Mathematics Course 421, Trinity Term 1995. Worked Solutions

1. (a) (**BOOKWORK**) A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

   (b) (**BOOKWORK**) Let $x \in V$. For each $y \in K$ there exist open subsets D_y and E_y of X and Y respectively such that $(x, y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that

   $$K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k},$$

   since K is compact. Set

   $$N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}.$$

   Then N_x is an open set in X. Moreover

   $$N_x \times K \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

   so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required.

   (c) (**BOOKWORK**) Let X and Y be topological spaces, and let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

   Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space Y under the continuous map from Y to $X \times Y$ which sends $y \in Y$ to (x, y), and the image of any compact set under a continuous map is itself compact. Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that

   $$\{x\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r.$$

   Let

   $$V_x = \{x' \in X : \{x'\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r\}.$$

   It follows from *(b)* that V_x is an open set in X. We have therefore shown that, for each point x in X, there exists an open set V_x in X containing the point x such that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the space X. It follows from the compactness of X that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$

Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $\mathcal{U}$. On combining these finite collections, we obtain a finite collection of open sets belonging to $\mathcal{U}$ which covers $X \times Y$. This shows that $X \times Y$ is compact.

2. (a) (**BOOKWORK**) A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed. A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a path in X from x_0 to x_1.

(b) (**BOOKWORK**) Suppose that X is connected. Let $f\colon X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let

$$U = \{x \in X : f(x) = n\}, \qquad V = \{x \in X : f(x) \neq n\}.$$

Then U and V are the preimages of the open subsets $\{n\}$ and $\mathbb{Z}\setminus\{n\}$ of $\mathbb{Z}$, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f\colon X \to \mathbb{Z}$ is constant, with value n.

Conversely suppose that every continuous function $f\colon X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f\colon X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. It follows from (*iii*) that the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected.

(c) (**BOOKWORK**) Let X be a path-connected topological space, and let $f\colon X \to \mathbb{Z}$ be a continuous integer-valued function on X. If x_0 and x_1 are any two points of X then there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. But then $f \circ \gamma\colon [0,1] \to \mathbb{Z}$ is a continuous integer-valued function on $[0,1]$. But $[0,1]$ is connected, therefore $f \circ \gamma$ is constant. It follows that $f(x_0) = f(x_1)$. Thus every continuous integer-valued function on X is constant. Therefore X is connected, by *(b)*

(d) Let b be a point of X and let Y be the set of all points of X that are endpoints of paths starting at b. Let y be a point of Y. Now X is open in $\mathbb{R}^n$, and therefore there exists some $\delta > 0$ such that $B(y,\delta) \subset X$, where $B(y,\delta) = \{x \in \mathbb{R}^n : |x - y| < \delta\}$. But then $B(y,\delta) \subset Y$, since a path in X from b to any point z of $B(y,\delta)$ can be constructed by concatenating a path in X from b to y with a path along the straight line segment joining y to z. Therefore Y is open in $\mathbb{R}^n$.

Now let w be a point of $X \setminus Y$. There exists some $\delta > 0$ such that $B(w, \delta) \subset X$. Now there is no path in X that joins b to w, since $w \in X \setminus Y$. It follows that there cannot exist a path joining b to any point z in $B(w, \delta)$, since any such path could be concatenated with a path along the line segment joining z to w to obtain a path in X from b to w. Thus $B(w, \delta) \subset X \setminus Y$. Therefore $X \setminus Y$ is open in $\mathbb{R}^n$.

A connected topological space cannot be expressed as the union of two disjoint open sets unless one of those sets is the whole space and the other is the empty set. Therefore $Y = X$, and therefore any point of X can be joined to b by a path in X. It follows that X is path-connected, as required.

3. (a) (**BOOKWORK**) An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) (**BOOKWORK**) Let V be open in $\tilde{X}$, and let $x \in p(V)$. Then $x = p(v)$ for some $v \in V$. Now there exists an open set U containing the point x which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains v; let $\tilde{U}$ be this open set, and let $N_x = p(V \cap \tilde{U})$. Now N_x is open in X, since $V \cap \tilde{U}$ is open in $\tilde{U}$ and $p|\tilde{U}$ is a homeomorphism from $\tilde{U}$ to U. Also $x \in N_x$ and $N_x \subset p(V)$. It follows that $p(V)$ is the union of the open sets N_x as x ranges over all points of $p(V)$, and thus $p(V)$ is itself an open set, as required.

(c) (**BOOKWORK**) Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}: \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0

is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

(d) The map $f\colon H \to \mathbb{C} \setminus \{0\}$ defined by $f(z) = z^4$ is not a covering map. One way of verifying this is to observe that the continuous path $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ defined by $\gamma(t) = \exp(4\pi i t)$ does not lift to a path in H. Indeed a lift of γ would have to be of the form $t \mapsto \exp(\pi i t)$, and such a path leaves H when $t \geq \frac{1}{2}$.

4. (a) (**BOOKWORK**) There exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. (This is a consequence of the Path Lifting Theorem, applied to the exponential map from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. We define

$$n(\gamma, w) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i}.$$

(b) Let $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ be a continuous path such that $\gamma(t) = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Then $\eta(t) = \exp(-\tilde{\gamma}(t))$ for all $t \in [0,1]$, and hence

$$n(\eta, 0) = \frac{-\tilde{\gamma}(1) - (-\tilde{\gamma}(0))}{2\pi i} = -\frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = -n(\gamma, 0).$$

(c) (**BOOKWORK**) Let $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be defined by $H(t,\tau) = \gamma_\tau(t) - w$. It follows from the Monodromy Theorem that there exists a continuous map $\tilde{H}\colon [0,1] \times [0,1] \to \mathbb{C}$ such that $H = \exp \circ \tilde{H}$. But then

$$\tilde{H}(1,\tau) - \tilde{H}(0,\tau) = 2\pi i n(\gamma_\tau, w)$$

for all $\tau \in [0,1]$, and therefore the function $\tau \mapsto n(\gamma_\tau, w)$ is a continuous function on the interval $[0,1]$ taking values in the set $\mathbb{Z}$ of integers. But such a function must be constant on $[0,1]$, since the interval $[0,1]$ is connected. Thus $n(\gamma_0, w) = n(\gamma_1, w)$, as required.

(d) (**BOOKWORK**) Let $\gamma_\tau(t) = (1-\tau)\gamma_0(t) + \tau\gamma_1(t)$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then

$$|\gamma_\tau(t) - \gamma_0(t)| = \tau|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|,$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, and thus the closed curve γ_τ does not pass through w. The result therefore follows from *(c)*.

(e) (**BOOKWORK**)

The Fundamental Theorem of Algebra
Let $P\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number z_0 such that $P(z_0) = 0$.

Proof: The result is trivial if $P(0) = 0$. Thus it suffices to prove the result when $P(0) \neq 0$.

For any $r > 0$, let the closed curve σ_r denote the circle about zero of radius r, traversed once in the anticlockwise direction, given by

$\sigma_r(t) = r\exp(2\pi i t)$ for all $t \in [0,1]$. Consider the winding number $n(P \circ \sigma_r, 0)$ of $P \circ \sigma_r$ about zero. We claim that this winding number is equal to m for large values of r, where m is the degree of the polynomial P.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_n$ are complex numbers, and where $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and

$$Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}.$$

Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then

$$\left|\frac{Q(z)}{P_m(z)}\right| = \frac{1}{|a_m z|}\left|\frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1}\right| < 1,$$

since $R \geq 1$, and thus $|P(z) - P_m(z)| < |P_m(z)|$. It follows from *(d)* that $n(P \circ \sigma_r, 0) = n(P_m \circ \sigma_r, 0) = m$ for all $r > R$.

Given $r > 0$, let $\gamma_\tau = P \circ \sigma_{\tau r}$ for all $\tau \in [0,1]$. Then $n(\gamma_0, 0) = 0$, since γ_0 is a constant curve with value $P(0)$. Thus if the polynomial P were everywhere non-zero, then it would follow from *(c)* that $n(\gamma_1, 0) = n(\gamma_0, 0) = 0$. But $n(\gamma_1, 0) = n(P \circ \sigma_r, 0) = m$ for all $r > R$, and $m > 0$. Therefore the polynomial P must have at least one zero in the complex plane.

5. (a) (**BOOKWORK**) A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f\colon \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

   (b) (**BOOKWORK**) We must show that any continuous function $f\colon \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(U)$ and $f^{-1}(V)$ of U and V are open in ∂D (since f is continuous), and $\partial D = f^{-1}(U) \cup f^{-1}(V)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(U)$ and $f^{-1}(V)$. Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that each point z_i is within a distance δ of its neighbours z_{i-1} and z_{i+1}, where $z_0 = z_n$. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets U and V.

   Let x_0 be some point of $U \cap V$. Now the sets U, V and $U \cap V$ are all path-connected. Therefore we can choose paths $\alpha_i\colon [0,1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0,1]) \subset U$ whenever $z_i \in U$, and $\alpha_i([0,1]) \subset V$ whenever $z_i \in V$. For convenience let $\alpha_0 = \alpha_n$.

   Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for any } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for any } t \in [0,1], \end{cases}$$

   Note that $F_i(\partial T_i) \subset U$ whenever the short arc joining z_{i-1} to z_i is mapped by f into U, and $F_i(\partial T_i) \subset V$ whenever this short arc is mapped into V. But U and V are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required.

(c) (**BOOKWORK**) Let $U = \{\mathbf{x} \in S^n : x_{n+1} > -\frac{1}{2}\}$ and $V = \{\mathbf{x} \in S^n : x_{n+1} < \frac{1}{2}\}$. Then U and V are homeomorphic to an n-dimensional ball, and are therefore simply-connected. Moreover $U \cap V$ is path-connected, provided that $n > 1$. It follows that S^n is simply-connected for all $n > 1$.

6. (**BOOKWORK**) We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p: \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F: [0, 1] \times [0, 1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0, 1]$. It follows from the Monodromy Theorem that this homotopy lifts to a continuous map $G: [0, 1] \times [0, 1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0, 1]$, since $p(G(0, \tau)) = b = p(G(1, \tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0, 1]$, since $p(G(t, 0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t, 1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0, 1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0, 1]$ to $G(0, \tau)$ and $G(1, \tau)$ are constant, as are the functions sending $t \in [0, 1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

$$G(0, 0) = G(0, 1), \qquad G(1, 0) = G(1, 1),$$

$$G(1, 0) - \tilde{\alpha}(1) = G(0, 0) - \tilde{\alpha}(0), \qquad G(1, 1) - \tilde{\beta}(1) = G(0, 1) - \tilde{\beta}(0).$$

On combining these results, we see that

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1, 0) - G(0, 0) = G(1, 1) - G(0, 1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

We conclude from this that there exists a well-defined function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha}: [0, 1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Define a continuous path $\sigma: [0, 1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \tilde{\beta}(2t - 1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\lambda([\alpha][\beta]) = \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) = \lambda([\alpha]) + \lambda([\beta]).$$

Thus $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F\colon [0,1] \times [0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t, \tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0,1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n\colon [0,1] \to S^1$ is given by $\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$ for all $t \in [0,1]$. We conclude that $\lambda\colon \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism, as required.

7. (a) (**BOOKWORK**) The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are said to be *geometrically independent* if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} \lambda_j \mathbf{v}_j &= \mathbf{0}, \\ \sum_{j=0}^{q} \lambda_j &= 0 \end{cases}$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

(b) (**BOOKWORK**) A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$.

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

(i) if σ is a simplex belonging to K then every face of σ also belongs to K,

(ii) if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2. The *polyhedron* of a simplicial complex is the topological space that is the union of all the simplices of the complex.

(c) (**BOOKWORK**) We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required.

8. (a) (**BOOKWORK**) Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

(b) (**BOOKWORK**) Every point of $|K|$ belongs to the interior of a unique simplex of K. It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K.

(c) (**BOOKWORK**) Let $s: K \to L$ be a simplicial approximation to $f: |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, since $\mathbf{x}$ is in the interior of σ. It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s: K \to L$ is a simplicial approximation to $f: |K| \to |L|$, then $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$.

Conversely let $s: \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}_j))$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s: \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f: |K| \to |L|$, as required.

(d) (**BOOKWORK**)

Simplicial Approximation Theorem
Let K and L be simplicial complexes, and let $f: |K| \to |L|$

be a continuous map. Then, for some sufficiently large integer j, there exists a simplicial approximation $s: K^{(j)} \to L$ to f defined on the jth barycentric subdivision $K^{(j)}$ of K.

Proof. The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{v}$ is a vertex of that simplex. It follows from the continuity of the map $f: |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$. It then follows from the Lebesgue Lemma that there exists some $\delta > 0$ with the property that every subset of $|K|$ whose diameter is less than δ is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$, since

$$\mu(K^{(j)}) \leq \left(\frac{\dim K}{\dim K + 1} \right)^j \mu(K)$$

for all j. Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s: \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. It follows from *(c)* that this is the desired simplicial approximation to f.

9. (a) (**BOOKWORK**) $\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$,

where $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle = \langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$.
We now show that $\partial_{q-1} \circ \partial_q = 0$, where $2 \leq q \leq \dim K$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned} &\partial_{q-1}\partial_q (\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\ &\quad = \sum_{j=0}^{q} (-1)^j \partial_{q-1} (\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle) \\ &\quad = \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\ &\qquad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\ &\quad = 0 \end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K.

(b) (**BOOKWORK**) $Z_q(K) = \ker(\partial_q \colon C_q(K) \to C_{q-1}(K))$, $B_q(K) = \partial_{q+1}(C_q(K))$, $H_q(K) = Z_q(K)/B_q(K)$.

(c) (**BOOKWORK**) There is a well-defined homomorphism

$$D_q \colon C_q(K) \to C_{q+1}(K)$$

characterized by the property that $D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends

$\sum_{r=1}^{s} n_r\langle \mathbf{v}_r\rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle)) \\
&\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle) \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus $\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$ for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required.

10. (a) By inspection the boundary of the 2-chain is given by

$$\begin{aligned}(a+d+g)\langle\mathbf{v}_1\mathbf{v}_2\rangle + (b+e+g)\langle\mathbf{v}_2\mathbf{v}_3\rangle + (c+f+g)\langle\mathbf{v}_3\mathbf{v}_1\rangle \\ + (c-a)\langle\mathbf{v}_1\mathbf{v}_4\rangle + (a-b)\langle\mathbf{v}_2\mathbf{v}_4\rangle + (b-c)\langle\mathbf{v}_3\mathbf{v}_4\rangle \\ + (f-d)\langle\mathbf{v}_1\mathbf{v}_4\rangle + (d-e)\langle\mathbf{v}_2\mathbf{v}_4\rangle + (e-f)\langle\mathbf{v}_3\mathbf{v}_4\rangle\end{aligned}$$

Thus the boundary of the 2-chain is zero if and only if $a = b = c$, $d = e = f$ and $a + d + g = 0$. It follows that the 2-chain is a 2-cycle if and only if it is of the form $mz_1 + nz_2$ for some integers m and n. (Indeed z_1 and z_2 are 2-cycles, and if the 2-chain of *(a)* is a 2-cycle then it is of the form $mz_1 + nz_2$ with $a = b = c = m$, $d = e = f = n$ and $g = -m - n$.)

Now $H_2(K) = Z_2(K)$ since $B_2(K) = 0$. The function sending $mz_1 + nz_2$ to (m, n) is an isomorphism from $Z_2(K)$ to $\mathbb{Z} \oplus \mathbb{Z}$. Thus $H_2(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b) The 1-chain $\langle\mathbf{v}_1\mathbf{v}_2\rangle + \langle\mathbf{v}_2\mathbf{v}_4\rangle + \langle\mathbf{v}_4\mathbf{v}_3\rangle + \langle\mathbf{v}_3\mathbf{v}_5\rangle + \langle\mathbf{v}_5\mathbf{v}_1\rangle$ is the boundary of

$$\langle\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\rangle - \langle\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4\rangle - \langle\mathbf{v}_3\mathbf{v}_1\mathbf{v}_5\rangle.$$

The 1-chain $2\langle\mathbf{v}_1\mathbf{v}_2\rangle + 3\langle\mathbf{v}_2\mathbf{v}_3\rangle + \langle\mathbf{v}_3\mathbf{v}_1\rangle$ is not a 1-boundary since it is not a 1-cycle:

$$\partial_1(2\langle\mathbf{v}_1\mathbf{v}_2\rangle + 3\langle\mathbf{v}_2\mathbf{v}_3\rangle + \langle\mathbf{v}_3\mathbf{v}_1\rangle) = 2\langle\mathbf{v}_3\rangle - \langle\mathbf{v}_1\rangle - \langle\mathbf{v}_2\rangle.$$

11. (a) (**BOOKWORK**) The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if image$(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i\colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i\colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}\colon C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i\colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \downarrow \partial_{i+2} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \\
0 & \longrightarrow & A_i & \xrightarrow{p_i} & B_i & \xrightarrow{q_i} & C_i & \longrightarrow & 0\,. \\
 & & \downarrow \partial_i & & \downarrow \partial_i & & \downarrow \partial_i & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \downarrow \partial_{i-1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

(b) (**BOOKWORK**) Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i\colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}\colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_{i-1}$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) &= \partial_i(b) + \partial_i(b' - b) = \partial_i(b') \\ &= p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_a(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism

$$\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*),$$

as required.