Course 421: Academic Year 1998-9
Part IV: The Topological Classification of Closed Surfaces

D. R. Wilkins

Contents

13 The Classification of Surfaces

13.1 Triangulated Closed Surfaces

We define a *triangulated closed surface* K in $\mathbb{R}^N$ to be a 2-dimensional simplicial complex (consisting of triangles, edges and vertices) which satisfies the following additional properties:—

- every point of $|K|$ belongs to at least one triangle (2-simplex) of K,
- every edge of K is an edge of exactly two triangles of K,
- if $\mathbf{v}$ is any vertex of K then the vertices $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ of K adjacent to $\mathbf{v}$ may be ordered so that $\mathbf{v}$, $\mathbf{w}_{i-1}$ and $\mathbf{w}_i$ span a triangle of K for $i = 2, 3, \ldots, m$ and $\mathbf{v}$, $\mathbf{w}_m$ and $\mathbf{w}_1$ also span a triangle of K (see Figure 5).

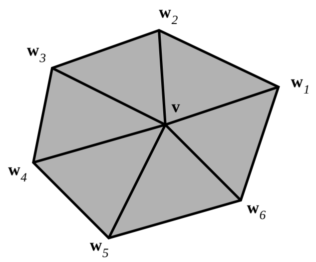

Figure 5

On considering separately points in the interiors of triangles, line segments and vertices, one can verify that every point of the polyhedron $|K|$ of a triangulated closed surface K is contained in an open set homeomorphic to the open unit disk.

A *topological closed surface* is a compact Hausdorff space X with the property that every element of X is contained in an open set homeomorphic to the unit disk. A deep theorem states that any topological closed surface is homeomorphic to the polyhedron of some triangulated closed surface.

Lemma 13.1 *Let K be a triangulated closed surface in $\mathbb{R}^N$. Then the polyhedron $|K|$ of K is a connected topological space if and only if given any two triangles σ and τ of K we can find a sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ of triangles of K with $\sigma = \sigma_1$ and $\tau = \sigma_k$, where σ_{i-1} and σ_i intersect in a common edge for $i = 2, 3, \ldots, k$.*

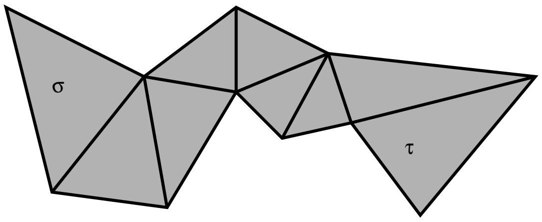

Figure 6

Proof Using the definition of a triangulated closed surface, one can readily verify that any two triangles of K that intersect in a common edge or vertex can be joined by such a sequence of triangles in K, where successive triangles in the sequence intersect in a common edge. It follows easily from this that any two triangles of K can be joined by such a sequence of triangles in K if and only if any two vertices of K can be joined by an edge path. But any two vertices of K can be joined by an edge path if and only if $|K|$ is connected. ∎

Lemma 13.2 *Let K be a triangulated closed surface whose polyhedron $|K|$ is a connected topological space. Then $|K|$ is homeomorphic to the topological space obtained from a filled polygon with an even number of edges by identifying edges in pairs (i.e., given any edge with endpoints $\mathbf{a}$ and $\mathbf{b}$, there exists exactly one other edge with endpoints $\mathbf{c}$ and $\mathbf{d}$ such that $(1-t)\mathbf{a}+t\mathbf{b}$ is identified with $(1-t)\mathbf{c}+t\mathbf{d}$ for all $t \in [0,1]$).*

Proof Suppose that we have constructed some subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon P_L by identifying some of the edges of that polygon in pairs. Let $q_L\colon P_L \to L$ denote the identification map. Suppose that e is an edge of P_L that is not identified to any other edge of L. Then e corresponds under the identification map to some edge e' of P_L. Moreover only one of the two triangles in K adjoining the edge e' belongs to L. Thus there is some triangle σ of $K \setminus L$ which has e' as one of its edges. Let M be the subcomplex of K obtained on adjoining to L the triangle σ, together with all its edges and vertices.

We now extend the polygon P_L by attaching a triangle T along the free edge e to obtain a filled polygon P_M, where $P_M = P_L \cup T$ and $P_L \cap T = e$. We also extend the identification map $q_L\colon P_L \to |L|$ over this attached triangle to obtain an identification map $q_M\colon P_M \to |M|$, where $q_M|P_L = q_L$ and $q_M|T$ is a simplicial homeomorphism mapping the triangle T onto σ. Then the new identification map $q_M\colon P_M \to M$ also identifies some of the edges of the polygon P_M in pairs. Thus, by successively adding triangles in this fashion, we eventually obtain a subcomplex L of K whose polyhedron is homeomorphic to the identification space obtained from a filled polygon on identifying all of the edges of that polygon in pairs. But then, given any two triangles of K that intersect along a common edge, either both triangles belong to L, or else neither triangle belongs to L. It now follows from Lemma 13.1 that $L = K$, and thus the polyhedron of K is an identification space of the prescribed type. ∎

13.2 The Topological Classification of Closed Surfaces

We wish to classify up to homeomorphism the identification spaces obtained from polygons by identifying edges in pairs.

Suppose that we are given a polygon with its edges identified in pairs. Choose an orientation (i.e., a 'direction') on each edge of the polygon in such a way that, for each pair of identified edges, the orientations on those edges correspond under the identification of these edges. Denote each pair of identified edges by some letter $a, b, c, \ldots$. Suppose that we travel round the boundary of the polygon in the anticlockwise direction, starting at some chosen vertex. We obtain a *surface symbol* consisting of a sequence of symbols taken from $a, a^{-1}, b, b^{-1}, c, c^{-1}$, ordered so as to represent the order in which the corresponding edges of the polygon are traversed (on travelling round the polygon in the anticlockwise direction), and where an edge represented by some letter x occurs in the surface symbol as 'x' if the chosen orientation on the edge agrees with the anticlockise orientation, or as 'x^{-1}' if the chosen orientation on the edge is opposite to the anticlockwise orientation. For example, the surface symbol $xyx^{-1}y^{-1}$ represents the torus (see Figure 7), and the surface symbol $xy^{-1}xy^{-1}$ represents the real projective plane (see Figure 8).

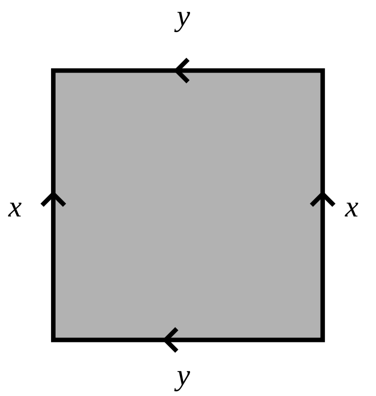

Figure 7: Torus

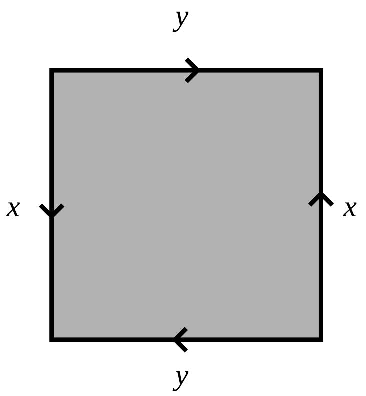

Figure 8: Real projective plane

Lemma 13.2 shows that any connected triangulated closed surface can be described by such a *surface symbol.* This is a finite sequence of symbols of the form $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots,$ $a, b, c, \ldots$ representing some suitable list of 'letters' that label the pairs of identified edges of the polygon representing the surface. Each 'letter' x present occurs exactly twice in the surface symbol, either as 'x' or as 'x^{-1}'. Conversely any surface symbol of this form determines a scheme for identifying in pairs the edges of a suitable polygon to obtain a closed surface. We wish to determine necessary and sufficient conditions for determining whether or not the surfaces obtained in this way from two such surface symbols are homeomorphic.

Let us use capital letters $A, B, C, \ldots$ to denote (possibly empty) sequences of symbols taken from the list $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots$. Also if A is such a sequence of symbols, given by $A = a_1 a_2 \cdots a_n$, then we write A^{-1} for the sequence $a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$ (where $(x^{-1})^{-1} \equiv x$ for any letter x.) Using these conventions, we now state three rules which enable one to transform one surface symbol into another in such a way that the two surface symbols represent surfaces that are homeomorphic.

- **Rule 1.** cyclically permute the symbols occurring in the surface symbol,
- **Rule 2.** Replace $ABxCDxE$ by $AyDB^{-1}yC^{-1}E$ (where y represents some letter not occurring in the first surface symbol).
- **Rule 3.** Replace $ABxCDx^{-1}E$ by $AyDCy^{-1}BE$.
- **Rule 4.** Replace $Axx^{-1}B$ or $Ax^{-1}xB$ by AB, provided that AB contains at least two letters (each occurring twice).

Lemma 13.3 *The application of Rules 1–4 to a surface symbol gives a new surface symbol such that the surfaces determined by the two surface symbols are homeomorphic.*

Proof Rule 1 corresponds to traversing the boundary of the polygon starting from a different vertex. Rules 2–4 are justified by the simple 'cut and paste' operations depicted in Figures 9, 10 and 11. ∎

Rules 1–4 allow the reduction of surface symbols to certain standard forms. Now each letter x in a surface symbol occurs exactly twice; if the letter x either occurs both times as 'x' or else occurs both times as 'x^{-1}', then we call the occurrence of the letter x in the surface symbol a *similar pair*;

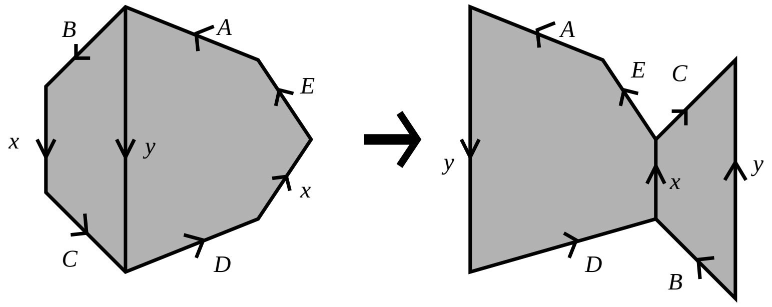

Figure 9: Rule 2

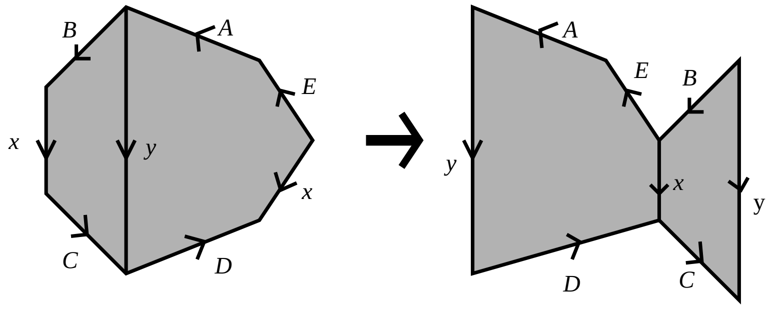

Figure 10: Rule 3

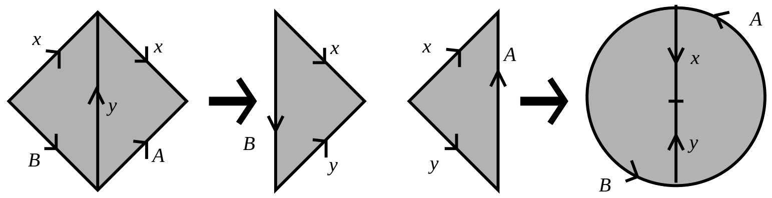

Figure 11: Rule 4

otherwise we call the occurrence of this letter a *reversed pair*. Two reversed pairs in some given surface symbol are said to *interlock* if they occur in the order $\cdots y \cdots z \cdots y^{-1} \cdots z^{-1} \cdots$ (after interchanging y and y^{-1}, or z and z^{-1}, if necessary).

Proposition 13.4 *Any surface symbol can be reduced, by suitable applications of the transformations described in Rules 1–4 above and their inverses, to one of the following canonical forms:—*

$$x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1} \qquad (g \geq 1),$$

$$x_1 x_1 x_2 x_2 \cdots x_h x_h \qquad (h \geq 2),$$

$$x x^{-1} y y^{-1}, \qquad x x y y^{-1}.$$

Proof First we note that if C is any sequence of the form $x_1 x_1 x_2 x_2 \ldots$, then a sequence of transformations

$$CDxExF \to CyD^{-1}yE^{-1}F \to CzzDE^{-1}F$$

of the type specified by Rule 2 will transform any surface symbol $CDxExF$ with a similar pair to one of the form $CzzDE^{-1}$. Repeated applications of this procedure reduce any surface symbol to one of the form AB, where A is of the form form $x_1 x_1 x_2 x_2 \cdots x_r x_r$ and B contains only reversed pairs (where either A or B may be empty).

One can now use Rule 3 in order to reduce a surface symbol of the form AB to one of the form ACD, where C is of the form

$$y_1 z_1 y_1^{-1} z_1^{-1} \cdots y_s z_s y_s^{-1} z_s^{-1},$$

and D contains only non-interlocking reversed pairs. Indeed if E is any surface symbol of the required form, then successive applications of Rule 3 show that any surface symbol of the form $EFaGbHa^{-1}Ib^{-1}J$ can be transformed to $Eefe^{-1}f^{-1}FIHGJ$ by the following sequence of transformations:

$$\begin{array}{rcl} EFaGbHa^{-1}Ib^{-1}J & \to & EcGbHc^{-1}FIb^{-1}J \\ & \to & EcGdFIHc^{-1}d^{-1}J \\ & \to & EeFIHGde^{-1}d^{-1}J \\ & \to & Eefe^{-1}f^{-1}FIHGJ. \end{array}$$

The stated reduction of AB to ACD now follows by induction on the number of interlocking reversed pairs.

If A is non-empty then one can reduce a surface symbol of the form ACD (where A, C and D are as above) to one of the form ED, where E is of the form $x_1x_1x_2x_2\cdots$ and D contains only non-interlocking reversed pairs. This follows from successive applications of the following sequence of transformations (which are inverses of transformations of the type specified by Rule 2):

$$Fxxaba^{-1}b^{-1}G \to Fyb^{-1}a^{-1}ya^{-1}b^{-1}G \to Fyay^{-1}accG \leftarrow FyyddccG.$$

Now consider D, which consists only of non-interlocking reversed pairs. Let $\cdots x \cdots x^{-1} \cdots$ be the closest reversed pair occurring in D. Then x and x^{-1} must be adjacent (since otherwise D would contain two interlocking reversed pairs). We can therefore 'cancel' xx^{-1}, by Rule 4, provided that the resultant symbol always contains at least two letters. It follows that any surface symbol with more than two letters can be reduced to one or other of the first two canonical forms specified (with $g \geq 1$ or $h \geq 2$).

Finally consider surface symbols with two letters occurring. The procedures described above reduce such a surface symbol to one of the following forms: $xyx^{-1}y^{-1}$, $xxyy$, $xxyy^{-1}$, $xx^{-1}yy^{-1}$, $xyy^{-1}x^{-1}$. The first four of these are included in the list of canonical forms. The symbol $xyy^{-1}x^{-1}$ reduces to $zz^{-1}yy^{-1}$ on cyclically permuting the symbol (according to Rule 1) and replacing x^{-1} and x by z and z^{-1} respectively, as required. ∎

Let M_g $(g \geq 1)$ be the space obtained from a regular $4g$-sided polygon by identifying the edges according to the sequence $x_1y_1x_1^{-1}y_1^{-1}\cdots x_gy_gx_g^{-1}y_g^{-1}$ (see Figure 12), and let N_h $(h \geq 2)$ be defined similarly using $x_1x_1\cdots x_hx_h$ (see Figure 13). Also let M_0 and N_1 be surfaces whose surface symbols are $xx^{-1}yy^{-1}$ and $xxyy^{-1}$ respectively. We have so far proved that the polyhedron of any connected triangulated closed surface is homeomorphic to one of the spaces M_g $(g \geq 0)$ or N_h $(h \geq 1)$.

Now the homology groups of any simplical complex are topological invariants of the polyhedron of that complex. Calculations using the Mayer-Vietoris sequence show that

$$H_0(M_g) \cong \mathbb{Z}, \qquad H_1(M_g) \cong \mathbb{Z}^{2g}, \qquad H_2(M_g) \cong \mathbb{Z} (g \geq 0),$$

$$H_0(N_h) \cong \mathbb{Z}, \qquad H_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}_2, \qquad H_2(N_h) = 0 (h \geq 1).$$

It follows that no two of the surfaces M_g and N_h are homeomorphic to each other. We thus obtain the following topological classification theorem for triangulated closed surfaces.

Theorem 13.5 *Every triangulated closed surface is homeomorphic to exactly one of the following surfaces:—*

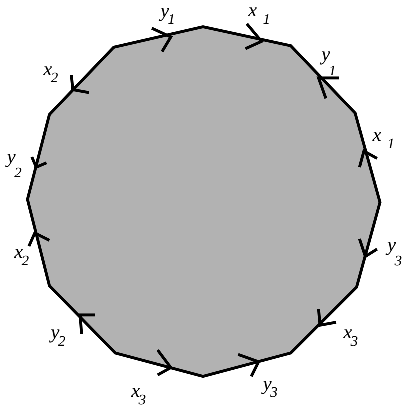

Figure 12: The surface M_g $(g = 3)$

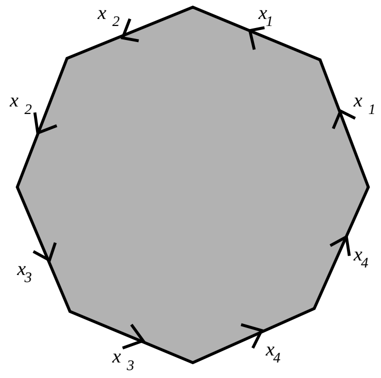

Figure 13: The surface N_h $(h = 4)$

- *The surface M_g, where the integer g satisfies $g \geq 0$,*
- *The surface N_h, where the integer h satisfies $h \geq 1$.*

The surface M_0 is homeomorphic to the standard 2-sphere. The surface M_g for $g > 0$ can be represented as a g-holed torus. The non-negative integer g is referred to as the *genus* of the surface.

The surface N_h can be constructed by taking a standard 2-sphere, removing h disjoint open disks from this sphere and then gluing a Möbius strip onto the boundary of each removed disk by identifying points on the boundary of the removed disk with corresponding points on the boundary of the Möbius strip. It is impossible to represent the surface N_h as an embedded surface in 3-dimensional Euclidean space $\mathbb{R}^3$ without self-intersections. The surface N_1 represents the real projective plane, and the surface N_2 represents the Klein bottle.