UNIVERSITY OF DUBLIN

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JS Mathematics
SS Mathematics

Trinity Term 1997

COURSE 421

Friday, May 30 Luce Hall 9.30 — 12.30

Dr. D. R. Wilkins

Credit will be given for the best 7 questions answered. Logarithmic tables will be available in the examination hall.

1. (a) What is a *topological space*? What is meant by saying that a function $f: X \to Y$ between topological spaces X and Y is *continuous*?

   (b) Give the definition of the *product topology* on a Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces $X_1, X_2, \cdots, X_n$.

   (c) Prove that the product topology on $\mathbb{R}^n$ coincides with the usual topology (i.e., the topology generated by the Euclidean distance function on $\mathbb{R}^n$).

2. (a) What is a *compact* topological space?

   (b) Let $f: X \to Y$ be a continuous map between topological spaces X and Y. Let A be a compact subset of X. Prove that $f(A)$ is compact.

   (c) Prove that any compact subset of $\mathbb{R}^n$ is bounded.

   (d) Let A be a subset of $\mathbb{R}^n$. The *convex hull* of A consists of all points of $\mathbb{R}^n$ that are of the form $(1-t)\mathbf{x} + t\mathbf{y}$ with $\mathbf{x} \in A, \mathbf{y} \in A$ and $t \in [0,1]$. Prove that if A is compact then so is the convex hull of A.

3. (a) What is a *connected* topological space? What is a *path-connected* topological space?

   (b) Prove that a topological space X is connected if and only if every continuous function $f: X \to \mathbb{Z}$ from X to the set of integers is constant.

   (c) Prove that every path-connected topological space is connected.

   (d) Let $X = \{(x,y,z) \in \mathbb{R}^3 : y^2 + z^2 = x^3 - x\}$. What are the connected components of X? [Justify your answer.]

4. (a) Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve in the complex plane $\mathbb{C}$, and let w be a complex number that does not lie on the curve γ. Give the definition of the *winding number* $n(\gamma, w)$ of the curve γ about w.

   (b) Let $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be a continuous map from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. Explain briefly why $n(f \circ \beta, 0) = 0$ for all closed curves $\beta : [0,1] \to \mathbb{C}$ in the complex plane.

   (c) Let $\gamma(t) = 3e^{6\pi i t} - 2e^{8\pi i t}$ for all $t \in [0,1]$. What is the winding number of the closed curve $\gamma : [0,1] \to \mathbb{C}$ about 0. [Justify your answer.]

5. Give the definition of the *fundamental group* $\pi_1(X, x_0)$ of a topological space X based at some point x_0 of X, define the group operation on $\pi_1(X, x_0)$, and verify that $\pi_1(X, x_0)$ is indeed a group by showing that the group operation is well-defined and associative and proving the existence of an identity element and appropriate inverses.

6. Prove that $\pi_1(S^1, b) \cong \mathbb{Z}$, where b is some point on the circle S^1. [You may use, without proof, the *Path Lifting Theorem* and the *Monodromy Theorem*.]

7. (a) What is a *simplex*? What is the *interior* of a simplex? What is a *simplicial complex*? What is the *polyhedron* of a simplicial complex.

   (b) Let K be a finite collection of simplices in $\mathbb{R}^k$ and let $|K|$ be the union of the simplices belonging to K. Suppose that every face of a simplex of K belongs to K. Prove that K is a simplicial complex if and only if every point of $|K|$ belongs to the interior of a unique simplex of K.

8. Let K be a simplicial complex.

   (a) What is meant by saying that two vertices of K can be joined by an edge path?

   (b) Prove that the polyhedron $|K|$ of K is connected if and only if any two vertices of K can be joined by an edge path.

   (c) Prove that if $|K|$ is connected then $H_0(K) \cong \mathbb{Z}$.

9. (a) Let $C_q(K)$ be the *qth* chain group of a simplicial complex K, and let $\delta_q : C_q(K) \to C_{q-1}(K)$ be the boundary homomorphism. Give the definition of $\delta_q(<\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q>)$ for an oriented q-simplex $<\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q>$ of K.

   (b) Define the group $Z_q(K)$ of q-*cycles*, the group $B_q(K)$ of q-*boundaries* and the qth *homology group* $H_q(K)$ of a simplicial complex K.

   (c) Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}$, and $\mathbf{u}$ be the vertices of a hexagon in the plane, and let K be the simplicial complex consisting of the six vertices $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}$, and $\mathbf{u}$, the eight edges $\mathbf{pq}, \mathbf{qr}, \mathbf{rp}, \mathbf{st}, \mathbf{tu}, \mathbf{us}, \mathbf{pu}$ and $\mathbf{rs}$ and the two triangles $\mathbf{pqr}$ and $\mathbf{stu}$. Let

$$c = n_1 <\mathbf{p}, \mathbf{q}> + n_2 <\mathbf{q}, \mathbf{r}> + n_3 <\mathbf{r}, \mathbf{p}> + n_4 <\mathbf{s}, \mathbf{t}>$$

$$+ n_5 <\mathbf{t}, \mathbf{u}> + n_6 <\mathbf{u}, \mathbf{s}> + n_7 <\mathbf{p}, \mathbf{u}> + n_8 <\mathbf{r}, \mathbf{s}>$$

   where the coefficients $n_1, n_2, \cdots, n_8$ are integers. Calculate $\delta_1(c)$. What conditions must the coefficients satisfy for c to be a 1-cycle or a 1-boundary? Show that $H_1(K) \cong \mathbb{Z}$.

10. (a) What is meant by saying that a sequence of Abelian groups and homomorphisms is *exact*?

(b) Let

$$\begin{array}{ccccccccc}
G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\
\downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 & & \downarrow \psi_4 & & \downarrow \psi_5 \\
H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5
\end{array}$$

be a commutative diagram of Abelian groups and homomorphisms whose rows are both exact sequences. Prove that if ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms then so is ψ_3 (the *Five-Lemma*).

11. Let K be a simplicial complex and let K' be its first barycentric division. Prove that the homology groups of K' are isomorphic to those of K. [You may use without proof the Five-Lemma and the fact that any two simplicial approximations to some continuous map induce the same homomorphisms of homology groups.]

12. Write an account of aspects of the theory of the topological classification of closed surfaces. [A detailed proof of the classification theorem is not required.]