UNIVERSITY OF DUBLIN

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JS Mathematics
SS Mathematics

Trinity Term 1995

COURSE 421

Wednesday, May 31 Sports Hall 14.00 — 17.00

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ATTEMPT UP TO **EIGHT** QUESTIONS

1. *(a)* What is meant by saying that a topological space is *compact*?

(b) Let X and Y be topological spaces, let K be a compact subset of Y, and let U be an open set in $X \times Y$. Let $V = \{x \in X : \{x\} \times K \subset U\}$. Prove that V is an open set in X.

(c) Prove that a Cartesian product of two compact topological spaces is compact.

2. *(a)* What is meant by saying that a topological space is *connected*? What is meant by saying that a topological space is *path-connected*?

(b) Prove that a topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.

(c) Prove that every path-connected topological space is connected.

(d) Let X be a non-empty connected open subset of $\mathbb{R}^n$ for some positive integer n. Prove that X is path-connected. [Hint: use the connectedness of X to show that, given a point b of X, the set of points of X that are endpoints of paths in X starting at b must be the whole of X.]

3. *(a)* Let $\tilde{X}$ and X be topological spaces, and let $p\colon \tilde{X} \to X$ be a continuous map. What is meant by saying that an open set U in X is *evenly covered* by the map p? What is meant by saying that the map $p\colon \tilde{X} \to X$ is a *covering map*?

(b) Let $p\colon \tilde{X} \to X$ be a covering map. Prove that $p(V)$ is open in X for every open set V in $\tilde{X}$.

(c) Let $p\colon \tilde{X} \to X$ be a covering map, let Z be a connected topological space, and let $g\colon Z \to \tilde{X}$ and $h\colon Z \to \tilde{X}$ be continuous maps. Suppose that $p \circ g = p \circ h$ and that $g(z) = h(z)$ for some $z \in Z$. Prove that $g = h$.

(d) Let $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, and let $f\colon H \to \mathbb{C} \setminus \{0\}$ be defined by $f(z) = z^4$ for all $z \in H$. Is the map f a covering map from H to $\mathbb{C} \setminus \{0\}$? [Justify your answer.]

4. *(a)* Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed curve in the complex plane $\mathbb{C}$, and let w be a complex number that does not lie on the curve γ. Give the definition of the *winding number* $n(\gamma, w)$ of the curve γ about w.
(b) Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed curve in the complex plane that does not pass through zero, and let $\eta\colon [0,1] \to \mathbb{C}$ be the closed curve in the complex plane defined by $\eta(t) = 1/\gamma(t)$ for all $t \in [0,1]$. Prove that $n(\eta, 0) = -n(\gamma, 0)$.
(c) Let w be a complex number and, for each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t,\tau) \in [0,1] \times [0,1]$ to $\gamma_\tau(t)$ is a continuous map from $[0,1] \times [0,1]$ to $\mathbb{C}$. Using the Monodromy Theorem, or otherwise, prove that $n(\gamma_0, w) = n(\gamma_1, w)$.
(d) Let $\gamma_0\colon [0,1] \to \mathbb{C}$ and $\gamma_1\colon [0,1] \to \mathbb{C}$ be closed curves in $\mathbb{C}$, and let w be a complex number which does not lie on the images of γ_0 and γ_1. Suppose that $|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|$ for all $t \in [0,1]$. Prove that $n(\gamma_0, w) = n(\gamma_1, w)$.
(e) State and prove the *Fundamental Theorem of Algebra*.

5. *(a)* What is meant by saying that a topological space X is *simply-connected*?
(b) Let X be a topological space, and let U and V be open subsets of X with $U \cup V = X$. Suppose that U and V are simply-connected, and that $U \cap V$ is non-empty and path-connected. Prove that X is simply-connected.
(c) Explain why the unit sphere S^n in $\mathbb{R}^{n+1}$ is simply-connected when $n > 1$.

6. Prove that $\pi_1(S^1, \mathbf{b}) \cong \mathbb{Z}$, where $\mathbf{b}$ is a point on the circle S^1. [You may use, without proof, the *Path Lifting Theorem* and the *Monodromy Theorem.*]

7. *(a)* What is meant by saying that points $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_q}$ in $\mathbb{R}^k$ are *geometrically independent* (or *affinely independent*)?
(b) Define the concepts of *simplex* and *simplicial complex*. What is the *polyhedron* of a simplicial complex?
(c) An *edge path* $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_m}$ joining vertices $\mathbf{v_0}$ and $\mathbf{v_m}$ of a simplicial complex is a finite sequence of vertices such that $\mathbf{v_{j-1}}$ and $\mathbf{v_j}$ are endpoints of an edge of K for $j = 1, 2, \ldots, m$. Prove that if the polyhedron $|K|$ of a simplicial complex is a connected topological space then any two vertices of K can be joined by an edge path.

8. *(a)* Let K be a simplicial complex and let $\mathbf{x}$ be a point of the polyhedron $|K|$ of K. What is the *star* $\mathrm{st_K}(\mathbf{x})$ of $\mathbf{x}$ in K?
(b) Let K be a simplicial complex and let $\mathbf{x}$ be a point of the polyhedron $|K|$ of K. Prove that the star $\mathrm{st_K}(\mathbf{x})$ is an open subset of $|K|$ and that $\mathbf{x} \in \mathrm{st_K}(\mathbf{x})$. [You may use, without proof, the result that every point of $|K|$ belongs to the interior of a unique simplex of K.]
(c) Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Prove that a function $s\colon \mathrm{VertK} \to \mathrm{VertL}$ between the vertex sets of simplicial complexes K and L is a simplicial map, and is a simplicial approximation to $f\colon |K| \to |L|$, if and only if $f(\mathrm{st_K}(\mathbf{v})) \subset \mathrm{st_L}(\mathrm{s}(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K.
(d) State and prove the *Simplicial Approximation Theorem.* [You may use, without proof, the result that the mesh of the jth barycentric subdivision of a simplical complex K converges to zero as $j \to +\infty$.]

9. *(a)* Let $C_q(K)$ be the qth chain group of a simplicial complex K, and let $\partial_q\colon C_q(K) \to C_{q-1}(K)$ be the boundary homomorphism. Write down the expression which defines $\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)$ for an oriented q-simplex $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle$ of K, and show that $\partial_{q-1} \circ \partial_q = 0$ when $2 \leq q \leq \dim K$.
(b) Define the group $Z_q(K)$ of *q-cycles*, the group $B_q(K)$ of *q-boundaries* and the qth homology group $H_q(K)$ of a simplicial complex K.
(c) Let K be a simplicial complex. Suppose that there exists a vertex $\mathbf{w}$ of K with the following property: vertices $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ do so. Prove that $H_0(K) \cong \mathbb{Z}$ and $H_q(K) = 0$ for all $q > 0$.

10. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the vertices of a triangle contained in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, and let $\mathbf{v}_4 = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ and $\mathbf{v}_5 = (\mathbf{0}, \mathbf{0}, -\mathbf{1})$. Let K be the 2-dimensional simplicial complex consisting of the triangular faces, edges and vertices of the tetrahedra $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ and $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_5$. (Thus the polyhedron of $|K|$ consists of the surfaces of the two tetrahedra which are glued together along a single common triangular face $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$.)
(a) Calculate the boundary of the 2-chain

$$a\langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_4\rangle + \mathbf{b}\langle \mathbf{v}_2\mathbf{v}_3\mathbf{v}_4\rangle + \mathbf{c}\langle \mathbf{v}_3\mathbf{v}_1\mathbf{v}_4\rangle + \mathbf{d}\langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_5\rangle + \mathbf{e}\langle \mathbf{v}_2\mathbf{v}_3\mathbf{v}_5\rangle + \mathbf{f}\langle \mathbf{v}_3\mathbf{v}_1\mathbf{v}_5\rangle + \mathbf{g}\langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\rangle$$

of K, where a, b, c, d, e, f and g are integers. Show that the group $Z_2(K)$ of 2-cycles of K is given by $Z_2(K) = \{mz_1 + nz_2 : m, n \in \mathbb{Z}\}$, where

$$\begin{aligned} z_1 &= \langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_4\rangle + \langle \mathbf{v}_2\mathbf{v}_3\mathbf{v}_4\rangle + \langle \mathbf{v}_3\mathbf{v}_1\mathbf{v}_4\rangle - \langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\rangle \\ z_2 &= \langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_5\rangle + \langle \mathbf{v}_2\mathbf{v}_3\mathbf{v}_5\rangle + \langle \mathbf{v}_3\mathbf{v}_1\mathbf{v}_5\rangle - \langle \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\rangle, \end{aligned}$$

and explain why $H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.
(b) For each of the following 1-chains of K determine whether or not that 1-chain of K is a 1-boundary of K and, if so, find a 2-chain of K of which it is the boundary:

(i) $\langle \mathbf{v}_1\mathbf{v}_2\rangle + \langle \mathbf{v}_2\mathbf{v}_4\rangle + \langle \mathbf{v}_4\mathbf{v}_3\rangle + \langle \mathbf{v}_3\mathbf{v}_5\rangle + \langle \mathbf{v}_5\mathbf{v}_1\rangle$;
(ii) $2\langle \mathbf{v}_1\mathbf{v}_2\rangle + \mathbf{3}\langle \mathbf{v}_2\mathbf{v}_3\rangle + \langle \mathbf{v}_3\mathbf{v}_1\rangle$.

11. *(a)* Define the following: an *exact sequence* of Abelian groups and homomorphisms; a *chain complex*; the *homology groups* of a chain complex; a *chain map*; a *short exact sequence of chain complexes.*
(b) Let $0 \to A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \to 0$ be a short exact sequence of chain complexes. Prove that there is a well-defined homomorphism $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$ which sends the homology class $[z]$ of an element z of $Z_i(C_*)$ to the homology class $[w]$ of any element w of $Z_{i-1}(A_*)$ with the property that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. (Here $Z_i(C_*)$ and $Z_{i-1}(A_*)$ denote the kernels of the homomorphisms $\partial_i\colon C_i \to C_{i-1}$ and $\partial_{i-1}\colon A_{i-1} \to A_{i-2}$ respectively.)

12. Write an account of aspects of the theory of the topological classification of closed surfaces.