UNIVERSITY OF DUBLIN

TRINITY COLLEGE

Faculty of Science

school of mathematics

JS Mathematics **Trinity Term 1993**
SS Mathematics
SS Two Sub Mod

Course 421

Thursday, May 27 Luce Hall 2.00 - 5.00

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ATTEMPT UP TO **EIGHT** QUESTIONS

1. Give the definition of the *fundamental group* $\pi_1(X, x_0)$ of a topological space X based at some point x_0 of X, define the group operation on $\pi_1(X, x_0)$, and verify that $\pi_1(X, x_0)$ is indeed a group by showing that the group operation is well-defined and associative and proving the existence of an identity element and appropriate inverses.

2. *(a)* Let $\tilde{X}$ and X be topological spaces, and let $p\colon \tilde{X} \to X$ be a continuous map. What is meant by saying that an open set U in X is *evenly covered* by the map p? What is meant by saying that the map $p\colon \tilde{X} \to X$ is a *covering map*?
(b) Let $p\colon \tilde{X} \to X$ be a covering map over a connected topological space X. Suppose that $p^{-1}(\{x\})$ is a finite set for at least one point x of X. Use the definition of covering maps to prove that there is a well-defined non-negative integer n with the property that $p^{-1}(\{x\})$ has exactly n elements for all $x \in X$.
(c) Determine which of the following continuous maps are covering maps:—

(i) the map $q\colon S^1 \to S^1$ sending $(\cos\theta, \sin\theta) \in S^1$ to $(\cos m\theta, \sin m\theta)$, where m is some non-zero integer and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,

(ii) the map $r\colon S^2 \to D$ sending $(x, y, z) \in S^2$ to (x, y), where

$$\begin{aligned} S^2 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}, \\ D &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. \end{aligned}$$

[Briefly justify your answers.]

3. Prove that $\pi_1(S^1, b) \cong \mathbb{Z}$, where b is some point on the circle S^1. [You may use, without proof, the *Path Lifting Theorem* and the *Monodromy Theorem*, also known as the *Homotopy Lifting Theorem* for covering maps.]

4. *(a)* What is meant by saying that points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in $\mathbb{R}^k$ are *geometrically independent* (or *affine independent*)?
(b) Define the concepts of *simplex* and *simplicial complex*.
(c) A subset K of $\mathbb{R}^k$ is said to be *convex* if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in K$ for all points $\mathbf{x}$ and $\mathbf{y}$ of K and real numbers λ satisfying $0 \leq \lambda \leq 1$.

Let σ be a simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Show that σ is convex. Show also that $\sigma \subset K$ for any convex subset K of $\mathbb{R}^k$ that contains the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ.

5. *(a)* Let K be a simplicial complex which is a subdivision of an n-dimensional simplex. What is a *Sperner labelling* of the vertices of K?
(b) State and prove *Sperner's Lemma*.
(c) Use Sperner's Lemma and the Simplicial Approximation Theorem to show that there is no continuous map $r\colon \Delta \to \partial\Delta$ from an n-simplex Δ to its boundary $\partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

6. Give an account of the manner in which the Brouwer Fixed Point Theorem can be applied to prove the existence of a Walras equilibrium in an exchange economy (i.e., prove the existence of prices that ensure that the supply of each commodity matches its demand).

7. *(a)* Let $C_q(K)$ be the qth chain group of a simplicial complex K, and let $\partial_q\colon C_q(K) \to C_{q-1}(K)$ be the boundary homomorphism. Write down the expression which defines $\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)$ for an oriented q-simplex $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ of K. [You are not required to prove that the boundary homomorphism is well-defined.]
(b) Define the group $Z_q(K)$ of *q-cycles*, the group $B_q(K)$ of *q-boundaries* and the qth homology group $H_q(K)$ of a simplicial complex K.
(c) Let K be the simplicial complex consisting of all the vertices, edges and triangular faces of a tetrahedron (or 3-simplex) in $\mathbb{R}^3$ with vertices P_0, P_1 P_2 and P_3. (The tetrahedron itself does not belong to the simplicial complex K.) Let

$$\tau_0 = \langle P_1, P_2, P_3 \rangle, \quad \tau_1 = \langle P_0, P_2, P_3 \rangle, \quad \tau_2 = \langle P_0, P_1, P_3 \rangle, \quad \tau_3 = \langle P_0, P_1, P_2 \rangle,$$

$$\rho_{01} = \langle P_0, P_1 \rangle, \quad \rho_{02} = \langle P_0, P_2 \rangle, \quad \rho_{03} = \langle P_0, P_3 \rangle,$$
$$\rho_{12} = \langle P_1, P_2 \rangle, \quad \rho_{13} = \langle P_1, P_3 \rangle, \quad \rho_{23} = \langle P_2, P_3 \rangle.$$

Let α and β be elements of $C_2(K)$ and $C_1(K)$ respectively given by

$$\begin{aligned} \alpha &= n_0\tau_0 + n_1\tau_1 + n_2\tau_2 + n_3\tau_3, \\ \beta &= m_{01}\rho_{01} + m_{02}\rho_{02} + m_{03}\rho_{03} + m_{12}\rho_{12} + m_{13}\rho_{13} + m_{23}\rho_{23}, \end{aligned}$$

where the coefficients n_i and m_{ij} are integers. Calculate $\partial_2(\alpha)$ in terms of the oriented edges ρ_{ij}, and find necessary and sufficient conditions on the coefficients n_i to ensure that $\partial_2(\alpha) = 0$. Calculate also $\partial_1(\beta)$, and find necessary and sufficient conditions on the coefficients m_{ij} to ensure that $\partial_1(\beta) = 0$. Show moreover that $\beta \in C_1(K)$ satisfies $\partial_1(\beta) = 0$ if and only if $\beta = \partial_2(\alpha)$ for some $\alpha \in C_2(K)$. [Hint: look for a solution α in which the coefficient of τ_0 is zero.] Use your results to calculate the groups $B_q(K)$ and $Z_q(K)$ for $q = 0, 1, 2$. Hence calculate the homology groups of the simplicial complex K.

8. Let K be a simplicial complex.
(a) What is meant by saying that two vertices of K can be joined by an edge path?
(b) Prove that the polyhedron $|K|$ of K is connected if and only if any two vertices of K can be joined by an edge path.
(c) Prove that if $|K|$ is connected then $H_0(K) \cong \mathbb{Z}$.

9. *(a)* What is meant by saying that a sequence of Abelian groups and homomorphisms is *exact*?
(b) Let $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$ be a short exact sequence of Abelian groups and homomorphisms, and let $\theta\colon G \to K$ be a homomorphism of Abelian groups. Suppose that $\theta \circ \alpha = 0$. Prove that there exists a unique homomorphism $\varphi\colon H \to K$ such that $\theta = \varphi \circ \beta$.

Let

$$\begin{array}{ccccccccc}
G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\
H_1 & \xrightarrow{\phi_1} & H_2 & \xrightarrow{\phi_2} & H_3 & \xrightarrow{\phi_3} & H_4 & \xrightarrow{\phi_4} & H_5
\end{array}$$

be a commutative diagram of Abelian groups and homomorphisms whose rows are both exact sequences.
(c) Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is a epimorphism. Prove that ψ_3 is an monomorphism.
(d) Suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. Prove that ψ_3 is an epimorphism.

[Recall that a *monomorphism* is an injective homomorphism, and an *epimorphism* is a surjective homomorphism.]

10. *(a)* Write down the *Mayer-Vietoris* exact sequence associated with the decomposition of a simplicial complex K as the union of two subcomplexes L and M.
(b) Let $0 \longrightarrow \mathbb{Z} \xrightarrow{h} G \xrightarrow{k} \mathbb{Z} \longrightarrow 0$ be an exact sequence of Abelian groups and homomorphisms. Explain why there exists a homomorphism $s\colon \mathbb{Z} \to G$ such that $k \circ s$ is the identity homomorphism of $\mathbb{Z}$, and show that the homomorphism sending $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ to $h(m) + s(n)$ is an isomorphism from G to $\mathbb{Z} \oplus \mathbb{Z}$.
(c) Let K be a simplicial complex whose polyhedron is homeomorphic to the union of the unit sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

in $\mathbb{R}^3$ and the closed disk

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } z = 0.\}$$

Use the Mayer-Vietoris exact sequence to calculate the homology groups of K. [You may make use of the topological invariance of homology groups together with the following well-known results: if $|L|$ is homeomorphic to a closed disk then $H_0(L) \cong \mathbb{Z}$ and $H_q(L) = 0$

for all $q > 0$; also if $|M|$ is homeomorphic to an n-sphere for some $n > 0$ then $H_q(M) \cong \mathbb{Z}$ for $q = 0$ and $q = n$, and $H_q(M) = 0$ for all other values of q.]

11. *(a)* Let K and L be simplicial complexes. What is meant by saying that two simplicial maps $s: K \to L$ and $t: K \to L$ from K to L are contiguous?
(b) Let K and M be simplicial complexes, and let $s: K \to L$ and $t: K \to L$ be simplicial approximations to some continuous map $f: |K| \to |L|$ from the polyhedron of K to that of L. Prove that the simplicial maps s and t are contiguous.
(c) Let $s: K \to L$ and $t: K \to L$ be contiguous simplicial maps. For each non-negative integer q, let $D_q: C_q(K) \to C_{q+1}(K)$ be the homomorphism defined so that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle .$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order with respect to some chosen ordering of the vertices of K. Show by direct calculation that $\partial_1 \circ D_0 = t_0 - s_0$ and

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. Use this result to prove that the induced homomorphisms s_* and t_* from $H_q(K)$ to $H_q(L)$ are equal for all non-negative integers q.

12. *(a)* The Möbius strip M is the topological space obtained from the unit square $[0,1] \times [0,1]$ by identifying $(0,t)$ with $(1, 1-t)$ for all $t \in [0,1]$. Show that $H_0(M) \cong \mathbb{Z}$, $H_1(M) \cong \mathbb{Z}$, and $H_q(M) = 0$ for all $q \geq 2$. [Hint: in order to study the homology of M, you may find it helpful to show that a certain continuous map from S^1 to M induces isomorphisms of homology groups.]
(b) Let $[z]$ and $[w]$ be generators of $H_1(M)$ and $H_1(\partial M)$ respectively, where ∂M is the boundary of M. Indicate briefly why $i_*([w]) = \pm 2[z]$, where $i_*: H_1(\partial M) \to H_1(M)$ is the homomorphism of homology groups induced by the inclusion map $i: \partial M \to M$. [You may use, without proof, the result that the homomorphism induced by any continuous map from S^1 to itself sends $\theta \in H_1(S^1)$ to $n\theta$, where n is the winding number of the map.]
(c) The real projective plane can be obtained by gluing together along their boundaries a Möbius strip and a closed disk. Calculate the homology groups of the real projective plane using the Mayer-Vietoris exact sequence associated with the resulting decomposition of the real projective plane as a union of a Möbius strip and a closed disk.