Course 421: Annual Examination
Course outline and worked solutions

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March 8, 2007

Course Type

This examination does not relate to any course of lectures given in the current academic year. Instead 'Course 421' is a reading course, and candidates are examined on the majority of the material on the course notes dating from the time the course was last taught, in the academic year 2004–05, and also on material on the topological classification of closed surfaces, that was included in the course in previous years.

In consequence, there is an even greater proportion of 'bookwork' in the examination. (Note that, in particular, the students will not have had the tutorials on, for example, applications of the Mayer-Vietoris Exact Sequence that students in previous years would have received.)

Course Website

The course website, with online lecture notes, problem sets. etc. is located at

`http://www.maths.tcd.ie/~dwilkins/Courses/421/`

Course Outline

2. Homotopies and Covering Maps

- **2.1** Homotopies
- **2.2** Covering Maps
- **2.3** Path Lifting and the Monodromy Theorem

3. The Fundamental Group

- **3.1** The Fundamental Group of a Topological Space
- **3.2** Simply-Connected Topological Spaces
- **3.3** The Fundamental Group of the Circle

4. Simplicial Complexes

- **4.1** Geometrical Independence
- **4.2** Simplicial Complexes in Euclidean Spaces
- **4.3** Simplicial Maps
- **4.4** Barycentric Subdivision of a Simplicial Complex

Style of questions

1. (a), (b) and (c) bookwork, (c) not bookwork.
2. Bookwork
3. Bookwork
4. Bookwork.
5. (a) and (b) bookwork, (c) and (d) not bookwork.
6. Bookwork.
7. Bookwork.
8. Bookwork.
9. Candidates to select suitable material relevant to the topological classification of closed surfaces (almost certainly utilizing material from printed lecture notes).

1. (a) [From printed lecture notes.] Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [From printed lecture notes.] The map $p\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $p(z) = \exp(z)$ is a covering map. Indeed, given any $\theta \in [-\pi, \pi]$ let us define

$$U_\theta = \{z \in \mathbb{C} \setminus \{0\} : \arg(-z) \neq \theta\}.$$

Then $p^{-1}(U_\theta)$ is the disjoint union of the open sets

$$\{z \in \mathbb{C} : |\mathrm{Im}\, z - \theta - 2\pi n| < \pi\},$$

for all integers n, and p maps each of these open sets homeomorphically onto U_θ. Thus U_θ is evenly covered by the map p.

(c) [From printed lecture notes.] Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}\colon \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

(d) This is not a covering map. Let $E = S^2 \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Then no point of E has an open neighbourhood evenly covered by the map p. One way to see this is to note that the map p violates the conclusions of the path-lifting theorem. If $\gamma\colon [a, b] \to D^2$ is a path with $\gamma(a) \in E$, if $\tilde{\gamma}\colon [a, b] \to S^2$ is a lift of γ satisfying $p \circ \tilde{\gamma} = \gamma$, and if $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the map sending $(x, y, z) \in \mathbb{R}^3$ to $(x, y, -z)$, then $\rho \circ \tilde{\gamma}$ is also a lift of γ. Thus if $\gamma([a, b])$ is not wholly contained in the set E then $\tilde{\gamma}$ and $\rho \circ \tilde{\gamma}$ are distinct lifts of γ. This could not happen were $p\colon S^2 \to D^2$ a covering map.

2. (a) A topological space X is said to be *simply-connected* if it is path-connected, and any continuous map $f\colon \partial D \to X$ mapping the boundary circle ∂D of a closed disc D into X can be extended continuously over the whole of the disk.

(b) [From printed lecture notes, with U and V replaced by V and W respectively.] We must show that any continuous function $f\colon \partial D \to X$ defined on the unit circle ∂D can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(V)$ and $f^{-1}(W)$ of V and W are open in ∂D (since f is continuous), and $\partial D = f^{-1}(V) \cup f^{-1}(W)$. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that any arc in ∂D whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(V)$ and $f^{-1}(W)$. Choose points $z_1, z_2, \ldots, z_n$ around ∂D such that the distance from z_i to z_{i+1} is less than δ for $i = 1, 2, \ldots, n-1$ and the distance from z_n to z_1 is also less than δ. Then, for each i, the short arc joining z_{i-1} to z_i is mapped by f into one or other of the open sets V and W.

Let x_0 be some point of $V \cap W$. Now the sets V, W and $V \cap W$ are all path-connected. Therefore we can choose paths $\alpha_i\colon [0,1] \to X$ for $i = 1, 2, \ldots, n$ such that $\alpha_i(0) = x_0$, $\alpha_i(1) = f(z_i)$, $\alpha_i([0,1]) \subset V$ whenever $f(z_i) \in V$, and $\alpha_i([0,1]) \subset W$ whenever $f(z_i) \in W$. For convenience let $\alpha_0 = \alpha_n$.

Now, for each i, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points z_{i-1} and z_i and by the short arc joining z_{i-1} to z_i. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary ∂T_i of T_i into a simply-connected space can be extended continuously over the whole of T_i. In particular, let F_i be the function on ∂T_i defined by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in T_i \cap \partial D, \\ \alpha_{i-1}(t) & \text{if } z = tz_{i-1} \text{ for any } t \in [0,1], \\ \alpha_i(t) & \text{if } z = tz_i \text{ for any } t \in [0,1], \end{cases}$$

Note that $F_i(\partial T_i) \subset V$ whenever the short arc joining z_{i-1} to z_i is mapped by f into V, and $F_i(\partial T_i) \subset W$ whenever this short arc is mapped into W. But V and W are both simply-connected. It follows that each of the functions F_i can be extended continuously over the whole of the sector T_i. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect, and can therefore be pieced together to yield a continuous map defined over the the whole of the closed disk D which extends the map f, as required.

(c) [From printed lecture notes, with U and V replaced by V and W respectively.] The n-dimensional sphere S^n is simply-connected for all $n > 1$, where $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$. Indeed let $V = \{\mathbf{x} \in S^n : x_{n+1} > -\frac{1}{2}\}$ and $W = \{\mathbf{x} \in S^n : x_{n+1} < \frac{1}{2}\}$. Then V and W are homeomorphic to an n-dimensional ball, and are therefore simply-connected. Moreover $V \cap W$ is path-connected, provided that $n > 1$. It follows that S^n is simply-connected for all $n > 1$.

3. [Based on lecture notes.] Let X be a topological space, and let x_0 and x_1 be points of X. A *path* in X from x_0 to x_1 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A *loop* in X based at x_0 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = \gamma(1) = x_0$.

   We can concatenate paths. Let $\gamma_1\colon [0,1] \to X$ and $\gamma_2\colon [0,1] \to X$ be paths in some topological space X. Suppose that $\gamma_1(1) = \gamma_2(0)$. We define the *product path* $\gamma_1.\gamma_2\colon [0,1] \to X$ by

$$(\gamma_1.\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   If $\gamma\colon [0,1] \to X$ is a path in X then we define the *inverse path* $\gamma^{-1}\colon [0,1] \to X$ by $\gamma^{-1}(t) = \gamma(1-t)$.

   Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$. We denote the equivalence class of a loop $\gamma\colon [0,1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$.

   Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0. We show $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0. This group is the *fundamental group* of the topological space X based at x_0.

   First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(t,\tau) = \begin{cases} F_1(2t,\tau) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ F_2(2t-1,\tau) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

   where $F_1\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_1 and γ_1', $F_2\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1.\gamma_2$ to $\gamma_1'.\gamma_2'$, and maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all

$\tau \in [0,1]$. Thus $[\gamma_1.\gamma_2] = [\gamma_1'.\gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

Next we show that the group operation on $\pi_1(X, x_0)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Thus the map $G\colon [0,1] \times [0,1] \to X$ defined by $G(t,\tau) = \alpha((1-\tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1.\gamma_2).\gamma_3 \simeq \gamma_1.(\gamma_2.\gamma_3)$ rel $\{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon.\gamma = \gamma \circ \theta_0$ and $\gamma.\varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \qquad \theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0,1]$. But the continuous map $(t,\tau) \mapsto \gamma((1-\tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon.\gamma \simeq \gamma \simeq \gamma.\varepsilon$ rel $\{0,1\}$, and hence $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Now the map $K\colon [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a homotopy between the loops $\gamma.\gamma^{-1}$ and ε, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\gamma.\gamma^{-1} \simeq \varepsilon \,\mathrm{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma.\gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required.

4. (a) [From printed lecture notes.] Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

   - for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
   - if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

   (b) [From printed lecture notes.] **Sperner's Lemma.** *Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.*

   Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

   We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

   Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2(N(0, 1, \ldots, n-1) - M),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required.

(c) [From printed lecture notes.] Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

5. (a) [From printed lecture notes.]

$$\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q. \rangle$$

We show that $\partial_{q-1} \circ \partial_q = 0$ when $2 \leq q \leq \dim K$.
Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned}
\partial_{q-1}\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle) \\
&= \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\
&= 0
\end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K. ∎

(b) [From printed lecture notes.] Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q : C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1} : C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$. Therefore $B_q(K) \subset Z_q(K)$. But these groups are subgroups of the Abelian group $C_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K. Note that $H_q(K) = 0$ if $q < 0$ or $q > \dim K$ (since $Z_q(K) = 0$ and $B_q(K) = 0$ in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

(c) Let p be this chain. Then

$$\partial_2 p = a(\langle \mathbf{v}_2\, \mathbf{v}_4 \rangle - \langle \mathbf{v}_1\, \mathbf{v}_4 \rangle + \langle \mathbf{v}_1\, \mathbf{v}_2 \rangle)$$

$$
\begin{aligned}
& + b\,(\langle \mathbf{v}_3\, \mathbf{v}_4 \rangle - \langle \mathbf{v}_2\, \mathbf{v}_4 \rangle + \langle \mathbf{v}_2\, \mathbf{v}_3 \rangle) \\
& + c\,(\langle \mathbf{v}_1\, \mathbf{v}_4 \rangle - \langle \mathbf{v}_3\, \mathbf{v}_4 \rangle + \langle \mathbf{v}_3\, \mathbf{v}_1 \rangle) \\
& + d\,(\langle \mathbf{v}_2\, \mathbf{v}_5 \rangle - \langle \mathbf{v}_1\, \mathbf{v}_5 \rangle + \langle \mathbf{v}_1\, \mathbf{v}_2 \rangle) \\
& + e\,(\langle \mathbf{v}_3\, \mathbf{v}_5 \rangle - \langle \mathbf{v}_2\, \mathbf{v}_5 \rangle + \langle \mathbf{v}_2\, \mathbf{v}_3 \rangle) \\
& + f\,(\langle \mathbf{v}_1\, \mathbf{v}_5 \rangle - \langle \mathbf{v}_3\, \mathbf{v}_5 \rangle + \langle \mathbf{v}_3\, \mathbf{v}_1 \rangle) \\
= \; & (a+d)\langle \mathbf{v}_1\, \mathbf{v}_2 \rangle + (b+e)\langle \mathbf{v}_2\, \mathbf{v}_3 \rangle + (c+f)\langle \mathbf{v}_3\, \mathbf{v}_1 \rangle \\
& + (c-a)\langle \mathbf{v}_1\, \mathbf{v}_4 \rangle + (a-b)\langle \mathbf{v}_2\, \mathbf{v}_4 \rangle + (b-c)\langle \mathbf{v}_3\, \mathbf{v}_4 \rangle \\
& + (f-d)\langle \mathbf{v}_1\, \mathbf{v}_5 \rangle + (d-e)\langle \mathbf{v}_2\, \mathbf{v}_5 \rangle + (e-f)\langle \mathbf{v}_3\, \mathbf{v}_5 \rangle
\end{aligned}
$$

(d) Let p be the 2-chain of c. Then $\partial_2 p = 0$ if and only if $a = b = c = -d = -e = -f$. But a, b, c, d, e and f are integers. It follows that $\partial_2 p = 0$ if and only if $p = mz$ for some integer m. Thus $Z_2(K) = \{mz : m \in \mathbb{Z}\}$. There are no 3-chains, and therefore $B_2(K) = 0$. It follows that $H_2(K) = Z_2(K)/B_2(K) \cong Z_2(K) \cong \mathbb{Z}$, as required.

6. (a) [Quoted from lecture notes.] Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

(b) [Quoted from lecture notes.] It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required.

(c) [Quoted from lecture notes.] Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle$$

for some integers $n_1, n_2, \ldots, n_r$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1 \rangle + n_2\langle \mathbf{u}_2 \rangle + \cdots + n_r\langle \mathbf{u}_r \rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z} \rangle)) = \varepsilon(\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m \rangle - \langle \mathbf{v}_0 \rangle = \partial_1 \left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j \rangle \right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path. We deduce that $\langle \mathbf{z} \rangle - \langle \mathbf{y} \rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j \langle \mathbf{u}_j \rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j (\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle)$. But $\langle \mathbf{u}_j \rangle - \langle \mathbf{u}_1 \rangle \in B_0(K)$. It follows that $c \in B_0(K)$. We conclude that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required.

7. (a) [From printed lecture notes.] A sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\mathrm{image}(p: F \to G) = \ker(q: G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

(b) $\phi \circ \psi_1 = \psi_2 \circ \theta$

(c) [Based on printed lecture notes.] First we prove that if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism, Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove that if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism. Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3\,(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism.

It follows that if ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms, then so is ψ_3.

8. (a) [Quoted from from printed lecture notes.] Two simplicial maps $s: K \to L$ and $t: K \to L$ between simplicial complexes K and L are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

(b) [Quoted from from printed lecture notes.] Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required.

(c) [Quoted from from printed lecture notes.] Choose an ordering of the vertices of K. Then there are well-defined homomorphisms $D_q: C_q(K) \to C_{q+1}(L)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle.$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order (with respect to the chosen ordering of the vertices of K). Then

$$\partial_1(D_0(\langle \mathbf{v} \rangle)) = \partial_1(\langle s(\mathbf{v}), t(\mathbf{v}) \rangle) = \langle t(\mathbf{v}) \rangle - \langle s(\mathbf{v}) \rangle,$$

and thus $\partial_1 \circ D_0 = t_0 - s_0$. Also

$$\begin{aligned}
&D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) \\
&\quad = \sum_{i=0}^{q} (-1)^i D_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle) \\
&\quad = \sum_{i=0}^{q} \sum_{j=0}^{i-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{i=0}^{q} \sum_{j=i+1}^{q} (-1)^{i+j-1} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle
\end{aligned}$$

and

$$\partial_{q+1}(D_q(\langle \mathbf{v}_0, \ldots \mathbf{v}_q \rangle))$$

$$
\begin{aligned}
&= \sum_{j=0}^{q}(-1)^j \partial_{q+1}(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle) \\
&= \sum_{j=0}^{q}\sum_{i=0}^{j-1}(-1)^{i+j}\langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\quad + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle + \sum_{j=1}^{q}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q)\rangle \\
&\quad - \sum_{j=0}^{q-1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle \\
&\quad + \sum_{j=0}^{q}\sum_{i=j+1}^{q}(-1)^{i+j+1}\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q)\rangle \\
&= -D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q\rangle)) + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q)\rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q)\rangle
\end{aligned}
$$

and thus

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K)$ to $H_q(L)$, as required.