only if the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

2. Show that any two continuous maps from a topological space $X$ to a simplex $\sigma$ are homotopic.

3. Prove that the boundary of an $n$-simplex is homeomorphic to an $(n-1)$-sphere for any $n > 0$. (The *boundary* of a simplex is the union of all the proper faces of the simplex.)

4. Let $\varphi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ be a bijection between the vertex sets of simplicial complexes $K$ and $L$. Suppose that vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of $K$ if and only if $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex of $L$. Show that the bijection $\varphi$ induces a homeomorphism between the polyhedra $|K|$ and $|L|$ of $K$ and $L$.

5. Let $s\colon K \to L$ and $t\colon K \to L$ be simplicial maps between simplicial complexes $K$ and $L$. Suppose that $t$ is a simplicial approximation to $s$. (More precisely, suppose that $t\colon K \to L$ is a simplicial approximation to the continuous map $s\colon |K| \to |L|$ induced by the simplicial map $s\colon K \to L$.) Show that $t = s$.

6. Let $K$ be a simplicial complex, let $L$ be the simplicial complex consisting of the proper faces of some $(n+1)$-simplex, where $n > 0$, and let $s\colon |K| \to |L|$ be a continuous map between the polyhedra of $K$ and $L$ that is induced by some simplicial map from $K$ to $L$. Suppose that $\dim K < n$. Show that the function $s\colon |K| \to |L|$ is homotopic to a constant map sending $|K|$ to a vertex of $L$. Use this result and the Simplicial Approximation Theorem to prove that any two continuous maps from $|K|$ to the $n$-dimensional sphere $S^n$ are homotopic when $\dim K < n$.

7. Given a simplicial complex $K$ we define the *Euler characteristic* $\chi(K)$ to be $\sum_{r=0}^{\dim K} (-1)^r n_r$, where $n_r$ is the number of simplices of $K$ of dimension $r$.

(a) Let $K_\sigma$ be the simplicial complex consisting of some $n$-simplex $\sigma$ together with all of its faces. Explain why the number of $r$-simplices of $K_\sigma$ is $\binom{n+1}{r+1}$ for any integer $r$ between 0 and $n$, and use this result to show that $\chi(K_\sigma) = 1$. [Hint: use the Binomial Theorem.]

(b) Prove that $\chi(K) = \chi(K')$ for any simplicial complex $K$, where $\chi(K)$ is the Euler characteristic of $K$, and $K'$ is the first barycentric subdivision of $K$. [Hint: it is possible to prove this using the result of part $(a)$, together with induction on the number of simplices in $K$.]

8. Suppose that there were to exist a continuous map $f\colon E^n \to E^n$ with no fixed point, where $E^n$ is the closed unit ball in $\mathbb{R}^n$. Let $q\colon E^n \to S^{n-1}$ be the map (utilized in the proof of the Brouwer Fixed Point Theorem) which sends a $\mathbf{x} \in E^n$ to the point $q(\mathbf{x})$ at which the half line starting at $f(\mathbf{x})$ and passing through $\mathbf{x}$ intersects the boundary sphere $S^{n-1}$ of $E^n$. Calculate a formula giving $q(\mathbf{x})$ in terms of $\mathbf{x}$ and $f(\mathbf{x})$, and verify that the map $q\colon E^n \to S^{n-1}$ would be continuous.

9. Give an example of a continuous function $f\colon S^n \to S^n$ mapping the $n$-sphere into itself which has no fixed point.

10. Let $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ be the vertices of a triangle in the plane $\mathbb{R}^2$, and let $T$ be the simplicial complex consisting of this triangle together with all its edges and vertices. Calculate

$$\partial_1 \left( n_0 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle + n_1 \langle \mathbf{u}_2, \mathbf{u}_0 \rangle + n_2 \langle \mathbf{u}_0, \mathbf{u}_1 \rangle \right)$$

where $n_0$, $n_1$ and $n_2$ are arbitrary integers, and use the result to show that $H_0(T) \cong \mathbb{Z}$ and that $H_q(T) = 0$ for all integers $q$ satisfying $q \neq 0$.

11. Let $OCT$ be the simplicial complex consisting of all the faces, edges and vertices of an octohedron in $\mathbb{R}^3$. Let $P_1, P_2, P_3, P_4, P_5, P_6$ be the vertices of this octohedron, let $e_1, e_2, \ldots, e_{12}$ be the edges of the octohedron, oriented so that

$$\begin{gathered} e_1 = \langle P_1, P_2 \rangle, \quad e_2 = \langle P_1, P_3 \rangle, \quad e_3 = \langle P_1, P_4 \rangle, \quad e_4 = \langle P_1, P_5 \rangle, \\ e_5 = \langle P_2, P_3 \rangle, \quad e_6 = \langle P_3, P_4 \rangle, \quad e_7 = \langle P_4, P_5 \rangle, \quad e_8 = \langle P_5, P_2 \rangle, \\ e_9 = \langle P_2, P_6 \rangle, \quad e_{10} = \langle P_3, P_6 \rangle, \quad e_{11} = \langle P_4, P_6 \rangle, \quad e_{12} = \langle P_5, P_6 \rangle, \end{gathered}$$

and let $\sigma_1, \sigma_2, \ldots, \cdots \sigma_8$ be the triangular faces of the octohedron, oriented so that

$$\sigma_1 = \langle P_1, P_2, P_3 \rangle, \quad \sigma_2 = \langle P_1, P_3, P_4 \rangle, \quad \sigma_3 = \langle P_1, P_4, P_5 \rangle,$$

$$\sigma_4 = \langle P_1, P_5, P_2 \rangle\,, \quad \sigma_5 = \langle P_6, P_3, P_2 \rangle\,, \quad \sigma_6 = \langle P_6, P_4, P_3 \rangle\,,$$
$$\sigma_7 = \langle P_6, P_5, P_4 \rangle\,, \quad \sigma_8 = \langle P_6, P_2, P_5 \rangle\,.$$

(a) Draw a diagram of this octohedron with all the vertices, edges and faces clearly labelled. (It is suggested that you take

$$P_1 = (0,0,1), \quad P_2 = (1,0,0), \quad P_3 = (0,1,0),$$

$$P_4 = (-1,0,0), \quad P_5 = (0,-1,0), \quad P_6 = (0,0,-1)$$

as the vertices). Put an arrow on each edge to indicate the chosen orientation on that edge. (Thus if the edge is listed above as $\langle P_i, P_j \rangle$ then draw an arrow on the edge pointing from $P_i$ to $P_j$.)

(b) Calculate $\partial_2(\sigma_i)$ for $i = 1, 2, \ldots, 8$ in terms of $e_1, e_2, \ldots, e_{12}$. Show that

$$\partial_2 \left( \sum_{i=1}^{8} n_i \sigma_i \right) = 0,$$

where $n_1, n_2, \ldots, n_8 \in \mathbb{Z}$, if and only if $n_1 = n_2 = \cdots = n_8$. Hence show that $H_2(OCT) = \mathbb{Z}$.

(c) Let $c = m_1 e_1 + m_2 e_2 + \cdots + m_{12} e_{12}$ be a typical 1-chain of $OCT$, where $m_1, m_2, \ldots, m_{12}$ are integers. Show that $c$ is a 1-cycle of $OCT$ if and only if $c$ is a 1-boundary of $OCT$. Hence write down $H_1(OCT)$.

(d) What is $H_0(OCT)$?

12. Let $K$ be a simplicial complex and let $L$ and $M$ be subcomplexes of $K$ with the property that $K = L \cup M$. Let

$$i_*\colon H_q(L \cap M) \to H_q(L), \qquad u_*\colon H_q(L) \to H_q(K),$$

$$j_*\colon H_q(L \cap M) \to H_q(M), \qquad v_*\colon H_q(M) \to H_q(K)$$

be the homomorphisms of homology groups induced by the inclusions

$$i_q\colon C_q(L \cap M) \hookrightarrow C_q(L), \qquad j_q\colon C_q(L \cap M) \hookrightarrow C_q(M),$$

$$u_q\colon C_q(L) \hookrightarrow C_q(K), \qquad v_q\colon C_q(M) \hookrightarrow C_q(K).$$

(a) Show that if $H_q(L \cap M) = 0$ and $H_{q-1}(L \cap M) = 0$ for some positive integer $q$, then $H_q(K) \cong H_q(L) \oplus H_q(M)$.

(b) Show that if $H_q(L) = 0$, $H_q(M) = 0$, $H_{q-1}(L) = 0$ and $H_{q-1}(M) = 0$ for some positive integer $q$, then $H_q(K) \cong H_{q-1}(L \cap M)$.

(c) Show that if $j_*\colon H_q(L \cap M) \to H_q(M)$ is an isomorphism for all $q \in \mathbb{Z}$ then so is $u_*\colon H_q(L) \to H_q(K)$.

13. Let $K$ be a 2-dimensional simplicial complex, let $L$ be a subcomplex of $K$ and let $\sigma$ be a 2-simplex (triangle) belonging to $K$. Let $M_\sigma$ be the simplicial complex consisting of the 2-simplex $\sigma$ together with all its edges and vertices. Suppose that $K = L \cup M_\sigma$ and that $L \cap M_\sigma$ is given by one of the following possibilities:—

(i) $L \cap M_\sigma$ consists of a single vertex of $\sigma$,

(ii) $L \cap M_\sigma$ consists of a single edge of $\sigma$ together with its endpoints,

(iii) $L \cap M_\sigma$ consists of two edges of $\sigma$ together with their endpoints,

Using the Mayer-Vietoris exact sequence, prove that the inclusion of the subcomplex $L$ in $K$ induces isomorphisms $H_q(L) \cong H_q(K)$ of homology groups for all $q$ in each of these three cases.

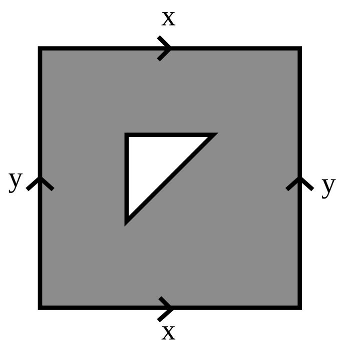

Figure 1

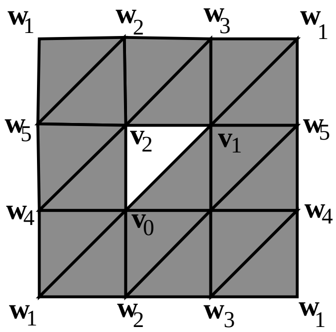

Figure 2

14. Let $L$ be a simplicial complex whose polyhedron is a square with a triangle removed, with top and bottom edges and left and right edges identified as depicted in Figure 1, triangulated as depicted in Figure 2.

Let $z$ be the 1-cycle of $L$ given by

$$z = \langle \mathbf{v}_0, \mathbf{v}_1 \rangle + \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{v}_2, \mathbf{v}_0 \rangle.$$

(a) Show that $z \in B_1(L)$.

Let $L_0$ be the subcomplex of $L$ consisting of the vertices $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ and $\mathbf{w}_5$ and the edges $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, $\langle \mathbf{w}_2, \mathbf{w}_3 \rangle$, $\langle \mathbf{w}_3, \mathbf{w}_1 \rangle$, $\langle \mathbf{w}_1, \mathbf{w}_4 \rangle$, $\langle \mathbf{w}_4, \mathbf{w}_5 \rangle$ and $\langle \mathbf{w}_5, \mathbf{w}_1 \rangle$.

(b) Using repeated applications of question 13, show that the inclusion of the subcomplex $L_0$ in $L$ induces isomorphisms $H_q(L_0) \cong H_q(L)$ of homology groups for all $q$.

(c) Show that $H_1(L_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and that $H_1(L_0)$ is generated by the homology classes of the 1-cycles $z_1$ and $z_2$, where

$$\begin{aligned} z_1 &= \langle \mathbf{w}_1, \mathbf{w}_2 \rangle + \langle \mathbf{w}_2, \mathbf{w}_3 \rangle + \langle \mathbf{w}_3, \mathbf{w}_1 \rangle, \\ z_2 &= \langle \mathbf{w}_1, \mathbf{w}_4 \rangle + \langle \mathbf{w}_4, \mathbf{w}_5 \rangle + \langle \mathbf{w}_5, \mathbf{w}_1 \rangle. \end{aligned}$$

What are $H_0(L_0)$ and $H_2(L_0)$?

(d) A triangulation $K$ of the torus is obtained on adding an extra triangle to the complex $L$ with edges $\langle \mathbf{v}_0, \mathbf{v}_1 \rangle$, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $\langle \mathbf{v}_2, \mathbf{v}_0 \rangle$ (thus filling in the 'missing triangle' in Figure 1). Use the Mayer-Vietoris exact sequence to calculate the homology groups of the torus (with this triangulation).

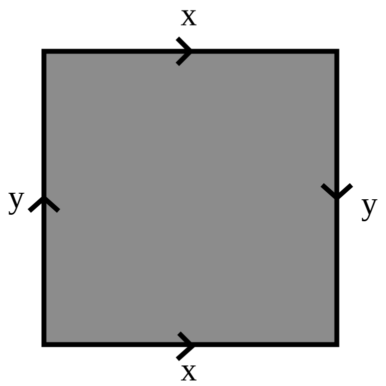

Figure 3: Klein bottle

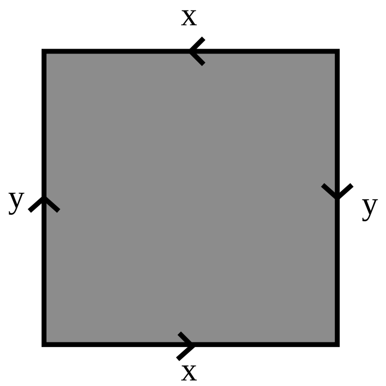

Figure 4: Real projective plane

15. The Klein bottle $\mathrm{Kl}^2$ and the real projective plane $\mathbb{R}P^2$ are obtained from the square by identifying edges as depicted in Figures 3 and 4. By applying the method of question 14 (i.e., removing a triangle, finding

the homology groups of the resultant subcomplex and making use of the Mayer-Vietoris exact sequence), show that

$$H_0(\mathrm{Kl}^2) \cong \mathbb{Z}, \quad H_1(\mathrm{Kl}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_2(\mathrm{Kl}^2) \cong 0,$$

$$H_0(\mathbb{R}P^2) \cong \mathbb{Z}, \quad H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2, \quad H_2(\mathbb{R}P^2) \cong 0.$$

[Note that, although all four corners of the square are identified together in the case of the Klein bottle, only diagonally opposite corners are identified together in the case of the real projective plane.]