Course 421: Academic Year 2002–3
Problems I

1. Let X be a non-empty set. Let $d: X \times X \to \mathbb{R}$ be defined so that $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Verify that the distance function d on X satisfies the metric space axioms.

2. Let X be a metric space with distance function d, and let A be a non-empty subset of X. Let $f: X \to \mathbb{R}$ be the function defined by $f(x) = \inf\{d(x, a) : a \in A\}$ (i.e., $f(x)$ is the largest real number with the property that $f(x) \leq d(x, a)$ for all $a \in A$). Use the Triangle Inequality to prove that $f(x) \leq f(y) + d(x, y)$ for all $x, y \in X$, and hence show that $|f(x) - f(y)| < d(x, y)$. (Note that this implies that the function $f: X \to \mathbb{R}$ is continuous.) Prove that A is closed in X if and only if $A = \{x \in X : f(x) = 0\}$.

3. Prove that the set
$$\{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } x^2 + y^2 < 1\}$$
is neither open nor closed in $\mathbb{R}^2$.

4. Explain why
$$\{(x, y) \in \mathbb{R}^2 : y^3 - 7 < \sin(3x^3 + x^{12} - 57x^{937})\}$$
is an open set in $\mathbb{R}^2$.

5. The *Zariski topology* on the real numbers $\mathbb{R}$ is the topology whose open sets are the empty set, the set $\mathbb{R}$ itself and those subsets of $\mathbb{R}$ whose complements are finite.

   (a) Verify that the Zariski topology is a well-defined topology on the set $\mathbb{R}$ of real numbers (i.e., show that the topological space axioms are satisfied).

   (b) Prove that any polynomial function from $\mathbb{R}$ to itself is continuous with respect to the Zariski topology in $\mathbb{R}$.

   (c) Give an example of a function from $\mathbb{R}$ to itself which is continuous with respect to the usual topology on $\mathbb{R}$ but is not continuous with respect to the Zariski topology on $\mathbb{R}$.

6. (a) Let $f: X \to Y$ be a function from a topological space X to a topological space Y, and let A and B be subsets of X for which $X = A \cup B$. Suppose that the restrictions $f|A$ and $f|B$ of f to the sets A and B are continuous. Is $f: X \to Y$ necessarily continuous on X? [Give proof or counterexample.]

   (b) Let $f: X \to Y$ be a function from a topological space X to a topological space Y, and let $\mathcal{F}$ be a (not necessarily finite) collection of closed subsets of X whose union is the whole of X. Suppose that the restriction $f|A$ of f to A is continuous for all closed sets A in the collection $\mathcal{F}$. Is $f: X \to Y$ necessarily continuous on X? [Give proof or counterexample.]

7. Let $f: X \to Y$ be a function from a topological space X to a topological space Y, and let $\mathcal{U}$ be a collection of open subsets of X whose union is the whole of X. Suppose that the restriction $f|W$ of f to W is continuous for all open sets W in the collection $\mathcal{U}$. Prove that $f: X \to Y$ is continuous on X.

8. Let X be a topological space, let A be a subset of X, and let B be the complement $X \setminus A$ of A in X. Prove that the interior of B is the complement of the closure of A.

9. Determine which of the following subsets of $\mathbb{R}^3$ are compact.

   (i) The x-axis $\{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$

   (ii) The surface of a tetrahedron in $\mathbb{R}^3$.

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 + y^2 - z^2 \leq 1\}$.

10. Let X be a topological space. Suppose that $X = A \cup B$, where A and B are path-connected subsets of X and $A \cap B$ is non-empty. Show that X is path-connected.

11. Let $f: X \to Y$ be a continuous map between topological spaces X and Y. Suppose that X is path-connected. Prove that the image $f(X)$ of the map f is also path-connected.

12. Let X and Y be path-connected topological spaces. Explain why the Cartesian product $X \times Y$ of X and Y is path-connected.

13. Determine the connected components of the following subsets of $\mathbb{R}^2$:

(i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,

(ii) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$,

(iii) $\{(x, y) \in \mathbb{R}^2 : y^2 = x(x^2 - 1)\}$,

(iv) $\{(x, y) \in \mathbb{R}^2 : (x - n)^2 + y^2 > \frac{1}{4}$ for all $n \in \mathbb{Z}\}$.

[Fully justify your answers.]

14. A topological space X is said to be *locally path-connected* if, given any point x of X there exists a path-connected open set U in X such that $x \in U$.

    (a) Let X be a locally path-connected topological space, and let p be a point of X. Let A be the set of all points x of X for which there exists a path from p to x, and let B be the complement of A in X. Prove that A and B are open in X.

    (b) Use the result of (a) to show that any connected and locally path-connected topological space is path-connected.

15. Let X be a convex subset of $\mathbb{R}^n$. (A subset X of $\mathbb{R}^n$ is said to be *convex* if $(1 - t)\mathbf{x} + t\mathbf{y} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and real numbers t satisfying $0 \leq t \leq 1$.)

    (a) Prove that any two continuous functions mapping some topological space into X are homotopic.

    (b) Prove that any two continuous functions mapping X into some path-connected topological space Y are homotopic.

16. Determine which of the following maps are covering maps:—

    (i) the map from $\mathbb{R}$ to $[-1, 1]$ sending θ to $\sin\theta$,

    (ii) the map from S^1 to S^1 sending $(\cos\theta, \sin\theta)$ to $(\cos n\theta, \sin n\theta)$, where n is some non-zero integer,

    (iii) the map from $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$ sending z to $\exp(z)$,

(iv) the map from $\{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ to $\{z \in \mathbb{C} : |z| > 0\}$ sending z to $\exp(z)$.

[Briefly justify your answers.]

17. A continuous function $f\colon X \to Y$ between topological spaces X and Y is said to be a *local homeomorphism* if, given any point x of X there exists an open set V in X containing the point x and an open set W in Y containing the point $f(x)$ such that the function f maps V homeomorphically onto W. Explain why any covering map is a local homeomorphism.

18. Determine which of the maps described in question 16 are local homeomorphisms.

19. (a) Let W, X, Y and Z be topological spaces, and let A be a subset of X. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps. Suppose that $f \simeq g \operatorname{rel} A$. Show that $h \circ f \simeq h \circ g \operatorname{rel} A$ for all continuous maps $h\colon Y \to Z$, and that $f \circ e \simeq g \circ e \operatorname{rel} e^{-1}(A)$ for all continuous maps $e\colon W \to X$.

    (b) Using (a), explain why, given any continuous map $f\colon X \to Y$ between topological spaces X and Y, there is a well-defined homomorphism $f_\#\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ of fundamental groups for any $x \in X$ which sends $[\gamma]$ to $[f \circ \gamma]$ for any loop γ based at the point x.

    (c) Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps satisfying $f(x) = g(x)$ and $f \simeq g \operatorname{rel}\{x\}$. Show that the homomorphisms $f_\#$ and $g_\#$ from $\pi_1(X, x)$ to $\pi_1(Y, f(x))$ induced by the maps f and g are equal.

20. (a) Let X and Y be topological spaces, let $f\colon X \to Y$ and $h\colon Y \to X$ continuous maps, and let x be a point of X. Suppose that $h(f(x)) = x$ and that $h \circ f \simeq 1_X \operatorname{rel}\{x\}$ and $f \circ h \simeq 1_Y \operatorname{rel}\{f(x)\}$, where 1_X and 1_Y denote the identity maps of the spaces X and Y. Explain why the fundamental groups $\pi_1(X, x)$ and $\pi_1(Y, f(x))$ are isomorphic.

    (b) Using (a), explain why the fundamental groups $\pi_1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{p})$ and $\pi^1(S^{n-1}, \mathbf{p})$ of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and the $(n-1)$-dimensional sphere S^{n-1} are isomorphic for all $n > 1$, where $\mathbf{p} \in S^{n-1}$.

21. Let X be a topological space, and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in X. We say that the path β is a *reparameterization* of the path α if there exists a strictly increasing continuous function $\sigma\colon [0,1] \to [0,1]$ such that $\sigma(0) = 0$, $\sigma(1) = 1$ and $\beta = \alpha \circ \sigma$. (Note that if β is a reparameterization of α then $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and the paths α and β have the same image in X.)

    (a) Show that there is a well-defined equivalence relation on the set of all paths in X, where a path α is related to a path β if and only if β is a reparameterization of a path α. [Hint: use the basic result of analysis which states that a strictly increasing continuous function mapping one interval onto another has a continuous inverse.]

    (b) Show that if the path β is a reparameterization of the path α, then $\beta \simeq \alpha \operatorname{rel}\{0,1\}$.

    Given paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in a topological space X, where $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, \ldots, n-1$, we define the *concatenation* $\gamma_1.\gamma_2.\cdots.\gamma_n$ of the paths by the formula $(\gamma_1.\gamma_2.\ldots.\gamma_n)(t) = \gamma_i(nt - i + 1)$ for all t satisfying $(i-1)/n \le t \le i/n$.

    (c) Show that the path $(\gamma_1.\ldots.\gamma_r).(\gamma_{r+1}.\ldots.\gamma_n)$ is a reparameterization of $\gamma_1.\gamma_2.\cdots.\gamma_n$ for any r between 1 and $n-1$.

    (d) By making repeated applications of (*c*), or otherwise, show that $(\gamma_1.\gamma_2).\gamma_3.(\gamma_4.\gamma_5)$ is a reparameterization of $\gamma_1.(\gamma_2.\gamma_3.\gamma_4).\gamma_5$ for all paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ in X satisfying $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, 3, 4$.

22. Let X be a topological space.

    (a) Show that, given any path $\alpha\colon [0,1] \to X$ in X, there is a well-defined homomorphism $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ of fundamental groups which sends the homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the homotopy class $[\alpha.\gamma.\alpha^{-1}]$ of the loop $\alpha.\gamma.\alpha^{-1}$, where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \gamma(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

    (i.e., $\alpha.\gamma.\alpha^{-1}$ represents 'α followed by γ followed by α reversed').

(b) Show that $\Theta_{\alpha.\beta} = \Theta_\alpha \circ \Theta_\beta$ for all paths α and β in X satisfying $\beta(0) = \alpha(1)$.

(c) Show that Θ_α is the identity homomorphism whenever α is a constant path.

(d) Let α and $\hat{\alpha}$ be paths in X satisfying $\alpha(0) = \hat{\alpha}(0)$ and $\alpha(1) = \hat{\alpha}(1)$. Suppose that $\alpha(0) \simeq \hat{\alpha}(0) \,\mathrm{rel}\{0, 1\}$. Show that $\Theta_\alpha = \Theta_{\hat{\alpha}}$.

(e) Explain why the homomorphism $\Theta_\alpha : \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ is an isomorphism for all paths α in X. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.)

23. Let X and Y be topological spaces, and let x_0 and y_0 be points of X and Y. Prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. [Hint: you should make use of the result that a function mapping a topological space into $X \times Y$ is continuous if and only if its components are continuous.]

24. (a) Let $p: \tilde{X} \to X$ be a covering map, let $\alpha: [0, 1] \to X$ and $\beta: [0, 1] \to X$ be paths in X, and let $\tilde{\alpha}: [0, 1] \to \tilde{X}$ and $\tilde{\beta}: [0, 1] \to \tilde{X}$ be lifts of α and β satisfying $p \circ \tilde{\alpha} = \alpha$, $p \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0)$. Suppose that $\alpha \simeq \beta \,\mathrm{rel}\{0, 1\}$. Prove that $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

    (b) Let $p: \tilde{X} \to X$ be a covering map. Suppose that $\tilde{X}$ is path-connected and X is simply-connected. Using the results of (a), or otherwise, prove that $p: \tilde{X} \to X$ is a bijection, and is thus a homeomorphism.

25. Real projective $\mathbb{R}P^n$ n-dimensional space may be described as the topological space obtained from the n-dimensional sphere S^n by identifying together each pair of antipodal points on S^n. There is thus an identification map $q: S^n \to \mathbb{R}P^n$ that sends each pair of antipodal points of S^n to the corresponding point of $\mathbb{R}^n$. One can easily verify that the map q is a covering map.

    Prove that the fundamental group of $\mathbb{R}P^n$ is isomorphic to $\mathbb{Z}_2$ when $n \geq 2$, where $\mathbb{Z}_2$ is the group with exactly two elements. [Hint: adapt the proof of the theorem concerning the fundamental group of the circle.]