Course 421: Annual Examination
Course outline and worked solutions

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Course Website

The course website, with online lecture notes, problem sets. etc. is located at

`http://www.maths.tcd.ie/~dwilkins/Courses/421/`

Course Outline

The class has been informed that Section 8 (Modules) is non-examinable in 2009.

The listing above covers the course material up to the end of Hilary Term. Additional examples and problems will be presented in the problems in Trinity Term.

1. (a) [Definitions. From printed lecture notes.] A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

   (i) the empty set $\emptyset$ and the whole set X are open sets,

   (ii) the union of any collection of open sets is itself an open set,

   (iii) the intersection of any *finite* collection of open sets is itself an open set.

   The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

   A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

(b) [Result is mentioned in the printed lecture notes, but a proof is not written out in the notes.] Let $x \in X$. Then

$$\begin{aligned}
x \in \bigcup\nolimits_{\alpha\in A} q^{-1}(V_\alpha) &\iff \text{there exists } \alpha \in A \text{ such that } x \in q^{-1}(V_\alpha)\\
&\iff \text{there exists } \alpha \in A \text{ such that } q(x) \in V_\alpha\\
&\iff q(x) \in \bigcup\nolimits_{\alpha\in A} V_\alpha\\
&\iff x \in q^{-1}\left(\bigcup\nolimits_{\alpha\in A} V_\alpha\right)\\
x \in \bigcap\nolimits_{\alpha\in A} q^{-1}(V_\alpha) &\iff \text{for all } \alpha \in A,\ x \in q^{-1}(V_\alpha)\\
&\iff \text{for all } \alpha \in A,\ q(x) \in V_\alpha\\
&\iff q(x) \in \bigcap\nolimits_{\alpha\in A} V_\alpha\\
&\iff x \in q^{-1}\left(\bigcap\nolimits_{\alpha\in A} V_\alpha\right)
\end{aligned}$$

The result now follows from the basic principle that two sets are equal if and only if they have the same elements.

(c) [Definition. From printed lecture notes.] Let X and Y be topological spaces and let $q: X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q: X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

(d) [Bookwork adapted from printed lecture notes] Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$.

It follows directly from (b) that, given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets. Therefore unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q: X \to Y$ is an identification map with respect to the topology τ. Moreover the definition of identification maps ensures that the open subsets of Y must be the subsets belonging to τ, and thus τ is the unique topology on Y for which the function $q: X \to Y$ is an identification map.

(e) [From printed lecture notes.] Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required.

2. (a) [Definitions. From printed lecture notes.] A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed. A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a path in X from x_0 to x_1.

(b) [Not bookwork. However the special case when $D = \mathbb{Z}$ is bookwork.] Suppose that X is connected. Let $f\colon X \to D$ be a continuous function. Choose $d \in f(X)$, and let

$$U = \{x \in X : f(x) = d\}, \qquad V = \{x \in X : f(x) \neq d\}.$$

Then U and V are the preimages of the open subsets $\{d\}$ and $D\backslash\{d\}$ of D, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $d \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f\colon X \to D$ is constant, with value d.

Conversely suppose that every continuous function $f\colon X \to D$ is constant. Let S be a subset of X which is both open and closed. Let u and v be distinct elements of D, and let $f\colon X \to D$ be defined by

$$f(x) = \begin{cases} u & \text{if } x \in S; \\ v & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of D under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. But then the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected.

(c) [From printed lecture notes.] Choose a point x_0 of X. Let Z be the subset of X consisting of all points x of X with the property that x can be joined to x_0 by a path. We show that the subset Z is both open and closed in X.

Now, given any point x of X there exists a path connected open set N_x in X such that $x \in N_x$. We claim that if $x \in Z$ then $N_x \subset Z$, and if $x \notin Z$ then $N_x \cap Z = \emptyset$.

Suppose that $x \in Z$. Then, given any point x' of N_x, there exists a path in N_x from x' to x. Moreover it follows from the definition of the set Z that there exists a path in X from x to x_0. These two paths can be concatenated to yield a path in X from x' to x_0, and therefore $x' \in Z$. This shows that $N_x \subset Z$ whenever $x \in Z$.

Next suppose that $x \notin Z$. Let $x' \in N_x$. If it were the case that $x' \in Z$, then we would be able to concatenate a path in N_x from x to x' with a path in X from x' to x_0 in order to obtain a path in X from x to x_0. But this is impossible, as $x \notin Z$. Therefore $N_x \cap Z = \emptyset$ whenever $x \notin Z$.

Now the set Z is the union of the open sets N_x as x ranges over all points of Z. It follows that Z is itself an open set. Similarly $X \setminus Z$ is the union of the open sets N_x as x ranges over all points of $X \setminus Z$, and therefore $X \setminus Z$ is itself an open set. It follows that Z is a subset of X that is both open and closed. Moreover $x_0 \in Z$, and therefore Z is non-empty. But the only subsets of X that are both open and closed are $\emptyset$ and X itself, since X is connected. Therefore $Z = X$, and thus every point of X can be joined to the point x_0 by a path in X. We conclude that X is path-connected, as required.

3. (a) [Definitions. From printed lecture notes.] Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p: \tilde{X} \to X$ is said to be a *covering map* if $p: \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [Not so far discussed, but result might be mentioned in Trinity Term.] Given $x \in X$, let $f(x)$ denote the number of elements in $p^{-1}(\{x\})$. Then $f: X \to \mathbb{Z}$ is a function on X whose values are positive integers. Let $x_0 \in X$. Then $x_0 \in U$ for some open set U in X which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, where each of these open sets is mapped homeomorphically (and thus bijectively) onto U by the covering map p. The number of such open sets must be $f(x_0)$, and moreover $f(x) = f(x_0)$ for all $x \in U$. Therefore the function f is constant around x_0. It follows that the function f is continuous on X. But every continuous integer-valued function on a connected topological space is constant. Therefore f is constant, and its value is the required positive integer n.

(c) [Example. Not bookwork.] If $(x, y, u, v) \in S$ then $u^2 - v^2 = x$ and $u^2 + v^2 = \sqrt{x^2 + y^2}$, and therefore $2u^2 = \sqrt{x^2 + y^2} + x$, and $2v^2 = \sqrt{x^2 + y^2} - x$. It follows that $u = 0$ if and only if $y = 0$ and $x \leq 0$, and $v = 0$ if and only if $y = 0$ and $x \geq 0$. Therefore $(x, y, u, v) \in S$ satisfies $u \neq 0$ if and only if $(x, y, u, v) \in U$. Also $(x, y, u, v) \in S$ satisfies $v \neq 0$ if and only if $(x, y, u, v) \in V$.

Now $p^{-1}(U) = \tilde{U}_+ \cup \tilde{U}_-$, where

$$\tilde{U}_+ = \{(x, y, u, v) \in S : u > 0\}, \quad \tilde{U}_- = \{(x, y, u, v) \in S : u < 0\}.$$

The sets $\tilde{U}_+$ and $\tilde{U}_-$ are disjoint and are open in S. Also $(x, y, u, v) \in \tilde{U}_\pm$ if and only if $(x, y) \in U$, $u = \pm\frac{1}{2}(\sqrt{x^2 + y^2} + x)$ and $v = y/2u$. It follows from this that p maps each of $\tilde{U}_+$ and $\tilde{U}_-$ homeomophically onto U. Thus U is evenly covered by the map p.

Similarly V is evenly covered by the map p, and $p^{-1}(V) = \tilde{V}_+ \cup \tilde{V}_-$, where

$$\tilde{V}_+ = \{(x, y, u, v) \in S : v > 0\}, \quad \tilde{V}_- = \{(x, y, u, v) \in S : v < 0\}.$$

Indeed $(x, y, u, v) \in \tilde{V}_\pm$ if and only if $(x, y) \in V$, $v = \pm\frac{1}{2}(\sqrt{x^2 + y^2} -$

x) and $u = y/2v$. It follows each of the sets $\tilde{V}_{\pm}$ is mapped homeomorphically onto V by p.

(d) [Not bookwork.] The map p_0 is a covering map, since its codomain is the union of the evenly-covered open sets U and V. The map p is not a covering map. Indeed let $f(x, y)$ denote the number of elements in $p^{-1}\{(x, y)\}$ for all $(x, y) \in \mathbb{R}^2$. Then $f(0, 0) = 1$, whereas $f(x, y) = 2$ when $(x, y) \neq (0, 0)$. Thus the result of (b) doesn't hold for the map p, and therefore p is not a covering map.

4. (a) [Extracted from non-contiguous material in lecture notes.] Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0, 1\}$. We denote the equivalence class of a loop $\gamma\colon [0, 1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$. Thus two loops γ_0 and γ_1 represent the same element of $\pi_1(X, x_0)$ if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0, 1\}$ (i.e., there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ between γ_0 and γ_1 which maps $(0, \tau)$ and $(1, \tau)$ to x_0 for all $\tau \in [0, 1]$). Then $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0. This group $\pi_1(X, x_0)$ is the *fundamental group* of the topological space X based at the point x_0.

(b) [Definition. From printed lecture notes.] A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f\colon \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

(c) [From printed lecture notes.] We must show that any continuous function $f\colon \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(U)$ and $f^{-1}(V)$ of U and V are open in ∂D (since f is continuous), and $\partial D = f^{-1}(U) \cup f^{-1}(V)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(U)$ and $f^{-1}(V)$. Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that the distance from z_i to z_{i+1} is less than δ for $i = 1, 2, \ldots, n-1$ and the distance from z_n to z_1 is also less than δ. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets U and V.

Let x_0 be some point of $U \cap V$. Now the sets U, V and $U \cap V$ are all path-connected. Therefore we can choose paths $\alpha_i\colon [0, 1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0, 1]) \subset U$ whenever $f(z_i) \in U$, and $\alpha_i([0, 1]) \subset V$ whenever $f(z_i) \in V$. For convenience let $\alpha_0 = \alpha_n$.

Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for any } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for any } t \in [0,1], \end{cases}$$

Note that $F_i(\partial T_i) \subset U$ whenever the short arc joining z_{i-1} to z_i is mapped by f into U, and $F_i(\partial T_i) \subset V$ whenever this short arc is mapped into V. But U and V are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required.

5. (a) [Definition. From printed lecture notes.] Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g: X \to X$, where the following conditions are satisfied:

   (i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

   (ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

   (iii) given any point x of X, there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

   (b) [From printed lecture notes.] Let x_0 and x_1 be the points of X given by

$$x_0 = \alpha(0) = \beta(0), \qquad x_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore there exists a homotopy $F: [0, 1] \times [0, 1] \to X$ such that

$$F(t, 0) = \alpha(t) \text{ and } F(t, 1) = \beta(t) \text{ for all } t \in [0, 1],$$

$$F(0, \tau) = x_0 \text{ and } F(1, \tau) = x_1 \text{ for all } \tau \in [0, 1].$$

It then follows from the Monodromy Theorem that there exists a continuous map $G: [0, 1] \times [0, 1] \to \tilde{X}$ such that $p \circ G = F$ and $G(0, 0) = \tilde{\alpha}(0)$. Then $p(G(0, \tau)) = x_0$ and $p(G(1, \tau)) = x_1$ for all $\tau \in [0, 1]$. Now any continuous lift of a constant path must itself be a constant path. Therefore $G(0, \tau) = \tilde{x}_0$ and $G(1, \tau) = \tilde{x}_1$ for all $\tau \in [0, 1]$, where

$$\tilde{x}_0 = G(0, 0) = \tilde{\alpha}(0), \qquad \tilde{x}_1 = G(1, 0).$$

However

$$G(0, 0) = G(0, 1) = \tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0),$$

$$p(G(t, 0)) = F(t, 0) = \alpha(t) = p(\tilde{\alpha}(t))$$

and

$$p(G(t, 1)) = F(t, 1) = \beta(t) = p(\tilde{\beta}(t))$$

for all $t \in [0, 1]$. Now the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points. It follows that $G(t, 0) = \tilde{\alpha}(t)$ and $G(t, 1) = \tilde{\beta}(t)$ for all $t \in [0, 1]$. In particular,

$$\tilde{\alpha}(1) = G(0, 1) = \tilde{x}_1 = G(1, 1) = \tilde{\beta}(1).$$

Moreover the map $G: [0,1] \times [0,1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta}$ rel $\{0,1\}$, as required.

(c) [From printed lecture notes.] Let $\gamma: [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = q(x_0)$. It follows from the Path Lifting Theorem for covering maps that there exists a unique path $\tilde{\gamma}: [0,1] \to X$ for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$. Now $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must belong to the same orbit, since $q(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = q(\tilde{\gamma}(1))$. Therefore there exists some element g of G such that $\tilde{\gamma}(1) = \theta_g(x_0)$. This element g is uniquely determined, since the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X, q(x_0))$. Indeed if σ is a loop in X/G based at $q(x_0)$, if $\tilde{\sigma}$ is the lift of σ starting at x_0 (so that $q \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = x_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, q(x_0))$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde{\gamma}(1) = \tilde{\sigma}(1)$. We conclude therefore that there exists a well-defined function

$$\lambda: \pi_1(X/G, q(x_0)) \to G,$$

which is characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$.

6. (a) [From printed lecture notes.] The homomorphism $\partial_q \colon C_q(K) \to C_{q-1}(K)$, is characterized by the property that

$$\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned}
&\partial_{q-1}\partial_q \left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \right) \\
&\qquad = \sum_{j=0}^{q} (-1)^j \partial_{q-1} \left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \right) \\
&\qquad = \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\
&\qquad = 0
\end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K.

(b) [Definitions. From printed lecture notes.] Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q \colon C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1} \colon C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, and therefore $B_q(K) \subset Z_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K.

(c) [From printed lecture notes.] There is a well-defined homomorphism $D_q \colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.
Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&= \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&= \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required.

7. (a) [Adapted from more general result in lecture notes.] We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any q-chain c of K can be expressed uniquely as a sum of the form $c = c_1 + c_2$, where c_1 is a q-chain of L and c_2 is a q-chain of M for $j = 1, 2, \ldots, r$. It follows that

$$C_q(K) \cong C_q(L) \oplus C_q(M).$$

Now let z be a q-cycle of K (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express z uniquely in the form $z = z_1 + z_2$, where z_1 is a q-chain of L and z_2 is a q-chain of M. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2).$$

Moreover $\partial_q(z_1)$ is a $(q-1)$-chain of L, and $\partial_q(z_2)$ is a $(q-1)$-chain of M. It follows that $\partial_q(z_1) = \partial_q(z_2) = 0$. Hence z_1 is a q-cycle of L and z_2 is a q-cycle of M, and thus

$$Z_q(K) \cong Z_q(L) \oplus Z_q(M).$$

Now let b be a q-boundary of K. Then $b = \partial_{q+1}(c)$ for some $(q+1)$-chain c of K. Moreover $c = c_1 + c_2$, where $c_1 \in C_{q+1}(L)$ and $c_2 \in C_{q+1}(M)$. Thus $b = b_1 + b_2$, where $b_1 \in B_q(L)$ and $b_2 \in B_q(M)$ are given by $b_1 = \partial_{q+1} c_1$ and $b_2 = \partial_{q+1} c_2$. We deduce that

$$B_q(K) \cong B_q(L) \oplus B_q(M).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu\colon H_q(K_1) \oplus H_q(K_2) \to H_q(K)$$

which maps $([z_1], [z_2])$ to $[z_1 + z_2]$ for all $z_1 \in Z_q(L)$ and $z_2 \in Z_q(M)$, where $[z_1]$ and $[z_2]$ denote the homology classes of z_1 and z_2 in $H_q(L)$ and $H_q(M)$ respectively.

(b) [Example. Not bookwork. Similar examples discussed in class.] Let p be this chain. Then

$$\begin{aligned}
\partial_2 p \;=\; & n_1 \left(\langle \mathbf{v}_2\,\mathbf{v}_4\rangle - \langle \mathbf{v}_1\,\mathbf{v}_4\rangle + \langle \mathbf{v}_1\,\mathbf{v}_2\rangle\right) \\
& + n_2 \left(\langle \mathbf{v}_3\,\mathbf{v}_4\rangle - \langle \mathbf{v}_2\,\mathbf{v}_4\rangle + \langle \mathbf{v}_2\,\mathbf{v}_3\rangle\right) \\
& + n_3 \left(\langle \mathbf{v}_1\,\mathbf{v}_4\rangle - \langle \mathbf{v}_3\,\mathbf{v}_4\rangle + \langle \mathbf{v}_3\,\mathbf{v}_1\rangle\right) \\
& + n_4 \left(\langle \mathbf{v}_2\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_2\rangle\right) \\
& + n_5 \left(\langle \mathbf{v}_3\,\mathbf{v}_5\rangle - \langle \mathbf{v}_2\,\mathbf{v}_5\rangle + \langle \mathbf{v}_2\,\mathbf{v}_3\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
& + n_6\left(\langle \mathbf{v}_1\,\mathbf{v}_5\rangle - \langle \mathbf{v}_3\,\mathbf{v}_5\rangle + \langle \mathbf{v}_3\,\mathbf{v}_1\rangle\right) \\
& + n_7\left(\langle \mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_4\rangle\right) \\
& + n_8\left(\langle \mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_2\,\mathbf{v}_5\rangle + \langle \mathbf{v}_2\,\mathbf{v}_4\rangle\right) \\
& + n_7\left(\langle \mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_3\,\mathbf{v}_5\rangle + \langle \mathbf{v}_3\,\mathbf{v}_4\rangle\right) \\
= \;& (n_1 + n_4)\langle \mathbf{v}_1\,\mathbf{v}_2\rangle + (n_2 + n_5)\langle \mathbf{v}_2\,\mathbf{v}_3\rangle + (n_3 + n_6)\langle \mathbf{v}_3\,\mathbf{v}_1\rangle \\
& + (n_3 - n_1 + n_7)\langle \mathbf{v}_1\,\mathbf{v}_4\rangle + (n_1 - n_2 + n_8)\langle \mathbf{v}_2\,\mathbf{v}_4\rangle \\
& + (n_2 - n_3 + n_9)\langle \mathbf{v}_3\,\mathbf{v}_4\rangle + (n_6 - n_4 - n_7)\langle \mathbf{v}_1\,\mathbf{v}_5\rangle \\
& + (n_7 - n_5 - n_8)\langle \mathbf{v}_2\,\mathbf{v}_5\rangle + (n_8 - n_6 - n_9)\langle \mathbf{v}_3\,\mathbf{v}_5\rangle \\
& + (n_7 + n_8 + n_9)\langle \mathbf{v}_4\,\mathbf{v}_5\rangle
\end{aligned}
$$

(c) [Example. Not bookwork.] Let p be the 2-chain of K defined in (c). Then $\partial_2 p = 0$ if and only if

$$n_4 = -n_1, \quad n_5 = -n_2, \quad n_6 = -n_3,$$

$$n_7 = n_1 - n_3, \quad n_8 = n_2 - n_1, \quad n_9 = n_3 - n_2.$$

Now the simplicial complex K is 2-dimensional, and therefore $B_2(K) = 0$. It follows that

$$H_2(K) \cong \mathbb{Z}_2(K) = \{n_1 z_1 + n_2 z_2 + n_3 z_3 : n_1, n_2, n_3 \in \mathbb{Z}\} \cong \mathbb{Z}^3,$$

where

$$
\begin{aligned}
z_1 &= \langle \mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_4\rangle - \langle \mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_5\rangle + \langle \mathbf{v}_1\,\mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_2\,\mathbf{v}_4\,\mathbf{v}_5\rangle. \\
z_2 &= \langle \mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4\rangle - \langle \mathbf{v}_3\,\mathbf{v}_3\,\mathbf{v}_5\rangle + \langle \mathbf{v}_2\,\mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_3\,\mathbf{v}_4\,\mathbf{v}_5\rangle, \\
z_3 &= \langle \mathbf{v}_3\,\mathbf{v}_1\,\mathbf{v}_4\rangle - \langle \mathbf{v}_3\,\mathbf{v}_1\,\mathbf{v}_5\rangle + \langle \mathbf{v}_3\,\mathbf{v}_4\,\mathbf{v}_5\rangle - \langle \mathbf{v}_1\,\mathbf{v}_4\,\mathbf{v}_5\rangle.
\end{aligned}
$$

8. (a) [Definitions. From printed lecture notes.] A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of R-modules, together with homomorphisms $\partial_i\colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient module $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i\colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1}\colon C_{i+1} \to C_i$.

Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i\colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

(b) [From printed lecture notes.] Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i\colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2}\colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$\begin{aligned} p_{i-1}(w + \partial_i(a)) &= p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) \\ &= \partial_i(b') = p_{i-1}(w'), \end{aligned}$$

and $p_{i-1}\colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}\colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i\colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism

$$\alpha_i\colon H_i(C_*) \to H_{i-1}(A_*),$$

as required.

9. (a) [Based on lecture notes.] The qth chain group $C_q(K, L)$ of the simplicial pair is defined to be the quotient group $C_q(K)/C_q(L)$, where $C_q(K)$ and $C_q(L)$ denote the groups of q-chains of K and L respectively.

The boundary homomorphism $\partial_q: C_q(K) \to C_{q-1}(L)$ maps the subgroup $C_q(L)$ into $C_{q-1}(L)$, and therefore induces a homomorphism $\partial_q: C_q(K, L) \to C_{q-1}(K, L)$. We define

$$H_q(K, L) = Z_q(K, L)/B_q(K, L),$$

where

$$\begin{aligned} Z_q(K, L) &= \ker(\partial_q: C_q(K, L) \to C_{q-1}(K, L)) \\ &= \{c + C_q(L) : c \in C_q(K) \text{ and } \partial_q c \in C_{q-1}(L)\}, \\ B_q(K, L) &= \operatorname{image}(\partial_{q+1}: C_{q+1}(K, L) \to C_q(K, L)) \\ &= \{\partial_{q+1}(e) + C_q(L) : e \in C_{q+1}(K)\}. \end{aligned}$$

(b) [Based on lecture notes.] The homology exact sequence of the simplicial pair (K, L) is the exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(L) \xrightarrow{i_*} H_q(K) \xrightarrow{u_*} H_q(K, L) \xrightarrow{\partial_*} H_{q-1}(L) \xrightarrow{i_*} H_{q-1}(K) \xrightarrow{u_*} \cdots$$

of homology groups is exact, where the homomorphism i_*, u_* and ∂_* are defined as in the examination question.

(c) [The essentials of the following argument may be covered in Trinity Term lectures, not yet delivered.] $H_0(L) \cong \mathbb{Z}$, since $|L|$ is connected. Moreover $H_0(L)$ is an infinite cyclic group generated by the homology class α determined by some vertex $\mathbf{v}$ of L. Also $H_0(K) \cong \mathbb{Z}^r$, where r is the number of connected components of $|K|$. Now $n i_*(\alpha) \neq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Therefore $n\alpha \in \ker i_*$ if and only if $n = 0$. We conclude that $i_*: H_0(L) \to H_0(K)$ is injective.

(d) [Not bookwork. Related examples will be discussed.] The image of the homomorphism $\partial_*: H_1(K, L) \to H_0(L)$ is the kernel of the homomorphism $i_*: H_0(L) \to H_0(K)$, by exactness of the homology exact sequence of the simplicial pair (K, L), and is therefore the zero group. Therefore $\partial_*: H_1(K, L) \to H_0(L)$ is the zero homomorphism. It follows from exactness that $u_*: H_1(K) \to H_1(K, L)$ is surjective. We then have the following exact sequence:

$$H_2(K, L) \xrightarrow{\partial_*} H_1(L) \xrightarrow{i_*} H_1(K) \xrightarrow{u_*} H_1(K, L) \longrightarrow 0$$

Now the kernel of $i_*: H_1(L) \to H_1(K)$ is the image $\partial_*(H_2(K, L))$ of $\partial_*: H_2(K, L) \to H_1(l)$, by exactness. Therefore

$$\begin{aligned} \ker(u_*: H_1(K) \to H_1(K, L)) &= i_*(H_1(L)) \\ &\cong H_1(L)/\partial_*(H_2(K, L)) = F. \end{aligned}$$

We thus have an exact sequence

$$0 \longrightarrow F \longrightarrow H_1(K) \xrightarrow{u_*} H_2(K, L) \longrightarrow .0$$

Now $H_2(K, L) \cong \mathbb{Z} \oplus \mathbb{Z}$, and thus there exist $\alpha_1, \alpha_2 \in H_1(K, L)$ such that

$$H_2(K, L) = \{n_1\alpha_1 + n_2\alpha_2 : n_1, n_2 \in \mathbb{Z}\}.$$

Also the homomorphism $u_*: H_1(K) \to H_1(K, L)$ is surjective. It follows that there exists a homomorphism $\theta: H_1(K, L) \to H_1(K)$ such that $u_* \circ \theta$ is the identity automorphism of $H_1(K, L)$. Thus the exact sequence

$$0 \longrightarrow F \longrightarrow H_1(K) \xrightarrow{u_*} H_2(K, L) \longrightarrow .0$$

splits, and $H_1(K) \cong F \oplus H_1(K, L) \cong F \oplus \mathbb{Z} \oplus \mathbb{Z}$, as required.

(e) [Not bookwork.] The image of $u_*: H_2(K) \to H_2(K, L)$ is the kernel of $\partial_*: H_2(K, L) \to H_1(L)$. It follows that $u_*: H_2(K) \to H_2(K, L)$ is surjective. Also $H_3(K, L) = 0$, because K is a 2-dimensional simplicial complex. Therefore the sequence

$$0 \longrightarrow H_2(L) \xrightarrow{i_*} H_2(K) \xrightarrow{u_*} H_2(K, L) \longrightarrow 0$$

is exact. Also $H_2(K, L) \cong \mathbb{Z}$. It follows that the above short exact sequence splits, and therefore

$$H_2(K) \cong H_2(L) \oplus H_2(K, L) \cong H_2(L) \oplus \mathbb{Z}.$$