Course 421: Algebraic Topology
Section 1: Topological Spaces

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Contents

1 Topological Spaces

1.1 Continuity and Topological Spaces

The concept of continuity is fundamental in large parts of contemporary mathematics. In the nineteenth century, precise definitions of continuity were formulated for functions of a real or complex variable, enabling mathematicians to produce rigorous proofs of fundamental theorems of real and complex analysis, such as the Intermediate Value Theorem, Taylor's Theorem, the Fundamental Theorem of Calculus, and Cauchy's Theorem.

In the early years of the Twentieth Century, the concept of continuity was generalized so as to be applicable to functions between metric spaces, and subsequently to functions between topological spaces.

1.2 Topological Spaces

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

1.3 Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y)+d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function* d, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$ defined above.

Definition Let (X, d) be a metric space. Given a point x of X and $r \geq 0$, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined by

$$B_X(x, r) \equiv \{x' \in X : d(x', x) < r\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some $\delta > 0$ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Lemma 1.1 *Let X be a metric space with distance function d, and let x_0 be a point of X. Then, for any $r > 0$, the open ball $B_X(x_0, r)$ of radius r about x_0 is an open set in X.*

Proof Let $x \in B_X(x_0, r)$. We must show that there exists some $\delta > 0$ such that $B_X(x, \delta) \subset B_X(x_0, r)$. Now $d(x, x_0) < r$, and hence $\delta > 0$, where $\delta = r - d(x, x_0)$. Moreover if $x' \in B_X(x, \delta)$ then

$$d(x', x_0) \leq d(x', x) + d(x, x_0) < \delta + d(x, x_0) = r,$$

by the Triangle Inequality, hence $x' \in B_X(x_0, r)$. Thus $B_X(x, \delta) \subset B_X(x_0, r)$, showing that $B_X(x_0, r)$ is an open set, as required. ∎

Proposition 1.2 *Let X be a metric space. The collection of open sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both open sets;*

(ii) *the union of any collection of open sets is itself an open set;*

(iii) *the intersection of any* finite *collection of open sets is itself an open set.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself an open set. Let $x \in U$. Then $x \in V$ for some open set V belonging to the collection $\mathcal{A}$. Therefore there exists some $\delta > 0$ such that $B_X(x, \delta) \subset V$. But $V \subset U$, and thus $B_X(x, \delta) \subset U$. This shows that U is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in X, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $x \in V$. Now $x \in V_j$ for all j, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(x, \delta) \subset B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(x, \delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

Any metric space may be regarded as a topological space. Indeed let X be a metric space with distance function d. We recall that a subset V of X is an *open set* if and only if, given any point v of V, there exists some $\delta > 0$ such that $\{x \in X : d(x, v) < \delta\} \subset V$. Proposition 1.2 shows that the topological space axioms are satisfied by the collection of open sets in any metric space. We refer to this collection of open sets as the *topology* generated by the distance function d on X.

1.4 Further Examples of Topological Spaces

Example Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

1.5 Closed Sets

Definition Let X be a topological space. A subset F of X is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

We recall that the complement of the union of some collection of subsets of some set X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets. The following result therefore follows directly from the definition of a topological space.

Proposition 1.3 *Let X be a topological space. Then the collection of closed sets of X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are closed sets,*

(ii) *the intersection of any collection of closed sets is itself a closed set,*

(iii) *the union of any finite collection of closed sets is itself a closed set.*

1.6 Hausdorff Spaces

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 1.4 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\varepsilon = \frac{1}{2}d(x, y)$. Then the open balls $B_X(x, \varepsilon)$ and $B_X(y, \varepsilon)$ of radius ε centred on the points x and y are open sets (see Lemma 1.1). If $B_X(x, \varepsilon) \cap B_X(y, \varepsilon)$ were non-empty then there would exist $z \in X$ satisfying $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(x, y) < 2\varepsilon$, contrary to the choice of ε. Thus $x \in B_X(x, \varepsilon)$, $y \in B_X(y, \varepsilon)$, $B_X(x, \varepsilon) \cap B_X(y, \varepsilon) = \emptyset$. This shows that the metric space X is a Hausdorff space. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example The *Zariski topology* on the set $\mathbb{R}$ of real numbers is defined as follows: a subset U of $\mathbb{R}$ is open (with respect to the Zariski topology) if and only if either $U = \emptyset$ or else $\mathbb{R} \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set $\mathbb{R}$ of real numbers is a topological space with respect to this Zariski topology. Now the intersection of any two non-empty open sets in this topology is always non-empty. (Indeed if U and V are non-empty open sets then $U = \mathbb{R} \setminus F_1$ and $V = \mathbb{R} \setminus F_2$, where F_1 and F_2 are finite sets of real numbers. But then $U \cap V = \mathbb{R} \setminus (F_1 \cup F_2)$, which is non-empty, since $F_1 \cup F_2$ is finite and $\mathbb{R}$ is infinite.) It follows immediately from this that $\mathbb{R}$, with the Zariski topology, is not a Hausdorff space.

1.7 Subspace Topologies

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A. (It is a straightforward exercise to verify that the topological space axioms are satisfied.) The topology τ_A on A is referred to as the *subspace topology* on A.

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Let X be a metric space with distance function d, and let A be a subset of X. It is not difficult to prove that a subset W of A is open with respect to the subspace topology on A if and only if, given any point w of W, there exists some $\delta > 0$ such that

$$\{a \in A : d(a, w) < \delta\} \subset W.$$

Thus the subspace topology on A coincides with the topology on A obtained on regarding A as a metric space (with respect to the distance function d).

Example Let X be any subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then the subspace topology on X coincides with the topology on X generated by the Euclidean distance function on X. We refer to this topology as the *usual topology* on X.

Let X be a topological space, and let A be a subset of X. One can readily verify the following:—

- a subset B of A is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X;

- if A is itself open in X then a subset B of A is open in A if and only if it is open in X;
- if A is itself closed in X then a subset B of A is closed in A if and only if it is closed in X.

1.8 Continuous Functions between Topological Spaces

Definition A function $f\colon X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 1.5 *Let X, Y and Z be topological spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Then the composition $g \circ f\colon X \to Z$ of the functions f and g is continuous.*

Proof Let V be an open set in Z. Then $g^{-1}(V)$ is open in Y (since g is continuous), and hence $f^{-1}(g^{-1}(V))$ is open in X (since f is continuous). But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Thus the composition function $g \circ f$ is continuous. ∎

Lemma 1.6 *Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is closed in X for every closed subset G of Y.*

Proof If G is any subset of Y then $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ (i.e., the complement of the preimage of G is the preimage of the complement of G). The result therefore follows immediately from the definitions of continuity and closed sets. ∎

1.9 A Criterion for Continuity

We now show that, if a topological space X is the union of a finite collection of closed sets, and if a function from X to some topological space is continuous on each of these closed sets, then that function is continuous on X.

Lemma 1.7 *Let X and Y be topological spaces, let $f\colon X \to Y$ be a function from X to Y, and let $X = A_1 \cup A_2 \cup \cdots \cup A_k$, where $A_1, A_2, \ldots, A_k$ are closed sets in X. Suppose that the restriction of f to the closed set A_i is continuous for $i = 1, 2, \ldots, k$. Then $f\colon X \to Y$ is continuous.*

Proof A function $f: X \to Y$ is continuous if and only if $f^{-1}(G)$ is closed in X for every closed set G in Y (Lemma 1.6). Let G be an closed set in Y. Then $f^{-1}(G) \cap A_i$ is relatively closed in A_i for $i = 1, 2, \ldots, k$, since the restriction of f to A_i is continuous for each i. But A_i is closed in X, and therefore a subset of A_i is relatively closed in A_i if and only if it is closed in X. Therefore $f^{-1}(G) \cap A_i$ is closed in X for $i = 1, 2, \ldots, k$. Now $f^{-1}(G)$ is the union of the sets $f^{-1}(G) \cap A_i$ for $i = 1, 2, \ldots, k$. It follows that $f^{-1}(G)$, being a finite union of closed sets, is itself closed in X. It now follows from Lemma 1.6 that $f: X \to Y$ is continuous. ∎

Example Let Y be a topological space, and let $\alpha: [0, 1] \to Y$ and $\beta: [0, 1] \to Y$ be continuous functions defined on the interval $[0, 1]$, where $\alpha(1) = \beta(0)$. Let $\gamma: [0, 1] \to Y$ be defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now $\gamma|[0, \frac{1}{2}] = \alpha \circ \rho$ where $\rho: [0, \frac{1}{2}] \to [0, 1]$ is the continuous function defined by $\rho(t) = 2t$ for all $t \in [0, \frac{1}{2}]$. Thus $\gamma|[0, \frac{1}{2}]$ is continuous, being a composition of two continuous functions. Similarly $\gamma|[\frac{1}{2}, 1]$ is continuous. The subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0, 1]$, and $[0, 1]$ is the union of these two subintervals. It follows from Lemma 1.7 that $\gamma: [0, 1] \to Y$ is continuous.

1.10 Homeomorphisms

Definition Let X and Y be topological spaces. A function $h: X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: Y \to X$),
- the function $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h: X \to Y$ from X to Y.

If $h: X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

1.11 Product Topologies

The *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$.

The sets $\mathbb{R}^2$ and $\mathbb{R}^3$ are the Cartesian products $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ respectively.

Definition Let $X_1, X_2, \ldots, X_n$ be topological spaces. A subset U of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is said to be *open* (with respect to the product topology) if, given any point p of U, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$.

Lemma 1.8 *Let $X_1, X_2, \ldots, X_n$ be topological spaces. Then the collection of open sets in $X_1 \times X_2 \times \cdots \times X_n$ is a topology on $X_1 \times X_2 \times \cdots \times X_n$.*

Proof Let $X = X_1 \times X_2 \times \cdots \times X_n$. The definition of open sets ensures that the empty set and the whole set X are open in X. We must prove that any union or finite intersection of open sets in X is an open set.

Let E be a union of a collection of open sets in X and let p be a point of E. Then $p \in D$ for some open set D in the collection. It follows from this that there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset D \subset E.$$

Thus E is open in X.

Let $U = U_1 \cap U_2 \cap \cdots \cap U_m$, where $U_1, U_2, \ldots, U_m$ are open sets in X, and let p be a point of U. Then there exist open sets V_{ki} in X_i for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$ for $k = 1, 2, \ldots, m$. Let $V_i = V_{1i} \cap V_{2i} \cap \cdots \cap V_{mi}$ for $i = 1, 2, \ldots, n$. Then

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$$

for $k = 1, 2, \ldots, m$, and hence $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$. It follows that U is open in X, as required. ∎

Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are topological spaces and X is given the product topology, and for each i, let $p_i \colon X \to X_i$ denote the projection function which sends $(x_1, x_2, \ldots, x_n) \in X$ to x_i. It can be shown that a function $f \colon Z \to X$ mapping a topological space Z into X is continuous if and only if $p_i \circ f \colon Z \to X_i$ is continuous for $i = 1, 2, \ldots, n$.

One can also prove that usual topology on $\mathbb{R}^n$ determined by the Euclidean distance function coincides with the product topology on $\mathbb{R}^n$ obtained on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ of n copies of the real line $\mathbb{R}$. (In other words, the collection of open sets in $\mathbb{R}^n$ defined using the Euclidean distance function coincides with the collection of open sets defined in accordance with the definition of the product topology on $\mathbb{R}^n$.) It follows from this that a function mapping a topological space into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous if and only if its components are continuous.

1.12 Identification Maps and Quotient Topologies

Definition Let X and Y be topological spaces and let $q\colon X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q\colon X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

It follows directly from the definition that any identification map is continuous. Moreover, in order to show that a continuous surjection $q\colon X \to Y$ is an identification map, it suffices to prove that if V is a subset of Y with the property that $q^{-1}(V)$ is open in X then V is open in Y.

Lemma 1.9 *Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjection. Then there is a unique topology on Y for which the function $q\colon X \to Y$ is an identification map.*

Proof Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$. If $\{V_\alpha : \alpha \in A\}$ is any collection of subsets of Y indexed by a set A, then it is a straightforward exercise to verify that

$$\bigcup_{\alpha \in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right), \qquad \bigcap_{\alpha \in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcap_{\alpha \in A} V_\alpha\right)$$

(i.e., given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets). It follows easily from this that unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q\colon X \to Y$ is an identification map with respect to the topology τ. Clearly τ is the unique topology on Y for which the function $q\colon X \to Y$ is an identification map. ∎

Let X be a topological space, let Y be a set, and let $q: X \to Y$ be a surjection. The unique topology on Y for which the function q is an identification map is referred to as the *quotient topology* (or *identification topology*) on Y.

Lemma 1.10 *Let X and Y be topological spaces and let $q: X \to Y$ be an identification map. Let Z be a topological space, and let $f: Y \to Z$ be a function from Y to Z. Then the function f is continuous if and only if the composition function $f \circ q: X \to Z$ is continuous.*

Proof Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, and let $q: [0,1] \to S^1$ be the map that sends $t \in [0,1]$ to $(\cos 2\pi t, \sin 2\pi t)$. Then $q: [0,1] \to S^1$ is an identification map, and therefore a function $f: S^1 \to Z$ from S^1 to some topological space Z is continuous if and only if $f \circ q: [0,1] \to Z$ is continuous.

Example Let S^n be the n-sphere, consisting of all points $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$). Let $q: S^n \to \mathbb{R}P^n$ denote the function which sends a point $\mathbf{x}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{x}$ and the origin. Note that each element of $\mathbb{R}P^n$ is the image (under q) of exactly two antipodal points $\mathbf{x}$ and $-\mathbf{x}$ of S^n. The function q induces a corresponding quotient topology on $\mathbb{R}P^n$ such that $q: S^n \to \mathbb{R}P^n$ is an identification map. The set $\mathbb{R}P^n$, with this topology, is referred to as *real projective n-space*. In particular $\mathbb{R}P^2$ is referred to as the *real projective plane*. It follows from Lemma 1.10 that a function $f: \mathbb{R}P^n \to Z$ from $\mathbb{R}P^n$ to any topological space Z is continuous if and only if the composition function $f \circ q: S^n \to Z$ is continuous.

1.13 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 1.11 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 1.12 (Heine-Borel) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{U}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{U}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{U}$, as required. ∎

Lemma 1.13 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 1.11 that A is compact, as required. ∎

Lemma 1.14 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 1.15 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof The range $f(X)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(X)$ is compact (Lemma 1.14) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on X, as required. ∎

Proposition 1.16 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 1.15. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x) - m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 1.15. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

Proposition 1.17 *Let A be a compact subset of a metric space X. Then A is closed in X.*

Proof Let p be a point of X that does not belong to A, and let $f(x) = d(x,p)$, where d is the distance function on X. It follows from Proposition 1.16 that there is a point q of A such that $f(a) \geq f(q)$ for all $a \in A$, since A is compact. Now $f(q) > 0$, since $q \neq p$. Let δ satisfy $0 < \delta \leq f(q)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f(x) < f(q)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X. ∎

Proposition 1.18 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 1.19 *A compact subset of a Hausdorff topological space is closed.*

Proof Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 1.18 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Proposition 1.20 *Let X be a Hausdorff topological space, and let K_1 and K_2 be compact subsets of X, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.*

Proof It follows from Proposition 1.18 that, for each point x of K_1, there exist open sets V_x and W_x such that $x \in V_x$, $K_2 \subset W_x$ and $V_x \cap W_x = \emptyset$. But then there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of K_1 such that

$$K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r},$$

since K_1 is compact. Define

$$U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \qquad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}.$$

Then U_1 and U_2 are open sets, $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$, as required. ∎

Lemma 1.21 *Let $f: X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 1.13), and therefore $f(K)$ is compact (Lemma 1.14). But any compact subset of a Hausdorff space is closed (Corollary 1.19). Thus $f(K)$ is closed in Y, as required. ∎

Remark If the Hausdorff space Y in Lemma 1.21 is a metric space, then Proposition 1.17 may be used in place of Corollary 1.19 in the proof of the lemma.

Theorem 1.22 *A continuous bijection $f: X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.*

Proof Let $g: Y \to X$ be the inverse of the bijection $f: X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y, by Lemma 1.21. But $f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$. It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g: Y \to X$ is continuous, and thus $f: X \to Y$ is a homeomorphism. ∎

We recall that a function $f: X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 1.23 *A continuous surjection $f: X \to Y$ from a compact topological space X to a Hausdorff space Y is an identification map.*

Proof Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f: X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 1.21. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(U)$ is open in X, since $f: X \to Y$ is continuous. Thus the surjection $f: X \to Y$ is an identification map. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $q\colon [0,1] \to S^1$ be defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0,1]$. It has been shown that the map q is an identification map. This also follows directly from the fact that $q\colon [0,1] \to S^1$ is a continuous surjection from the compact space $[0,1]$ to the Hausdorff space S^1.

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the *Tube Lemma*.

Lemma 1.24 *Let X and Y be topological spaces, let K be a compact subset of Y, and U be an open set in $X \times Y$. Let $V = \{x \in X : \{x\} \times K \subset U\}$. Then V is an open set in X.*

Proof Let $x \in V$. For each $y \in K$ there exist open subsets D_y and E_y of X and Y respectively such that $(x,y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that $K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}$, since K is compact. Set $N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}$. Then N_x is an open set in X. Moreover

$$N_x \times K \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required. ∎

Theorem 1.25 *A Cartesian product of a finite number of compact spaces is itself compact.*

Proof It suffices to prove that the product of two compact topological spaces X and Y is compact, since the general result then follows easily by induction on the number of compact spaces in the product.

Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space Y under the continuous map from Y to $X \times Y$ which sends $y \in Y$ to (x,y), and the image of any compact set under a continuous map is itself compact (Lemma 1.14). Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that $\{x\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Let V_x denote the set of all points x' of X for which $\{x'\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Then $x \in V_x$, and Lemma 1.24 ensures That V_x is an open set in X. Note that

$V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the space X. It follows from the compactness of X that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that $X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}$. Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $\mathcal{U}$. On combining these finite collections, we obtain a finite collection of open sets belonging to $\mathcal{U}$ which covers $X \times Y$. This shows that $X \times Y$ is compact. ∎

Theorem 1.26 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 1.19). For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 1.12), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Theorem 1.25 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 1.13. Thus K is compact, as required. ∎

1.14 The Lebesgue Lemma and Uniform Continuity

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Lemma 1.27 (Lebesgue Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that* *every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f: X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 1.28 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f: X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 1.27) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f: X \to Y$ is uniformly continuous, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 1.26) and Theorem 1.28 that any continuous function $f: K \to \mathbb{R}^k$ is uniformly continuous.

1.15 Connected Topological Spaces

Definition A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Lemma 1.29 *A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $X = U \cup V$, then $U \cap V$ is non-empty.*

Proof If U is a subset of X that is both open and closed, and if $V = X \setminus U$, then U and V are both open, $U \cup V = X$ and $U \cap V = \emptyset$. Conversely if U and V are open subsets of X satisfying $U \cup V = X$ and $U \cap V = \emptyset$, then $U = X \setminus V$, and hence U is both open and closed. Thus a topological space X is connected if and only if there do not exist non-empty open sets U and V such that $U \cup V = X$ and $U \cap V = \emptyset$. The result follows. ∎

Let $\mathbb{Z}$ be the set of integers with the usual topology (i.e., the subspace topology on $\mathbb{Z}$ induced by the usual topology on $\mathbb{R}$). Then $\{n\}$ is open for all $n \in \mathbb{Z}$, since

$$\{n\} = \mathbb{Z} \cap \{t \in \mathbb{R} : |t - n| < \tfrac{1}{2}\}.$$

It follows that every subset of $\mathbb{Z}$ is open (since it is a union of sets consisting of a single element, and any union of open sets is open). It follows that a function $f: X \to \mathbb{Z}$ on a topological space X is continuous if and only if $f^{-1}(V)$ is open in X for any subset V of $\mathbb{Z}$. We use this fact in the proof of the next theorem.

Proposition 1.30 *A topological space X is connected if and only if every continuous function $f: X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.*

Proof Suppose that X is connected. Let $f: X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let

$$U = \{x \in X : f(x) = n\}, \qquad V = \{x \in X : f(x) \neq n\}.$$

Then U and V are the preimages of the open subsets $\{n\}$ and $\mathbb{Z} \setminus \{n\}$ of $\mathbb{Z}$, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f: X \to \mathbb{Z}$ is constant, with value n.

Conversely suppose that every continuous function $f: X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f: X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. But then the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected. ∎

Lemma 1.31 *The closed interval $[a, b]$ is connected, for all real numbers a and b satisfying $a \leq b$.*

Proof Let $f: [a, b] \to \mathbb{Z}$ be a continuous integer-valued function on $[a, b]$. We show that f is constant on $[a, b]$. Indeed suppose that f were not constant. Then $f(\tau) \neq f(a)$ for some $\tau \in [a, b]$. But the Intermediate Value Theorem would then ensure that, given any real number c between $f(a)$ and $f(\tau)$, there would exist some $t \in [a, \tau]$ for which $f(t) = c$, and this is clearly impossible, since f is integer-valued. Thus f must be constant on $[a, b]$. We now deduce from Proposition 1.30 that $[a, b]$ is connected. ∎

Example Let $X = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed if $f: X \to \mathbb{Z}$ is defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

then f is continuous on X but is not constant.

A concept closely related to that of connectedness is *path-connectedness.* Let x_0 and x_1 be points in a topological space X. A *path* in X from x_0 to x_1 is defined to be a continuous function $\gamma: [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a path in X from x_0 to x_1.

Proposition 1.32 *Every path-connected topological space is connected.*

Proof Let X be a path-connected topological space, and let $f: X \to \mathbb{Z}$ be a continuous integer-valued function on X. If x_0 and x_1 are any two points of X then there exists a path $\gamma: [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. But then $f \circ \gamma: [0, 1] \to \mathbb{Z}$ is a continuous integer-valued function on $[0, 1]$. But $[0, 1]$ is connected (Lemma 1.31), therefore $f \circ \gamma$ is constant (Proposition 1.30). It follows that $f(x_0) = f(x_1)$. Thus every continuous integer-valued function on X is constant. Therefore X is connected, by Proposition 1.30. ∎

The topological spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are all path-connected. Indeed, given any two points of one of these spaces, the straight line segment joining these two points is a continuous path from one point to the other. Also the n-sphere S^n is path-connected for all $n > 0$. We conclude that these topological spaces are connected.

Let A be a subset of a topological space X. Using Lemma 1.29 and the definition of the subspace topology, we see that A is connected if and only if the following condition is satisfied:

- if U and V are open sets in X such that $A \cap U$ and $A \cap V$ are non-empty and $A \subset U \cup V$ then $A \cap U \cap V$ is also non-empty.

Lemma 1.33 *Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.*

Proof It follows from the definition of the closure of A that $\overline{A} \subset F$ for any closed subset F of X for which $A \subset F$. On taking F to be the complement of some open set U, we deduce that $\overline{A} \cap U = \emptyset$ for any open set U for which

$A \cap U = \emptyset$. Thus if U is an open set in X and if $\overline{A} \cap U$ is non-empty then $A \cap U$ must also be non-empty.

Now let U and V be open sets in X such that $\overline{A} \cap U$ and $\overline{A} \cap V$ are non-empty and $\overline{A} \subset U \cup V$. Then $A \cap U$ and $A \cap V$ are non-empty, and $A \subset U \cup V$. But A is connected. Therefore $A \cap U \cap V$ is non-empty, and thus $\overline{A} \cap U \cap V$ is non-empty. This shows that $\overline{A}$ is connected. ∎

Lemma 1.34 *Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then $f(A)$ is connected.*

Proof Let $g\colon f(A) \to \mathbb{Z}$ be any continuous integer-valued function on $f(A)$. Then $g \circ f\colon A \to \mathbb{Z}$ is a continuous integer-valued function on A. It follows from Proposition 1.30 that $g \circ f$ is constant on A. Therefore g is constant on $f(A)$. We deduce from Proposition 1.30 that $f(A)$ is connected. ∎

Lemma 1.35 *The Cartesian product $X \times Y$ of connected topological spaces X and Y is itself connected.*

Proof Let $f\colon X \times Y \to \mathbb{Z}$ be a continuous integer-valued function from $X \times Y$ to Z. Choose $x_0 \in X$ and $y_0 \in Y$. The function $x \mapsto f(x, y_0)$ is continuous on X, and is thus constant. Therefore $f(x, y_0) = f(x_0, y_0)$ for all $x \in X$. Now fix x. The function $y \mapsto f(x, y)$ is continuous on Y, and is thus constant. Therefore

$$f(x, y) = f(x, y_0) = f(x_0, y_0)$$

for all $x \in X$ and $y \in Y$. We deduce from Proposition 1.30 that $X \times Y$ is connected. ∎

We deduce immediately that a finite Cartesian product of connected topological spaces is connected.

Proposition 1.36 *Let X be a topological space. For each $x \in X$, let S_x be the union of all connected subsets of X that contain x. Then*

(i) *S_x is connected,*

(ii) *S_x is closed,*

(iii) *if $x, y \in X$, then either $S_x = S_y$, or else $S_x \cap S_y = \emptyset$.*

Proof Let $f\colon S_x \to \mathbb{Z}$ be a continuous integer-valued function on S_x, for some $x \in X$. Let y be any point of S_x. Then, by definition of S_x, there exists some connected set A containing both x and y. But then f is constant on A, and thus $f(x) = f(y)$. This shows that the function f is constant on S_x. We deduce that S_x is connected. This proves (i). Moreover the closure $\overline{S_x}$ is connected, by Lemma 1.33. Therefore $\overline{S_x} \subset S_x$. This shows that S_x is closed, proving (ii).

Finally, suppose that x and y are points of X for which $S_x \cap S_y \neq \emptyset$. Let $f\colon S_x \cup S_y \to \mathbb{Z}$ be any continuous integer-valued function on $S_x \cup S_y$. Then f is constant on both S_x and S_y. Moreover the value of f on S_x must agree with that on S_y, since $S_x \cap S_y$ is non-empty. We deduce that f is constant on $S_x \cup S_y$. Thus $S_x \cup S_y$ is a connected set containing both x and y, and thus $S_x \cup S_y \subset S_x$ and $S_x \cup S_y \subset S_y$, by definition of S_x and S_y. We conclude that $S_x = S_y$. This proves (iii). ∎

Given any topological space X, the connected subsets S_x of X defined as in the statement of Proposition 1.36 are referred to as the *connected components* of X. We see from Proposition 1.36, part (iii) that the topological space X is the disjoint union of its connected components.

Example The connected components of $\{(x,y) \in \mathbb{R}^2 : x \neq 0\}$ are

$$\{(x,y) \in \mathbb{R}^2 : x > 0\} \text{ and } \{(x,y) \in \mathbb{R}^2 : x < 0\}.$$

Example The connected components of

$$\{t \in \mathbb{R} : |t - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

are the sets J_n for all $n \in \mathbb{Z}$, where $J_n = (n - \frac{1}{2}, n + \frac{1}{2})$.