Course 421 examination, 1997. Worked Solutions

D. R. Wilkins

All questions are bookwork unless otherwise stated.

1. (a) A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

   (i) the empty set $\emptyset$ and the whole set X are open sets,

   (ii) the union of any collection of open sets is itself an open set,

   (iii) the intersection of any *finite* collection of open sets is itself an open set.

   A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

   $$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

   (b) Let $X_1, X_2, \ldots, X_n$ be topological spaces. A subset U of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is said to be *open* with respect to the product topology if, given any point p of U, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$.

   (c) We must show that a subset U of $\mathbb{R}^n$ is open with respect to the usual topology if and only if it is open with respect to the product topology.

   Let U be a subset of $\mathbb{R}^n$ that is open with respect to the usual topology, and let $\mathbf{u} \in U$. Then there exists some $\delta > 0$ such that $B(\mathbf{u}, \delta) \subset U$, where

   $$B(\mathbf{u}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\}.$$

Let $I_1, I_2, \ldots, I_n$ be the open intervals in $\mathbb{R}$ defined by

$$I_i = \{t \in \mathbb{R} : u_i - \frac{\delta}{\sqrt{n}} < t < u_i + \frac{\delta}{\sqrt{n}}\} \qquad (i = 1, 2, \ldots, n),$$

Then $I_1, I_2, \ldots, I_n$ are open sets in $\mathbb{R}$. Moreover

$$\{\mathbf{u}\} \subset I_1 \times I_2 \times \cdots \times I_n \subset B(\mathbf{u}, \delta) \subset U,$$

since

$$|\mathbf{x} - \mathbf{u}|^2 = \sum_{i=1}^{n} (x_i - u_i)^2 < n \left(\frac{\delta}{\sqrt{n}}\right)^2 = \delta^2$$

for all $\mathbf{x} \in I_1 \times I_2 \times \cdots \times I_n$. This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the usual topology on $\mathbb{R}^n$ is also open with respect to the product topology on $\mathbb{R}^n$.

Conversely suppose that U is a subset of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$, and let $\mathbf{u} \in U$. Then there exist open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ containing $u_1, u_2, \ldots, u_n$ respectively such that $V_1 \times V_2 \times \cdots \times V_n \subset U$. Now we can find $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i > 0$ and $(u_i - \delta_i, u_i + \delta_i) \subset V_i$ for all i. Let $\delta > 0$ be the minimum of $\delta_1, \delta_2, \ldots, \ldots, \delta_n$. Then

$$B(\mathbf{u}, \delta) \subset V_1 \times V_2 \times \cdots V_n \subset U,$$

for if $\mathbf{x} \in B(\mathbf{u}, \delta)$ then $|x_i - u_i| < \delta_i$ for $i = 1, 2, \ldots, n$. This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$ is also open with respect to the usual topology on $\mathbb{R}^n$.

2. (a) An *open cover* of a topological space X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

(b) Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact.

(c) Let K be a compact subset of $\mathbb{R}^n$. For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

(d) (**NOT BOOKWORK**) Consider the function $f\colon A \times A \times [0,1] \to \mathbb{R}^n$, where $f(\mathbf{x}, \mathbf{y}, t) = (1-t)\mathbf{x} = t\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in A$ and $t \in [0,1]$. Then the convex hull of A is $f(A \times A \times [0,1])$. The product space $A \times A \times [0,1]$ is compact, being the product of compact spaces, and the function f is continuous. It follows that the convex hull of A is compact, being the image of a compact set under a continuous map.

3. (a) A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

   Let u and v be points in a topological space X. A *path* in X from u to v is defined to be a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = u$ and $\gamma(1) = v$. A topological space X is said to be *path-connected* if and only if, given any two points u and v of X, there exists a path in X from u to v.

   (b) Suppose that X is connected. Let $f\colon X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let $U = \{x \in X : f(x) = n\}$ and $V = \{x \in X : f(x) \neq n\}$. Then U and V are both open in X, since they are the preimages of the open subsets $\{n\}$ and $\mathbb{Z} \setminus \{n\}$ of $\mathbb{Z}$. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f\colon X \to \mathbb{Z}$ is constant, with value n.

   Conversely suppose that every continuous function $f\colon X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f\colon X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

   Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. It follows that the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected.

(c) Let $f\colon X \to \mathbb{Z}$ be a continuous integer-valued function on a path-connected topological space X. Given points x_0 and x_1 of X, there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $f \circ \gamma\colon [0,1] \to \mathbb{Z}$ is a continuous integer-valued function on $[0,1]$. This function must be constant, since $[0,1]$ is connected. It follows that $f(x_0) = f(x_1)$. Thus every continuous integer-valued function on X is constant. Therefore X is connected.

(d) (**NOT BOOKWORK**) The connected components are A and B, where $A = \{(x,y,z) \in X : x \geq 1\}$ and $B = \{(x,y,z) \in X : -1 \leq x \leq 0\}$. Note that $x^3 - x = x(x+1)(x-1)$. Thus if a point (x,y,z) belongs to X then $x^3 - x \geq 0$, and hence either $x \geq 1$ or $-1 \leq x \leq 0$. If $a \geq 1$ or $-1 \leq a \leq 0$ then the set X_a is a circle and is therefore connected, where $X_a = \{(x,y,z) \in X : x = a\}$. Therefore each connected component of X is a union of some of the sets X_a. Let $f(t) = (t, \sqrt{t^3 - t}, 0)$ for $t \in [-1,0]$ and $t \in [1,+\infty)$. Then $f([-1,0])$ is a connected set that intersects X_a for all $a \in [-1,0]$. Therefore A is a connected set. Similarly B is a connected set. The sets A and B are both open and closed in X. They must therefore be the connected components of X.

4. (a) Let $\gamma\colon [0,1] \to \mathbb{C}$ be a continuous closed curve in the complex plane which is defined on some closed interval $[0,1]$ (so that $\gamma(0) = \gamma(1)$), and let w be a complex number which does not belong to the image of the closed curve γ. It then follows from the Path Lifting Theorem that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Let us define

$$n(\gamma, w) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i}.$$

Now $\exp(\tilde{\gamma}(1)) = \gamma(1) - w = \gamma(0) - w = \exp(\tilde{\gamma}(0))$ (since γ is a closed curve). It follows from this that $n(\gamma, w)$ is an integer. This integer is known as the *winding number* of the closed curve γ about w.

(b) (**NOT BOOKWORK**) For each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{C}$ be the closed curve defined by $\gamma_\tau(t) = f(\tau\beta(t))$. Then $n(\gamma_\tau, 0)$ is a continuous integer-valued function of τ and is therefore constant for $\tau \in [0,1]$. It follows that

$$n(f \circ \beta, 0) = n(\gamma_1, 0) = n(\gamma_0, 0) = 0.$$

(c) (**NOT BOOKWORK**) $n(\gamma, 0) = 3$. Let $\beta(t) = 3e^{6\pi i t}$ for all $t \in [0,1]$. Then $|\gamma(t) - \beta(t)| < |\beta(t)|$ for all $t \in [0,1]$. It follows from a standard result in the lecture notes that $n(\gamma, 0) = n(\beta, 0) = 3$.

5. Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0, 1\}$. We denote the equivalence class of a loop $\gamma\colon [0,1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$. Thus two loops γ_0 and γ_1 represent the same element of $\pi_1(X, x_0)$ if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0, 1\}$ (i.e., there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ between γ_0 and γ_1 which maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$).

   First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1] \times [0,1] \to X$ be defined by
$$F(t, \tau) = \begin{cases} F_1(2t, \tau) & \text{if } 0 \le t \le \frac{1}{2}, \\ F_2(2t - 1, \tau) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$
   where $F_1\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_1 and γ_1', $F_2\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1.\gamma_2$ to $\gamma_1'.\gamma_2'$, and maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$. Thus $[\gamma_1.\gamma_2] = [\gamma_1'.\gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

   Next we show that the group operation on $\pi_1(X, x_0)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \theta$, where
$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \le t \le \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \le t \le \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \le t \le 1. \end{cases}$$
   Thus the map $G\colon [0,1] \times [0,1] \to X$ defined by $G(t, \tau) = \alpha((1 - \tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1.\gamma_2).\gamma_3 \simeq \gamma_1.(\gamma_2.\gamma_3)$ rel $\{0, 1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon.\gamma = \gamma \circ \theta_0$ and $\gamma.\varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2}, \\ 2t-1 & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \qquad \theta_1(t) = \begin{cases} 2t & \text{if } 0 \le t \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

for all $t \in [0,1]$. But the continuous map $(t,\tau) \mapsto \gamma((1-\tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon.\gamma \simeq \gamma \simeq \gamma.\varepsilon$ rel $\{0,1\}$, and hence $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Now the map $K\colon [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

is a homotopy between the loops $\gamma.\gamma^{-1}$ and ε, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\gamma.\gamma^{-1} \simeq \varepsilon \,\mathrm{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma.\gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required.

Let x_0 be a point of some topological space X. The group $\pi_1(X, x_0)$ is referred to as the *fundamental group* of X based at the point x_0.

6. We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1,0)$. Now the map $p\colon \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

   Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F\colon [0,1] \times [0,1] \to S^1$ such that $F(t,0) = \alpha(t)$ and $F(t,1) = \beta(t)$ for all $t \in [0,1]$, and $F(0,\tau) = F(1,\tau) = b$ for all $\tau \in [0,1]$. It follows from the Monodromy Theorem that this homotopy lifts to a continuous map $G\colon [0,1] \times [0,1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0,\tau)$ and $G(1,\tau)$ are integers for all $\tau \in [0,1]$, since $p(G(0,\tau)) = b = p(G(1,\tau))$. Also $G(t,0) - \tilde{\alpha}(t)$ and $G(t,1) - \tilde{\beta}(t)$ are integers for all $t \in [0,1]$, since $p(G(t,0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t,1)) = \beta(t) = p(\tilde{\beta}(t))$.

Now any continuous integer-valued function on $[0,1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0,1]$ to $G(0,\tau)$ and $G(1,\tau)$ are constant, as are the functions sending $t \in [0,1]$ to $G(t,0) - \tilde{\alpha}(t)$ and $G(t,1) - \tilde{\beta}(t)$. Thus

$$G(0,0) = G(0,1), \qquad G(1,0) = G(1,1),$$

$$G(1,0) - \tilde{\alpha}(1) = G(0,0) - \widetilde{\alpha(0)}, \qquad G(1,1) - \tilde{\beta}(1) = G(0,1) - \widetilde{\beta(0)}.$$

On combining these results, we see that

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1,0) - G(0,0) = G(1,1) - G(0,1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

We conclude from this that there exists a well-defined function

$$\lambda \colon \pi_1(S^1, b) \to \mathbb{Z}$$

characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha} \colon [0,1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define a continuous path $\sigma \colon [0,1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \tilde{\beta}(2t-1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\begin{aligned} \lambda([\alpha][\beta]) &= \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) \\ &= \lambda([\alpha]) + \lambda([\beta]). \end{aligned}$$

Thus $\lambda \colon \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F \colon [0,1] \times [0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t,\tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore

$F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0, 1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n\colon [0, 1] \to S^1$ is given by

$$\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$$

for all $t \in [0, 1]$. We conclude that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism.

7. (a) A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \le t_j \le 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-simplex σ in some Euclidean space $\mathbb{R}^k$. We define the *interior* of the simplex σ to be the set of all points of σ that are of the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_j > 0$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$.

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The polyhedron $|K|$ of a simplicial complex is the union of the simplices of K.

(b) Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x}$ belongs to the interior of a face σ of some simplex of K (since every point of a simplex belongs to the interior of some face). But then $\sigma \in K$, since K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that $\mathbf{x}$ were to belong to the interior of two distinct simplices σ and τ of K. Then $\mathbf{x}$ would belong to some common face $\sigma \cap \tau$ of σ

and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that $\mathbf{x}$ belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing $\mathbf{x}$ in its interior is uniquely determined, as required.

Conversely, we must show that any collection of simplices satisfying the given conditions is a simplicial complex. Since K contains the faces of all its simplices, it only remains to verify that if σ and τ are any two simplices of K with non-empty intersection then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω of K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. We deduce that the simplices σ and τ have vertices in common, and that every point of $\sigma \cap \tau$ belongs to the common face ρ of σ and τ spanned by these common vertices. But this implies that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of both σ and τ, as required.

8. (a) Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

(b) It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$

and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required.

(c) Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle$$

for some integers $n_1, n_2, \ldots, n_r$. There is therefore a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z}\rangle)) = \varepsilon(\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.
Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m\rangle - \langle \mathbf{v}_0\rangle = \partial_1\left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j\rangle\right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path. We deduce that $\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j\langle \mathbf{u}_j\rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle)$. But $(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle) \in B_0(K)$. It follows that $c \in B_0(K)$. Thus We conclude that that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.
Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required.

9. (a)

$$\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle$$

where $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle$ is the oriented $(q-1)$-face

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q\rangle$$

of the simplex obtained on omitting $\mathbf{v}_j$ from the set of vertices of the simplex.

(b) A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q\colon C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1}\colon C_{q+1}(K) \to C_q(K)$. Now $B_q(K) \subset Z_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K.

(c) (**NOT BOOKWORK**)

$$\begin{aligned} \partial_1(c) &= (n_3 - n_7 - n_1)\mathbf{p} + (n_1 - n_2)\mathbf{q} + (n_2 - n_3 - n_8)\mathbf{r} \\ &\quad + (n_6 + n_8 - n_4)\mathbf{s} + (n_4 - n_5)\mathbf{t} + (n_5 - n_6 + n_7)\mathbf{u}. \end{aligned}$$

It follows that c is a 1-cycle if and only if $n_2 = n_1$, $n_5 = n_4$, $n_3 = n_1 + n_7$, $n_8 = -n_7$ and $n_6 = n_4 + n_7$. Thus

$$Z_1(K) = \{n_1 z' + n_4 z'' + n_7 z''' : n_1, n_4, n_7 \in \mathbb{Z}\},$$

where

$$\begin{aligned} z' &= \langle \mathbf{p}, \mathbf{q}\rangle + \langle \mathbf{q}, \mathbf{r}\rangle + \langle \mathbf{r}, \mathbf{p}\rangle, \\ z'' &= \langle \mathbf{s}, \mathbf{t}\rangle + \langle \mathbf{t}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{s}\rangle, \\ z''' &= \langle \mathbf{p}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{s}\rangle + \langle \mathbf{s}, \mathbf{r}\rangle + \langle \mathbf{r}, \mathbf{p}\rangle. \end{aligned}$$

Also c is a 1-boundary if and only if $n_1 = n_2 = n_3$, $n_4 = n_5 = n_6$ and $n_7 = n_8 = 0$. Thus

$$Z_1(K) = \{n_1 z' + n_4 z'' : n_1, n_4 \in \mathbb{Z}\}.$$

Clearly each coset of $B_1(K)$ in $Z_1(K)$ contains $n_7 z'''$ for some integer n_7 that determines and is determined by the coset. Thus $H_1(K) = \{n_7 z''' + B_1(K) : n_1 \in \mathbb{Z}\}$, and therefore $H_1(K) \cong \mathbb{Z}$.

10. (a) The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\mathrm{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

(b) First we prove that if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism. Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove that if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism. Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required.

The result now follows.

11. There exist simplicial approximations $\zeta: K' \to K$ to the identity map of $|K|$: such a simplicial approximation can be obtained by choosing, for each $\sigma \in K$, a vertex $\mathbf{v}_\sigma$ of σ, and defining $\zeta(\hat{\sigma}) = \mathbf{v}_\sigma$.

    Suppose that $\zeta: K' \to K$ and $\theta: K' \to K$ are both simplicial approximations to the identity map of $|K|$. Then ζ and θ are contiguous, and therefore the homomorphisms ζ_* and θ_* of homology groups induced by ζ and θ must coincide. It follows that there is a well-defined natural homomorphism $\nu_K: H_q(K') \to H_q(K)$ of homology groups which coincides with ζ_* for any simplicial approximation $\zeta: K' \to K$ to the identity map of $|K|$.

    Theorem. *The natural homomorphism $\nu_K: H_q(K') \to H_q(K)$ is an isomorphism for any simplicial complex K.*

    Let M be the simplicial complex consisting of some simplex σ together with all of its faces. Then $H_0(M) \cong \mathbb{Z}$, $H_0(M') \cong \mathbb{Z}$, and $H_q(M) = 0 = H_q(M')$ for all $q > 0$. Let $\mathbf{v}$ be a vertex of M. If $\theta: M' \to M$ is any simplicial approximation to the identity map of $|M|$ then $\theta(\mathbf{v}) = \mathbf{v}$. But the homology class of $\langle \mathbf{v} \rangle$ generates both $H_0(M)$ and $H_0(M')$. It follows that $\theta_*: H_0(M') \to H_0(M)$ is an isomorphism, and thus $\nu_M: H_q(M') \to H_q(M)$ is an isomorphism for all q.

    We now use induction on the number of simplices in K to prove the theorem in the general case. It therefore suffices to prove that the required result holds for a simplicial complex K under the additional assumption that the result is valid for all proper subcomplexes of K.

    Let σ be a simplex of K whose dimension equals the dimension of K. Then σ is not a face of any other simplex of K, and therefore $K \setminus \{\sigma\}$ is a subcomplex of K. Let M be the subcomplex of K consisting of the simplex σ, together with all of its faces. We have already proved the result in the special case when $K = M$. Thus we only need to verify the result in the case when M is a proper subcomplex of K.

    Let $\zeta: K' \to K$ be a simplicial approximation to the identity map of $|K|$. Then the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively. Therefore the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & C_q(L' \cap M') & \longrightarrow & C_q(L') \oplus C_q(M') & \longrightarrow & C_q(K') & \longrightarrow 0 \\
& \Big\downarrow \zeta|L' \cap M' & & \Big\downarrow (\zeta|L') \oplus (\zeta|M') & & \Big\downarrow \zeta & \\
0 \longrightarrow & C_q(L \cap M) & \longrightarrow & C_q(L) \oplus C_q(M) & \longrightarrow & C_q(K) & \longrightarrow 0
\end{array}$$

    commutes, and its rows are short exact sequences. But the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively, and therefore induce the natural homomorphisms ν_K, ν_M and $\nu_{L\cap M}$. We therefore obtain a commutative diagram

$$
\begin{array}{ccccccccc}
H_q(L'\cap M') & \longrightarrow & H_q(L')\oplus H_q(M') & \longrightarrow & H_q(K') & \xrightarrow{\alpha_q} & H_{q-1}(L'\cap M') & \longrightarrow & H_{q-1}(L')\oplus H_{q-1}(M') \\
\downarrow \nu_{L\cap M} & & \downarrow \nu_L \oplus \nu_M & & \downarrow \nu_K & & \downarrow \nu_{L\cap M} & & \downarrow \nu_L \oplus \nu_M \\
H_q(L\cap M) & \longrightarrow & H_q(L)\oplus H_q(M) & \longrightarrow & H_q(K) & \xrightarrow{\alpha_q} & H_{q-1}(L\cap M) & \longrightarrow & H_{q-1}(L)\oplus H_{q-1}(M)
\end{array}
$$

in which the rows are exact sequences, and are the Mayer-Vietoris sequences corresponding to the decompositions $K = L \cup M$ and $K' = L' \cup M'$ of K and K'. But the induction hypothesis ensures that the homomorphisms ν_L, ν_M and $\nu_{L\cap M}$ are isomorphisms, since L, M and $L \cap M$ are all proper subcomplexes of K. It now follows directly from the Five-Lemma that $\nu_K: H_q(K') \to H_q(K)$ is also an isomorphism, as required.