Course 421 - Trinity Term 1999: Worked Solutions

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Many of the questions represent bookwork (and the worked solutions presented here tend to follow the printed lecture notes). The exceptions are the following questions or parts of questions:—

Question 3 (d) This is novel.

Question 5 (d) This is novel, though problem sheets and previous examination papers have included questions involving winding numbers of products of closed curves and of reciprocals of closed curves).

Question 6 (c) This is new, though it is an simple application of (b), and the lecture notes include an analogous discussion of the simple-connectedness of the n-sphere for $n \geq 2$.

Question 9 This exercise is new, but analogous exercises for different simplicial complexes have occurred regularly on previous examination papers (and on a problem sheet).

Question 12 (b) The calculation of the homology groups of the torus using the Mayer-Vietoris sequence is not included in the lecture notes, but in a series of related problems (including calculations of the homology groups of the torus, the Klein bottle, and the projective plane) in a problem sheet, to be discussed in class. This examination question is based on the essential features of one of those problems (conclusions of earlier problems that are prerequisites for this particular problem being included in the information provided in the examination question).

1. (a) A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

   - the empty set $\emptyset$ and the whole set X are open sets,
   - the union of any collection of open sets is itself an open set,
   - the intersection of any *finite* collection of open sets is itself an open set.

   The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

   A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

   (b) A subset U of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is said to be *open* with respect to the product topology if, given any point p of U, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$. The collection of open sets defined in this way constitutes the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

   We now verify that the topological space axioms are satisfied.

   Let $X = X_1 \times X_2 \times \cdots \times X_n$. The definition of open sets ensures that the empty set and the whole set X are open in X. We must prove that any union or finite intersection of open sets in X is an open set.

   Let E be a union of a collection of open sets in X and let p be a point of E. Then Then $p \in D$ for some open set D in the collection. It follows from this that there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset D \subset E.$$

   Thus E is open in X.

   Let $U = U_1 \cap U_2 \cap \cdots \cap U_m$, where $U_1, U_2, \ldots, U_m$ are open sets in X, and let p be a point of U. Then there exist open sets V_{ki} in X_i for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$ for $k = 1, 2, \ldots, m$. Let $V_i = V_{1i} \cap V_{2i} \cap \cdots \cap V_{mi}$ for $i = 1, 2, \ldots, n$. Then

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$$

for $k = 1, 2, \ldots, m$, and hence $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$. It follows that U is open in X, as required.

(c) Let $X = X_1 \times X_2 \times \cdots \times X_n$, and let $p_i \colon X \to X_i$ denote the projection function that sends $(x_1, x_2, \ldots, x_n)$ to x_i, for $i = 1, 2, \ldots, n$.

Let V be an open set in X_i. Then

$$p_i^{-1}(V) = X_1 \times \cdots \times X_{i-1} \times V \times X_{i+1} \times \cdots \times X_n,$$

and therefore $p_i^{-1}(V)$ is open in X. Thus $p_i \colon X \to X_i$ is continuous for all i.

Let $f \colon Z \to X$ be continuous. Then, for each i, $p_i \circ f \colon Z \to X_i$ is a composition of continuous functions, and is thus itself continuous.

Conversely suppose that $f \colon Z \to X$ is a function with the property that $p_i \circ f$ is continuous for all i. Let U be an open set in X. We must show that $f^{-1}(U)$ is open in Z.

Let z be a point of $f^{-1}(U)$, and let $f(z) = (u_1, u_2, \ldots, u_n)$. Now U is open in X, and therefore there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ respectively such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. Let

$$N_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n),$$

where $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$. Now $f_i^{-1}(V_i)$ is an open subset of Z for $i = 1, 2, \ldots, n$, since V_i is open in X_i and $f_i \colon Z \to X_i$ is continuous. Thus N_z, being a finite intersection of open sets, is itself open in Z. Moreover

$$f(N_z) \subset V_1 \times V_2 \times \cdots \times V_n \subset U,$$

so that $N_z \subset f^{-1}(U)$. It follows that $f^{-1}(U)$ is the union of the open sets N_z as z ranges over all points of $f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in Z. This shows that $f \colon Z \to X$ is continuous, as required.

2. (a) A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

(b) Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(K)$. Then K is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of K that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$K \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(K) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(K)$ is compact.

(c) Let A be a closed subset of a compact topological space X, and let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. Therefore A is compact, as required.

(d) For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required.

(e) Let K be a compact subset of a Hausdorff topological space X. It follows immediately from the result of (c) that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed.

(f) A function $f\colon X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

(g) Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f\colon X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence K is compact. But then $f(K)$ is compact, since continuous functions map compact sets to compact sets. Then $f(K)$ is closed in Y, by the result of (d). It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(Y)$ is open in X, since $f\colon X \to Y$ is continuous. Thus the surjection $f\colon X \to Y$ is an identification map.

3. (a) A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

(b) Suppose that X is connected. Let $f: X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let

$$U = \{x \in X : f(x) = n\}, \qquad V = \{x \in X : f(x) \neq n\}.$$

Then U and V are the preimages of the open subsets $\{n\}$ and $\mathbb{Z} \setminus \{n\}$ of $\mathbb{Z}$, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f: X \to \mathbb{Z}$ is constant, with value n.

Conversely suppose that every continuous function $f: X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f: X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. But then the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected.

(c) Let $g: f(A) \to \mathbb{Z}$ be any continuous integer-valued function on $f(A)$. Then $g \circ f: A \to \mathbb{Z}$ is a continuous integer-valued function on A. It follows from the result of (b) that $g \circ f$ is constant on A. Therefore g is constant on $f(A)$. We deduce from the result of (b) that $f(A)$ is connected.

(d) Suppose that no such integer m were to exist. Then there would exist an integer k and points x_1 and x_2 of X such that $f(x_1) < k < f(x_2)$. Let

$$U = \{x \in X : f(x) < k\}, \quad V = \{x \in X : f(x) < k\}.$$

Then U and V would be open sets in X, and $U \cap V = \emptyset$. Moreover $U \cup V = X$ since the function f does not take on any integer value on X. Thus the open sets U and V would be complements of each other, and thus would be closed. They would therefore be subsets of X that were both open and closed, but neither would be the empty set or the whole of X, contradicting the connectedness of X. Therefore the required integer m must exist.

4. (a) An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) Given any $\theta \in [-\pi, \pi]$ let us define

$$U_\theta = \{z \in \mathbb{C} \setminus \{0\} : \arg(-z) \neq \theta\}.$$

Then $\exp^{-1}(U_\theta)$ is the disjoint union of the open sets

$$\{z \in \mathbb{C} : |\operatorname{Im} z - \theta - 2\pi n| < \pi\},$$

for all integers n, and exp maps each of these open sets homeomorphically onto U_θ. Thus U_θ is evenly covered by the map $exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$.

(c) Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}\colon \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

(d) The open set U is evenly covered by the covering map p, and therefore $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(a)$ for some $a \in A$; let this set be denoted by $\tilde{U}$. Let $\sigma: U \to \tilde{U}$ be the inverse of the homeomorphism $p|\tilde{U}: \tilde{U} \to U$, and let $\tilde{f} = \sigma \circ f$. Then $p \circ \tilde{f} = f$. Also $p \circ \tilde{f}|A = p \circ g$ and $\tilde{f}(a) = g(a)$. It follows from the result of (c) that $\tilde{f}|A = g$, since A is connected. Thus $\tilde{f}: Z \to \tilde{X}$ is the required map.

5. (a) The Path Lifting Theorem ensures that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. We define

$$n(\gamma, w) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i}.$$

Now $\exp(\tilde{\gamma}(1)) = \gamma(1) - w = \gamma(0) - w = \exp(\tilde{\gamma}(0))$ (since γ is a closed curve). It follows from this that $n(\gamma, w)$ is an integer. This integer is known as the *winding number* of the closed curve γ about w.

(b) Let $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be defined by $H(t,\tau) = \gamma_\tau(t) - w$. It follows from the Monodromy Theorem that there exists a continuous map $\tilde{H}\colon [0,1] \times [0,1] \to \mathbb{C}$ such that $H = \exp \circ \tilde{H}$. But then

$$\tilde{H}(1,\tau) - \tilde{H}(0,\tau) = 2\pi i n(\gamma_\tau, w)$$

for all $\tau \in [0,1]$, and therefore the function $\tau \mapsto n(\gamma_\tau, w)$ is a continuous function on the interval $[0,1]$ taking values in the set $\mathbb{Z}$ of integers. But such a function must be constant on $[0,1]$, since the interval $[0,1]$ is connected. Thus $n(\gamma_0, w) = n(\gamma_1, w)$, as required.

(c) For each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{C}$ be the closed curve given by $\gamma_\tau(t) = \gamma(t) - \sigma(\tau)$. Then the closed curves γ_τ do not pass through 0 (since the curves γ and σ do not intersect), and the map from $[0,1] \times [0,1]$ to $\mathbb{C}$ sending (t,τ) to $\gamma_\tau(t)$ is continuous. It follows from the result of (b) that

$$n(\gamma, \sigma(0)) = n(\gamma_0, 0) = n(\gamma_1, 0) = n(\gamma, \sigma(1)),$$

as required.

(d) It follows from the Path Lifting Theorem that there exist paths $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ and $\tilde{\eta}\colon [0,1] \to \mathbb{C}$ in the complex plane such that $\gamma(t) = \exp(\tilde{\gamma}(t))$ and $\eta(t) = \exp(\tilde{\eta}(t))$ for all $t \in [0,1]$. Let $\tilde{\sigma}(t) = 2\tilde{\gamma}(t) + \tilde{\eta}(t)$ and $\sigma(t) = \exp(\tilde{\sigma}(t)$ for all $t \in [0,1]$. Then $\sigma\colon [0,1] \to \mathbb{C}$ is a closed curve in the complex plane that does not pass through 0, and $\sigma(t) = \gamma(t)^2\eta(t)$, for all $t \in [0,1]$. But the real part of σ is strictly positive for all t. It follows that $n(\sigma, 0) = 0$ by an easy application of the result of (b), since the line segment joining a point on the curve σ to 1 does not pass through 0. But

$$\begin{aligned} 2\pi i n(\sigma, 0) &= \tilde{\sigma}(1) - \tilde{\sigma}(0) = 2(\tilde{\gamma}(1) - \tilde{\gamma}(0)) + (\tilde{\eta}(1) - \tilde{\eta}(0)) \\ &= 2n(\gamma, 0) + n(\eta, 0). \end{aligned}$$

Thus $2n(\gamma, 0) + n(\eta, 0) = 0$, which yields the required result.

6. (a) A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f: \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

(b) We must show that any continuous function $f: \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(U)$ and $f^{-1}(V)$ of U and V are open in ∂D (since f is continuous), and $\partial D = f^{-1}(U) \cup f^{-1}(V)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(U)$ and $f^{-1}(V)$. Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that the distance from z_i to z_{i+1} is less than δ for $i = 1, 2, \ldots, n-1$ and the distance from z_n to z_1 is also less than δ. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets U and V.

Let x_0 be some point of $U \cap V$. Now the sets U, V and $U \cap V$ are all path-connected. Therefore we can choose paths $\alpha_i: [0,1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0,1]) \subset U$ whenever $f(z_i) \in U$, and $\alpha_i([0,1]) \subset V$ whenever $f(z_i) \in V$. For convenience let $\alpha_0 = \alpha_n$.

Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for any } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for any } t \in [0,1], \end{cases}$$

Note that $F_i(\partial T_i) \subset U$ whenever the short arc joining z_{i-1} to z_i is mapped by f into U, and $F_i(\partial T_i) \subset V$ whenever this short arc is mapped into V. But U and V are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required.

(c) Apply the result of (b) with

$$\begin{aligned} U &= \{(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\} : z > -\sqrt{x^2+y^2}\}, \\ V &= \{(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\} : z < \sqrt{x^2+y^2}\}. \end{aligned}$$

Indeed U and V are open sets in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, and are each homeomorphic to $\mathbb{R}^2 \times (0,+\infty)$ under the homeomorphisms

$$(x,y,z) \mapsto (x,y,z+\sqrt{x^2+y^2}), \quad (x,y,z) \mapsto (x,y,z-\sqrt{x^2+y^2})$$

respectively. The convexity of $\mathbb{R}^2 \times (0,+\infty)$ ensures that this set is simply-connected, and thus U and V are both simply connected. Their intersection is path-connected, since any point of $U \cap V$ can be joined by a straight line segment to a point of the path-connected subset $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \{0\}$.

7. (a) Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

   - for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
   - if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

   (b) (Sperner's Lemma) *Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.*

   Proof.
   Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

   We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

   Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2(N(0, 1, \ldots, n-1) - M),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required.

(c) Suppose that such a map $r: \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem that there would exist a simplicial approximation $s: K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s: K \to L$ is a simplicial approximation to $r: \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r: \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

8. (a)

$$\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle,$$

where

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle$$

denotes the oriented $(q-1)$ face of the simplex obtained by omitting the vertex $\mathbf{v}_j$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned}
&\partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) \\
&\quad = \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle\right) \\
&\quad = \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&\qquad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle \\
&\quad = 0
\end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The identity $\partial_{q-1} \circ \partial_q = 0$ now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K.

(b) $Z_q(K)$ is the kernel of $\partial_q\colon C_q(K) \to C_{q-1}(K)$.
$B_q(K)$ is the image of $\partial_{q+1}\colon C_{q+1}(K) \to C_q(K)$.
$H_q(K)$ is the quotient group $Z_q(K)/B_q(K)$.

(c) There is a well-defined homomorphism $D_q\colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r\rangle - \left(\sum_{r=1}^{s} n_r\right) \langle \mathbf{w}\rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r\rangle - \langle \mathbf{w}\rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required.

9. (a) A 1-chain of K is of the form

$$\begin{aligned} c &= m_1\langle \mathbf{d}, \mathbf{a}\rangle + m_2\langle \mathbf{d}, \mathbf{b}\rangle + m_3\langle \mathbf{d}, \mathbf{c}\rangle \\ &\quad + n_1\langle \mathbf{b}, \mathbf{c}\rangle + n_2\langle \mathbf{c}, \mathbf{a}\rangle + n_3\langle \mathbf{a}, \mathbf{b}\rangle, \end{aligned}$$

and

$$\begin{aligned} \partial_1(c) &= (m_1 + n_2 - n_3)\langle \mathbf{a}\rangle + (m_2 + n_3 - n_1)\langle \mathbf{b}\rangle \\ &\quad + (m_3 + n_1 - n_2)\langle \mathbf{c}\rangle - (m_1 + m_2 + m_3)\langle \mathbf{d}\rangle. \end{aligned}$$

It follows that $\partial_1(c) = 0$ if and only if

$$m_1 = n_3 - n_2, \quad m_2 = n_1 - n_3, \quad m_3 = n_2 - n_1,$$

in which case

$$c = n_1 z_1 + n_2 z_2 + n_3 z_3,$$

since $\langle \mathbf{a}, \mathbf{d}\rangle = -\langle \mathbf{d}, \mathbf{a}\rangle$ etc. Thus the 1-cycles of K are of the required form.

(b) Any 2-chain of K is of the form $m\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ for some integer m, since **abc** is the only 2-simplex of K. But

$$\partial_2(m\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle) = m\langle \mathbf{b}, \mathbf{c}\rangle + m\langle \mathbf{c}, \mathbf{a}\rangle + m\langle \mathbf{a}, \mathbf{b}\rangle = m(z_1 + z_2 + z_3).$$

Thus $n_1 z_1 + n_2 z_2 + n_3 z_3$ is a 1-boundary of K if and only if $n_1 = n_2 = n_3$.

(c) Consider the homomorphism $\epsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\epsilon(l_1\langle \mathbf{a}\rangle + l_2\langle \mathbf{b}\rangle + l_3\langle \mathbf{c}\rangle + l_4\langle \mathbf{d}\rangle) = l_1 + l_2 + l_3 + l_4.$$

It is easily seen (from the calculation in (a)) the boundary of a 1-chain belongs to the kernel of ϵ. On the other hand, if $l_1 + l_2 + l_3 + l_4 = 0$ then

$$l_1\langle \mathbf{a}\rangle + l_2\langle \mathbf{b}\rangle + l_3\langle \mathbf{c}\rangle + l_4\langle \mathbf{d}\rangle = \partial_1(l_1\langle \mathbf{d}, \mathbf{a}\rangle + l_2\langle \mathbf{d}, \mathbf{b}\rangle + l_3\langle \mathbf{d}, \mathbf{c}\rangle).$$

Thus every element of the kernel of ϵ is the boundary of a 1-chain. Thus $\epsilon\colon C_0(K) \to \mathbb{Z}$ is a surjective homomorphism whose kernel is $B_0(K)$. Thus

$$H_0(K) = C_0(K)/B_0(K) = C_0(K)/\ker\epsilon \cong \epsilon(C_0(K)) = \mathbb{Z}.$$

[Alternatively, one can deduce this result by a known theorem from the connectedness of the polyhedron of K.]

Now consider the homomorphism $\eta\colon Z_1(K) \to \mathbb{Z} \oplus \mathbb{Z}$ defined by

$$\eta(n_1 z_1 + n_2 z_2 + n_3 z_3) = (n_1 - n_3, n_2 - n_3).$$

The homomorphism η is obviously surjective, and it follows from (b) that its kernel is $B_1(K)$. Therefore

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K)/\ker\eta \cong \eta(Z_1(K)) = \mathbb{Z} \oplus \mathbb{Z}.$$

Any 2-chain of K is of the form $m\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$, and it follows from the identity

$$\partial_2(m\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle) = m(z_1 + z_2 + z_3)$$

that this 2-chain is a 2-cycle if and only if $m = 0$. Therefore $Z_2(K) = 0$, and hence $H_2(K) = 0$, as required.

10. (a) The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\text{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i\colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i\colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}\colon C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i\colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

(b) Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i\colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}\colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then

$q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_{i-1}$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) &= p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) \\ &= \partial_i(b') = p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*)$, as required.

11. (a) The continuous map $f\colon |K| \to |L|$ has a simplicial approximation $s\colon K^{(j)} \to L$, defined on the jth barycentric subdivision of the simplicial complex K, for some sufficiently large value of j. This simplicial approximation induces homomorphisms

$$s_*\colon H_q(K^{(j)}) \to H_q(L)$$

of homology groups. Now it can be proved that any simplicial approximation to the identity map induces an isomorphism

$$\nu_{K,i}\colon H_q(K^{(i)}) \to H_q(K)$$

between the homology groups $H_q(K^{(j)})$ and $H_q(K)$, and moreover any two simplicial approximations to the identity map induce the same isomorphism. We define

$$f_*\colon H_q(K) \to H_q(L)$$

to be the composition $s_* \circ \nu_{K,i}^{-1}$. Using the fact that any two simplicial approximations to the same continuous map are contiguous, and therefore induce the same homomorphisms of homology groups, one can easily show that the homomorphism $f_*\colon H_q(K) \to H_q(L)$ does not depend on either the choice of j or on the choice of the simplicial approximation $s\colon K^{(j)} \to L$.

(b) Let $F\colon |K| \times [0,1] \to |L|$ be a homotopy with $F(\mathbf{x}, 0) = f(\mathbf{x})$ and $F(\mathbf{x}, 1) = g(\mathbf{x})$, and let $\varepsilon > 0$ be given. Using the fact that any continuous function from a compact metric spaces to a metric space is uniformly continuous, we see that there exists some $\delta > 0$ such that if $|s - t| < \delta$ then the distance from $F(\mathbf{x}, s)$ to $F(\mathbf{x}, t)$ is less than ε. Let $f_i(\mathbf{x}) = F(\mathbf{x}, t_i)$ for $i = 0, 1, \ldots, r$, where $t_0, t_1, \ldots, t_r$ have been chosen such that $0 = t_0 < t_1 < \cdots < t_r = 1$ and $t_i - t_{i-1} < \delta$ for all i. Then $f_{i-1}(\mathbf{x})$ is within a distance ε of $f_i(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Using the result stated in the question, we see that the maps f_{i-1} and f_i from $|K|$ to $|L|$ have a common simplicial approximation, and thus f_{i-1} and f_i induce the same homomorphisms of homology groups, provided that $\varepsilon > 0$ has been chosen sufficiently small. It follows that the maps f and g induce the same homomorphisms of homology groups, as required.

(c) A continuous map $f\colon X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y. The spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence from X to Y.

(d) Let $f\colon |K| \to |L|$ be a homotopy equivalence between the polyhedra of simplicial complexes K and L. There exists a continuous map $g\colon |L| \to |K|$ such that $g \circ f$ is homotopic to the identity map of $|K|$ and $f \circ g$ is homotopic to the identity map of $|L|$. It follows that the induced homomorphisms $(g \circ f)_*\colon H_q(K) \to H_q(K)$ and $(f \circ g)_*\colon H_q(L) \to H_q(L)$ are the identity automorphisms of $H_q(K)$ and $H_q(L)$ for each q. But $(g \circ f)_* = g_* \circ h_*$ and $(f \circ g)_* = g_* \circ h_*$. It follows that $f_*\colon H_q(K) \to H_q(L)$ is an isomorphism with inverse $g_*\colon H_q(L) \to H_q(K)$.

12. (a) Let

$$i_q\colon C_q(L\cap M)\to C_q(L),\quad j_q\colon C_q(L\cap M)\to C_q(M),$$
$$u_q\colon C_q(L)\to C_q(K),\quad v_q\colon C_q(M)\to C_q(K),$$

be the homomorphisms induced by the inclusions of the relevant subcomplexes, let $k_q\colon C_q(L\cap M)\to C_q(L)\oplus C_q(M)$ be defined by $k_q(c)=(i_q(c),-j_q(c))$, and let $w_q\colon C_q(L)\oplus C_q(M)\to C_q(K)$ be defined by $w_q(c',c'')=u_q(c')+v_q(c'')$. Let i_*, j_*, k_*, u_*, v_*, w_* be the homomorphisms of homology groups induced by the respective chain maps. Then there is an exact sequence

$$\cdots\to H_q(L\cap M)\xrightarrow{k_*}H_q(L)\oplus H_q(M)\xrightarrow{w_*}H_q(K)\to H_{q-1}(L\cap M)\to\cdots$$

This is the Mayer-Vietoris exact sequence.

(b) The groups $H_0(L\cap M)$, $H_0(L)$, $H_0(M)$ and $H_0(K)$ are all isomorphic to the group $\mathbb{Z}$ of integers, under the isomorphism that sends a 0-chain to the sum of its coefficients, since the polyhedra of $L\cap M$, L, M and K are all connected. Moreover the homomorphism $k_*\colon H_0(L\cap M)\to H_0(L)\oplus H_0(M)$ corresponds under these isomorphisms to the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}\oplus\mathbb{Z}$ that sends each integer n to $(n,-n)$, and is therefore injective. The exactness of the Mayer-Vietoris sequence then ensures that the homomorphism in that sequence from $H_1(K)$ to $H_0(L\cap M)$ is the zero homomorphism. Also the groups $H_1(M)$, $H_2(M)$ and $H_2(L)$ are all trivial. We therefore obtain from the Mayer-Vietoris sequence the following exact sequence:

$$0\to H_2(K)\to H_1(L\cap M)\xrightarrow{i_*}H_1(L)\xrightarrow{u_*}H_1(K)\to 0.$$

But the fact that any 1-cycle of $L\cap M$ is the boundary of a 2-chain of L ensures that $i_*\colon H_1(L\cap M)\to H_1(L)$ is the zero homomorphism. Its kernel is therefore $H_1(L\cap M)$, and its image is the zero subgroup of $H_1(L)$. We therefore obtain the following exact sequences:

$$0\to H_2(K)\to H_1(L\cap M)\to 0;$$
$$0\to H_1(L)\xrightarrow{u_*}\to H_1(K)\to 0.$$

Thus

$$H_2(K)\cong H_1(L\cap M)\cong\mathbb{Z},\quad H_1(K)\cong H_1(L)\cong\mathbb{Z}\oplus\mathbb{Z}.$$

We have already noted that $H_0(K)=0$.