Course 421: Academic Year 1998-9
Part III: Simplicial Homology Theory

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Contents

9 Simplicial Complexes

9.1 Geometrical Independence

Definition Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *geometrically independent* (or *affine independent*) if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^q \lambda_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^q \lambda_j & = & 0 \end{array} \right.$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

It is straightforward to verify that $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent if and only if the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent. It follows from this that any set of geometrically independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Note also that if a set consists of geometrically independent points in $\mathbb{R}^k$, then so does every subset of that set.

Definition A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^q t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^q t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex.

Note that a 0-simplex in $\mathbb{R}^k$ is a single point of $\mathbb{R}^k$, a 1-simplex in $\mathbb{R}^k$ is a line segment in $\mathbb{R}^k$, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^q t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^q t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum\limits_{j=0}^q s_j \mathbf{v}_j = \sum\limits_{j=0}^q t_j \mathbf{v}_j$ and $\sum\limits_{j=0}^q s_j = 1 = \sum\limits_{j=0}^q t_j$, then $\sum\limits_{j=0}^q (t_j - s_j)\mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^q (t_j - s_j) = 0$, hence $t_j - s_j = 0$ for all j, since $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent. We refer to $t_0, t_1, \ldots, t_q$ as the *barycentric coordinates* of the point $\mathbf{x}$ of σ.

Lemma 9.1 *Let q be a non-negative integer, let σ be a q-simplex in $\mathbb{R}^m$, and let τ be a q-simplex in $\mathbb{R}^n$, where $m \geq q$ and $n \geq q$. Then σ and τ are homeomorphic.*

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of τ. The required homeomorphism $h\colon \sigma \to \tau$ is given by

$$h\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \mathbf{w}_j$$

for all $t_0, t_1, \ldots, t_q$ satisfying $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum\limits_{j=0}^{q} t_j = 1$. ∎

A homeomorphism between two q-simplices defined as in the above proof is referred to as a *simplicial homeomorphism*.

9.2 Simplicial Complexes in Euclidean Spaces

Definition Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

Definition A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

Example Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 9.2 *Let K be a simplicial complex, and let X be a topological space. A function $f: |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Proof If a topological space can be expressed as a finite union of closed subsets, then a function is continuous on the whole space if and only if its restriction to each of the closed subsets is continuous on that closed set. The required result is a direct application of this general principle. ∎

We shall denote by $\mathrm{Vert}\, K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Definition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-simplex σ in some Euclidean space $\mathbb{R}^k$. We define the *interior* of the simplex σ to be the set of all points of σ that are of the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_j > 0$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. One can readily verify that the interior of the simplex σ consists of all points of σ that do not belong to any proper face of σ. (Note that, if $\sigma \in \mathbb{R}^k$, then the interior of a simplex defined in this fashion will not coincide with the topological interior of σ unless $\dim \sigma = k$.)

Note that any point of a simplex σ belongs to the interior of a unique face of σ. Indeed let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $\mathbf{x} \in \sigma$. Then $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$.

Lemma 9.3 *Let K be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—*

- *K contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of K.*

Proof Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x}$ belongs to the interior of a face σ of some simplex of K (since every point of a simplex belongs to the interior of some face). But then $\sigma \in K$, since K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that $\mathbf{x}$ were to belong to the interior of two distinct simplices σ and τ of K. Then $\mathbf{x}$ would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that $\mathbf{x}$ belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing $\mathbf{x}$ in its interior is uniquely determined, as required.

Conversely, we must show that any collection of simplices satisfying the given conditions is a simplicial complex. Since K contains the faces of all its simplices, it only remains to verify that if σ and τ are any two simplices of K with non-empty intersection then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω of K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. We deduce that the simplices σ and τ have vertices in common, and that every point of $\sigma \cap \tau$ belongs to the common face ρ of σ and τ spanned by these common vertices. But this implies that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of both σ and τ, as required. ∎

Definition A *triangulation* (K, h) of a topological space X consists of a simplicial complex K in some Euclidean space, together with a homeomorphism $h\colon |K| \to X$ mapping the polyhedron $|K|$ of K onto X.

The polyhedron of a simplicial complex is a compact Hausdorff space. Thus if a topological space admits a triangulation then it must itself be a compact Hausdorff space.

Lemma 9.4 *Let X be a Hausdorff topological space, let K be a simplicial complex, and let $h\colon |K| \to X$ be a bijection mapping $|K|$ onto X. Suppose that the restriction of h to each simplex of K is continuous on that simplex. Then the map $h\colon |K| \to X$ is a homeomorphism, and thus (K, h) is a triangulation of X.*

Proof Each simplex of K is a closed subset of $|K|$, and the number of simplices of K is finite. It follows from Lemma 9.2 that $h\colon |K| \to X$ is continuous. Also the polyhedron $|K|$ of K is a compact topological space. But every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Thus (K, h) is a triangulation of X. ∎

9.3 Simplicial Maps

Definition A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes K and L is a function $\varphi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

A simplicial map $\varphi\colon K \to L$ also induces in a natural fashion a continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. The continuity of this map follows immediately from a straightforward application of Lemma 9.2. Note that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

9.4 Barycentric Subdivision of a Simplicial Complex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of σ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

Let σ and τ be simplices in some Euclidean space. If σ is a proper face of τ then we denote this fact by writing $\sigma < \tau$.

A simplicial complex K_1 is said to be a *subdivision* of a simplicial complex K if $|K_1| = |K|$ and each simplex of K_1 is contained in a simplex of K.

Definition Let K be a simplicial complex in some Euclidean space $\mathbb{R}^k$. The *first barycentric subdivision* K' of K is defined to be the collection of simplices in $\mathbb{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of K with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of K' is the set of all the barycentres of all the simplices of K.

Note that every simplex of K' is contained in a simplex of K. Indeed if $\sigma_0, \sigma_1, \ldots, \sigma_r \in K$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ then the simplex of K' spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$, is contained in the simplex σ_r of K.

Proposition 9.5 *Let K be a simplicial complex in some Euclidean space, and let K' be the first barycentric subdivision of K. Then K' is itself a simplicial complex, and $|K'| = |K|$.*

Proof We prove the result by induction on the number of simplices in K. The result is clear when K consists of a single simplex, since that simplex must then be a point and therefore $K' = K$. We prove the result for a simplicial complex K, assuming that it holds for all complexes with fewer simplices.

It is clear from the definition of the barycentric subdivision K' that any face of a simplex of K' must itself belong to K'. We must verify that any two simplices of K' are disjoint or else intersect in a common face.

Choose a simplex σ of K for which $\dim \sigma = \dim K$, and let $L = K \setminus \{\sigma\}$. Then L is a subcomplex of K, since σ is not a proper face of any simplex of K. Now L has fewer simplices than K. It follows from the induction hypothesis that L' is a simplicial complex and $|L'| = |L|$. Also it follows from the definition of K' that K' consists of the following simplices:—

- the simplices of L',
- the barycentre $\hat{\sigma}$ of σ,
- simplices $\hat{\sigma}\rho$ whose vertex set is obtained by adjoining $\hat{\sigma}$ to the vertex set of some simplex ρ of L', where the vertices of ρ are barycentres of proper faces of σ.

By checking all possible intersections of simplices of the above types, it is easy to verify that any two simplices of K' intersect in a common face. Indeed any two simplices of L' intersect in a common face, since L' is a simplicial complex. If ρ_1 and ρ_2 are simplices of L' whose vertices are barycentres of proper faces of σ, then $\rho_1 \cap \rho_2$ is a common face of ρ_1 and ρ_2 which is of this type, and $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2 = \hat{\sigma}(\rho_1 \cap \rho_2)$. Thus $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2$ is a common face of $\hat{\sigma}\rho_1$ and $\hat{\sigma}\rho_2$. Also any simplex τ of L' is disjoint from the barycentre $\hat{\sigma}$ of σ, and $\hat{\sigma}\rho \cap \tau = \rho \cap \tau$. We conclude that K' is indeed a simplicial complex.

It remains to verify that $|K'| = |K|$. Now $|K'| \subset |K|$, since every simplex of K' is contained in a simplex of K. Let $\mathbf{x}$ be a point of the chosen simplex σ. Then there exists a point $\mathbf{y}$ belonging to a proper face of σ and some $t \in [0,1]$ such that $\mathbf{x} = (1-t)\hat{\sigma} + t\mathbf{y}$. But then $\mathbf{y} \in |L|$, and $|L| = |L'|$ by the induction hypothesis. It follows that $\mathbf{y} \in \rho$ for some simplex ρ of L' whose vertices are barycentres of proper faces of σ. But then $\mathbf{x} \in \hat{\sigma}\rho$, and therefore $\mathbf{x} \in |K'|$. Thus $|K| \subset |K'|$, and hence $|K'| = |K|$, as required. ∎

We define (by induction on j) the jth barycentric subdivision $K^{(j)}$ of K to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.

Lemma 9.6 *Let σ be a q-simplex and let τ be a face of σ. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of σ and τ respectively. If all the 1-simplices (edges) of σ have length not exceeding d for some $d > 0$ then*

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ. Let $\mathbf{x}$ and $\mathbf{y}$ be points of σ. We can write $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. Now

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= \left| \sum_{i=0}^{q} t_i (\mathbf{x} - \mathbf{v}_i) \right| \leq \sum_{i=0}^{q} t_i |\mathbf{x} - \mathbf{v}_i| \\ &\leq \text{maximum}\left(|\mathbf{x} - \mathbf{v}_0|, |\mathbf{x} - \mathbf{v}_1|, \ldots, |\mathbf{x} - \mathbf{v}_q| \right). \end{aligned}$$

Applying this result with $\mathbf{x} = \hat{\sigma}$ and $\mathbf{y} = \hat{\tau}$, we find that

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q| \right).$$

But

$$\hat{\sigma} = \frac{1}{q+1}\mathbf{v}_i + \frac{q}{q+1}\mathbf{z}_i$$

for $i = 0, 1, \ldots, q$, where $\mathbf{z}_i$ is the barycentre of the $(q-1)$-face of σ opposite to $\mathbf{v}_i$, given by

$$\mathbf{z}_i = \frac{1}{q} \sum_{j \neq i} \mathbf{v}_j.$$

Moreover $\mathbf{z}_i \in \sigma$. It follows that

$$|\hat{\sigma} - \mathbf{v}_i| = \frac{q}{q+1} |\mathbf{z}_i - \mathbf{v}_i| \leq \frac{qd}{q+1}$$

for $i = 1, 2, \ldots, q$, and thus

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum} \left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q| \right) \leq \frac{qd}{q+1},$$

as required. ∎

The *mesh* $\mu(K)$ of a simplicial complex K is the length of the longest edge of K.

Lemma 9.7 *Let K be a simplicial complex in $\mathbb{R}^k$ for some k, and let n be the dimension of K. Let K' be the first barycentric subdivision of K. Then*

$$\mu(K') \leq \frac{n}{n+1} \mu(K).$$

Proof A 1-simplex of K' is of the form $(\hat{\tau}, \hat{\sigma})$, where σ is a q-simplex of K for some $q \leq n$ and τ is a proper face of σ. Then

$$|\hat{\tau} - \hat{\sigma}| \leq \frac{q}{q+1} \mu(K) \leq \frac{n}{n+1} \mu(K)$$

by Lemma 9.6, as required. ∎

It follows directly from the above lemma that $\lim_{j \to +\infty} \mu(K^{(j)}) = 0$, where $K^{(j)}$ is the jth barycentric subdivision of K.

9.5 The Simplicial Approximation Theorem

Definition Let $f \colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s \colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

Note that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$ then s and f are homotopic. Indeed the map from $|K| \times [0,1]$ to $|L|$ sending $(\mathbf{x}, t)$ to $(1-t)f(\mathbf{x}) + ts(\mathbf{x})$ is a well-defined homotopy between f and s.

Definition Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

Lemma 9.8 *Let K be a simplicial complex and let $\mathbf{x} \in |K|$. Then the star $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ is open in $|K|$, and $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$.*

Proof Every point of $|K|$ belongs to the interior of a unique simplex of K (Lemma 9.3). It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K. ∎

Proposition 9.9 *A function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ between the vertex sets of simplicial complexes K and L is a simplicial map, and a simplicial approximation to some continuous map $f\colon |K| \to |L|$, if and only if $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K.*

Proof Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, since $\mathbf{x}$ is in the interior of σ. It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$.

Conversely let $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}_j))$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$.

We conclude that the function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required. ∎

Corollary 9.10 *If $s\colon K \to L$ and $t\colon L \to M$ are simplicial approximations to continuous maps $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$, where K, L and M are simplicial complexes, then $t \circ s\colon K \to M$ is a simplicial approximation to $g \circ f\colon |K| \to |M|$.*

Theorem 9.11 (Simplicial Approximation Theorem) *Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then, for some sufficiently large integer j, there exists a simplicial approximation $s\colon K^{(j)} \to L$ to f defined on the jth barycentric subdivision $K^{(j)}$ of K.*

Proof The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ (Lemma 9.8) and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$. It then follows from the Lebesgue Lemma that there exists some $\delta > 0$ with the property that every subset of $|K|$ whose diameter is less than δ is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$, since

$$\mu(K^{(j)}) \leq \left(\frac{\dim K}{\dim K + 1}\right)^j \mu(K)$$

for all j (Lemma 9.7). Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. It follows directly from Proposition 9.9 that this is the desired simplicial approximation to f. ∎

9.6 The Brouwer Fixed Point Theorem

Definition Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

- for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
- if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

Lemma 9.12 (Sperner's Lemma) *Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.*

Proof Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2(N(0, 1, \ldots, n-1) - M),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required. ∎

Proposition 9.13 *Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.*

Proof Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem (Theorem 9.11) that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma (Lemma 9.12) guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. ∎

Theorem 9.14 (Brouwer Fixed Point Theorem) *A continuous map $f\colon E^n \to E^n$ sending a closed n-dimensional ball E^n into itself has at least one fixed point (i.e., there exists $\mathbf{x} \in E^n$ for which $f(\mathbf{x}) = \mathbf{x}$).*

Proof Suppose that the map $f\colon E^n \to E^n$ had no fixed point. For each $\mathbf{x} \in E$, let $q(\mathbf{x})$ be the point at which the half line starting at $f(\mathbf{x})$ and passing through $\mathbf{x}$ intersects the boundary sphere S^{n-1} of E^n. Then $q\colon E^n \to S^{n-1}$ would be continuous, and $q(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$. But the closed n-dimensional ball E^n is homeomorphic to an n-simplex Δ. Therefore the map $q\colon E^n \to S^{n-1}$ would correspond under some homeomorphism $h\colon \Delta \to E^n$ to a continuous map $r\colon \Delta \to \partial\Delta$ mapping Δ onto its boundary $\partial\Delta$, where $h(r(\mathbf{y})) = q(h(\mathbf{y}))$ for all $\mathbf{y} \in \Delta$. Moreover $r(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in \partial\Delta$. However Proposition 9.13 shows that there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with this property. Therefore the map f must have at least one fixed point, as required. ∎

10 Simplicial Homology Groups

10.1 The Chain Groups of a Simplicial Complex

Let K be a simplicial complex. For each non-negative integer q, let $\Delta_q(K)$ be the additive group consisting of all formal sums of the form

$$n_1(\mathbf{v}_0^1, \mathbf{v}_1^1, \ldots, \mathbf{v}_q^1) + n_2(\mathbf{v}_0^2, \mathbf{v}_1^2, \ldots, \mathbf{v}_q^2) + \cdots + n_s(\mathbf{v}_0^s, \mathbf{v}_1^s, \ldots, \mathbf{v}_q^s),$$

where $n_1, n_2, \ldots, n_s$ are integers and $\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r$ are (not necessarily distinct) vertices of K that span a simplex of K for $r = 1, 2, \ldots, s$. (In more formal language, the group $\Delta_q(K)$ is the *free Abelian group* generated by the set of all $(q+1)$-tuples of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.)

We recall some basic facts concerning *permutations*. A *permutation* of a set S is a bijection mapping S onto itself. The set of all permutations of some set S is a group; the group multiplication corresponds to composition of permutations. A *transposition* is a permutation of a set S which interchanges two elements of S, leaving the remaining elements of the set fixed. If S is finite and has more than one element then any permutation of S can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and j for some j.

Associated to any permutation π of a finite set S is a number ϵ_π, known as the *parity* or *signature* of the permutation, which can take on the values ± 1. If π can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if π can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set S to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi\rho} = \epsilon_\pi \epsilon_\rho$ for all permutations π and ρ of the set S). Note in particular that the parity of any transposition is -1.

Definition The qth *chain group* $C_q(K)$ of the simplicial complex K is defined to be the quotient group $\Delta_q(K)/\Delta_q^0(K)$, where $\Delta_q^0(K)$ is the subgroup of $\Delta_q(K)$ generated by elements of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, and by elements of the form

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \epsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

where π is some permutation of $\{0, 1, \ldots, q\}$ with parity ϵ_π. For convenience, we define $C_q(K) = \{0\}$ when $q < 0$ or $q > \dim K$, where $\dim K$ is the dimension of the simplicial complex K. An element of the chain group $C_q(K)$ is referred to as q-*chain* of the simplicial complex K.

We denote by $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ the element $\Delta_q^0(K) + (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of $C_q(K)$ corresponding to $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$. The following results follow immediately from the definition of $C_q(K)$.

Lemma 10.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices of a simplicial complex K that span a simplex of K. Then*

- $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0$ *if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct,*
- $\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle = \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ *for any permutation π of the set $\{0, 1, \ldots, q\}$.*

Example If $\mathbf{v}_0$ and $\mathbf{v}_1$ are the endpoints of some line segment then

$$\langle \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0 \rangle.$$

If $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices of a triangle in some Euclidean space then

$$\begin{aligned} \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle &= \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0 \rangle = \langle \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0 \rangle \\ &= -\langle \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2 \rangle. \end{aligned}$$

Definition An *oriented q-simplex* is an element of the chain group $C_q(K)$ of the form $\pm\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct and span a simplex of K.

An oriented simplex of K can be thought of as consisting of a simplex of K (namely the simplex spanned by the prescribed vertices), together with one of two possible 'orientations' on that simplex. Any ordering of the vertices determines an orientation of the simplex; any even permutation of the ordering of the vertices preserves the orientation on the simplex, whereas any odd permutation of this ordering reverses orientation.

Any q-chain of a simplicial complex K can be expressed as a sum of the form

$$n_1\sigma_1 + n_2\sigma_2 + \cdots + n_s\sigma_s$$

where $n_1, n_2, \ldots, n_s$ are integers and $\sigma_1, \sigma_2, \ldots, \sigma_s$ are *oriented* q-simplices of K. If we reverse the orientation on one of these simplices σ_i then this reverses the sign of the corresponding coefficient n_i. If $\sigma_1, \sigma_2, \ldots, \sigma_s$ represent distinct simplices of K then the coefficients $n_1, n_2, \ldots, n_s$ are uniquely determined.

Example Let $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ be the vertices of a triangle in some Euclidean space. Let K be the simplicial complex consisting of this triangle, together with its edges and vertices. Every 0-chain of K can be expressed uniquely in the form

$$n_0\langle \mathbf{v}_0\rangle + n_1\langle \mathbf{v}_1\rangle + n_2\langle \mathbf{v}_2\rangle$$

for some $n_0, n_1, n_2 \in \mathbb{Z}$. Similarly any 1-chain of K can be expressed uniquely in the form

$$m_0\langle \mathbf{v}_1, \mathbf{v}_2\rangle + m_1\langle \mathbf{v}_2, \mathbf{v}_0\rangle + m_2\langle \mathbf{v}_0, \mathbf{v}_1\rangle$$

for some $m_0, m_1, m_2 \in \mathbb{Z}$, and any 2-chain of K can be expressed uniquely as $n\langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\rangle$ for some integer n.

Lemma 10.2 *Let K be a simplicial complex, and let A be an additive group. Suppose that, to each $(q+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices spanning a simplex of K, there corresponds an element $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of A, where*

- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ *unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct,*
- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ *changes sign on interchanging any two adjacent vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$.*

Then there exists a well-defined homomorphism from $C_q(K)$ to A which sends $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle$ to $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. This homomorphism is uniquely determined.

Proof The given function defined on $(q+1)$-tuples of vertices of K extends to a well-defined homomorphism $\alpha\colon \Delta_q(K) \to A$ given by

$$\alpha\left(\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)\right) = \sum_{r=1}^{s} n_r\alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)$$

for all $\displaystyle\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r) \in \Delta_q(K)$. Moreover $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. Also

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \varepsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$$

for all permutations π of $\{0, 1, \ldots, q\}$, since the permutation π can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ with j for some j and leave the rest of the set fixed, and the parity ε_π of π is given by $\varepsilon_\pi = +1$ when the number of such transpositions is even, and by $\varepsilon_\pi = -1$ when the number of such transpositions is odd. Thus the generators of $\Delta_q^0(K)$ are contained in $\ker\alpha$, and hence $\Delta_q^0(K) \subset \ker\alpha$. The required homomorphism $\tilde\alpha\colon C_q(K) \to A$ is then defined by the formula

$$\tilde\alpha\left(\sum_{r=1}^{s} n_r\langle \mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r\rangle\right) = \sum_{r=1}^{s} n_r\alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r). \quad \blacksquare$$

10.2 Boundary Homomorphisms

Let K be a simplicial complex. We introduce below *boundary homomorphisms* $\partial_q: C_q(K) \to C_{q-1}(K)$ between the chain groups of K. If σ is an oriented q-simplex of K then $\partial_q(\sigma)$ is a $(q-1)$-chain which is a formal sum of the $(q-1)$-faces of σ, each with an orientation determined by the orientation of σ.

Let σ be a q-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. For each integer j between 0 and q we denote by $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ the oriented $(q-1)$-face

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$$

of the simplex σ obtained on omitting $\mathbf{v}_j$ from the set of vertices of σ. In particular

$$\langle \hat{\mathbf{v}}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \equiv \langle \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle, \qquad \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1}, \hat{\mathbf{v}}_q \rangle \equiv \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1} \rangle.$$

Similarly if j and k are integers between 0 and q, where $j < k$, we denote by

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots \mathbf{v}_q \rangle$$

the oriented $(q-2)$-face $\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_q \rangle$ of the simplex σ obtained on omitting $\mathbf{v}_j$ and $\mathbf{v}_k$ from the set of vertices of σ.

We now define a 'boundary homomorphism' $\partial_q: C_q(K) \to C_{q-1}(K)$ for each integer q. Define $\partial_q = 0$ if $q \leq 0$ or $q > \dim K$. (In this case one or other of the groups $C_q(K)$ and $C_{q-1}(K)$ is trivial.) Suppose then that $0 < q \leq \dim K$. Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ spanning a simplex of K, let

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Inspection of this formula shows that $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign whenever two adjacent vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are interchanged.

Suppose that $\mathbf{v}_j = \mathbf{v}_k$ for some j and k satisfying $j < k$. Then

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle + (-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle,$$

since the remaining terms in the expression defining $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ contain both $\mathbf{v}_j$ and $\mathbf{v}_k$. However $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q)$ can be transformed to $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q)$ by making $k - j - 1$ transpositions which interchange $\mathbf{v}_j$ successively with the vertices $\mathbf{v}_{j+1}, \mathbf{v}_{j+2}, \ldots, \mathbf{v}_{k-1}$. Therefore

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{k-j-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Thus $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. It now follows immediately from Lemma 10.2 that there is a well-defined homomorphism $\partial_q\colon C_q(K) \to C_{q-1}(K)$, characterized by the property that

$$\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Lemma 10.3 *$\partial_{q-1} \circ \partial_q = 0$ for all integers q.*

Proof The result is trivial if $q < 2$, since in this case $\partial_{q-1} = 0$. Suppose that $q \geq 2$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned} \partial_{q-1}\partial_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle\right) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1}\left(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle\right) \\ &= \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q\rangle \\ &\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q\rangle \\ &= 0 \end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K. ∎

10.3 The Homology Groups of a Simplicial Complex

Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q\colon C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1}\colon C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, by Lemma 10.3. Therefore $B_q(K) \subset Z_q(K)$. But these groups are subgroups of the Abelian group $C_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K. Note that $H_q(K) = 0$ if $q < 0$

or $q > \dim K$ (since $Z_q(K) = 0$ and $B_q(K) = 0$ in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

The element $[z] \in H_q(K)$ of the homology group $H_q(K)$ determined by $z \in Z_q(K)$ is referred to as the *homology class* of the q-cycle z. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1, z_2 \in Z_q(K)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q+1)$-chain c.

Proposition 10.4 *Let K be a simplicial complex. Suppose that there exists a vertex $\mathbf{w}$ of K with the following property:*

- *if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.*

Then $H_0(K) \cong \mathbb{Z}$, and $H_q(K)$ is the zero group for all $q > 0$.

Proof Using Lemma 10.2, we see that there is a well-defined homomorphism $D_q \colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required. ∎

Example The hypotheses of the proposition are satisfied for the complex K_σ consisting of a simplex σ together with all of its faces: we can choose $\mathbf{w}$ to be any vertex of the simplex σ. They are also satisfied for the first barycentric subdivision K'_σ of K_σ: in this case we must choose $\mathbf{w}$ to be the barycentre $\hat{\sigma}$ of the simplex σ. Thus the groups $H_0(K_\sigma)$ and $H_0(K'_\sigma)$ are both isomorphic of $\mathbb{Z}$, and the groups $H_q(K_\sigma)$ and $H_q(K'_\sigma)$ are the zero group for all $q > 0$.

10.4 Simplicial Maps and Induced Homomorphisms

Any simplicial map $\varphi: K \to L$ between simplicial complexes K and L induces well-defined homomorphisms $\varphi_q: C_q(K) \to C_q(L)$ of chain groups, where

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. (The existence of these induced homomorphisms follows from a straightforward application of Lemma 10.2.) Note that $\varphi_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) = 0$ unless $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ are all distinct.

Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer q. Therefore $\varphi_q(Z_q(K)) \subset Z_q(L)$ and $\varphi_q(B_q(K)) \subset B_q(L)$ for all integers q. It follows that any simplicial map $\varphi: K \to L$ induces well-defined homomorphisms $\varphi_*: H_q(K) \to H_q(L)$ of homology groups, where $\varphi_*([z]) = [\varphi_q(z)]$ for all q-cycles $z \in Z_q(K)$. It is a trivial exercise to verify that if K, L and M are simplicial complexes and if $\varphi: K \to L$ and $\psi: L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

10.5 Connectedness and $H_0(K)$

Lemma 10.5 *Let K be a simplicial complex. Then K can be partitioned into pairwise disjoint subcomplexes $K_1, K_2, \ldots, K_r$ whose polyhedra are the connected components of the polyhedron $|K|$ of K.*

Proof Let $X_1, X_2, \ldots, X_r$ be the connected components of the polyhedron of K, and, for each j, let K_j be the collection of all simplices σ of K for which $\sigma \subset X_j$. If a simplex belongs to K_j for all j then so do all its faces. Therefore $K_1, K_2, \ldots, K_r$ are subcomplexes of K. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, \ldots, X_r$ of $|K|$ are pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some j, since σ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_r$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_r|$, as required. ∎

The *direct sum* $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ of additive Abelian groups $A_1, A_2, \ldots, A_r$ is defined to be the additive group consisting of all r-tuples $(a_1, a_2, \ldots, a_r)$ with $a_i \in A_i$ for $i = 1, 2, \ldots, r$, where

$$(a_1, a_2, \ldots, a_r) + (b_1, b_2, \ldots, b_r) \equiv (a_1 + b_1, a_2 + b_2, \ldots, a_r + b_r).$$

Lemma 10.6 *Let K be a simplicial complex. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_r$, where $K_1, K_2, \ldots K_r$ are pairwise disjoint. Then*

$$H_q(K) \cong H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r)$$

for all integers q.

Proof We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any q-chain c of K can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_r$, where c_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. It follows that

$$C_q(K) \cong C_q(K_1) \oplus C_q(K_2) \oplus \cdots \oplus C_q(K_r).$$

Now let z be a q-cycle of K (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express z uniquely in the form $z = z_1 + z_2 + \cdots + z_r$, where z_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_r),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of K_j for $j = 1, 2, \ldots, r$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, r$. Hence each z_j is a q-cycle of K_j, and thus

$$Z_q(K) \cong Z_q(K_1) \oplus Z_q(K_2) \oplus \cdots \oplus Z_q(K_r).$$

Now let b be a q-boundary of K. Then $b = \partial_{q+1}(c)$ for some $(q+1)$-chain c of K. Moreover $c = c_1 + c_2 + \cdots c_r$, where $c_j \in C_{q+1}(K_j)$. Thus $b =$

$b_1 + b_2 + \cdots b_r$, where $b_j \in B_q(K_j)$ is given by $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, r$. We deduce that

$$B_q(K) \cong B_q(K_1) \oplus B_q(K_2) \oplus \cdots \oplus B_q(K_r).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu \colon H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r) \to H_q(K)$$

which maps $([z_1], [z_2], \ldots, [z_r])$ to $[z_1 + z_2 + \cdots + z_r]$, where $[z_j]$ denotes the homology class of a q-cycle z_j of K_j for $j = 1, 2, \ldots, r$. ∎

Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

Lemma 10.7 *The polyhedron $|K|$ of a simplicial complex K is a connected topological space if and only if any two vertices of K can be joined by an edge path.*

Proof It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required. ∎

Theorem 10.8 *Let K be a simplicial complex. Suppose that the polyhedron $|K|$ of K is connected. Then $H_0(K) \cong \mathbb{Z}$.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle$$

for some integers $n_1, n_2, \ldots, n_r$. There is therefore a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z}\rangle)) = \varepsilon(\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m\rangle - \langle \mathbf{v}_0\rangle = \partial_1\left(\sum_{j=1}^{m}\langle \mathbf{v}_{j-1}, \mathbf{v}_j\rangle\right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path (Lemma 10.7). We deduce that $\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j\langle \mathbf{u}_j\rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle)$. But $(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle) \in B_0(K)$. It follows that $c \in B_0(K)$. Thus We conclude that that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required. ∎

On combining Theorem 10.8 with Lemmas 10.5 and 10.6 we obtain immediately the following result.

Corollary 10.9 *Let K be a simplicial complex. Then*

$$H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \qquad (r\ \textit{times}),$$

where r is the number of connected components of $|K|$.

11 Introduction to Homological Algebra

11.1 Exact Sequences

In homological algebra we consider sequences

$$\cdots \longrightarrow F \xrightarrow{p} G \xrightarrow{q} H \xrightarrow{\cdots}$$

where F, G, H etc. are Abelian groups and p, q etc. are homomorphisms. We denote the trivial group $\{0\}$ by 0, and we denote by $0 \longrightarrow G$ and $G \longrightarrow 0$ the zero homomorphisms from 0 to G and from G to 0 respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial group 0.)

Definition The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\operatorname{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *monomorphism* is an injective homomorphism. An *epimorphism* is a surjective homomorphism. An *isomorphism* is a bijective homomorphism.

The following result follows directly from the relevant definitions.

Lemma 11.1 *Let $h\colon G \to H$ be a homomorphism of Abelian groups.*

- *$h\colon G \to H$ is a monomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H$ is an exact sequence.*
- *$h\colon G \to H$ is an epimorphism if and only if $G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*
- *$h\colon G \to H$ is an isomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*

Let F be a subgroup of an Abelian group G. Then the sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \longrightarrow 0,$$

is exact, where G/F is the quotient group, $i\colon F \hookrightarrow G$ is the inclusion homomorphism, and $q\colon G \to G/F$ is the quotient homomorphism. Conversely, given any exact sequence of the form

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 0,$$

we can regard F as a subgroup of G (on identifying F with $i(F)$), and then H is isomorphic to the quotient group G/F. Exact sequences of this type are referred to as *short exact sequences.*

We now introduce the concept of a *commutative diagram.* This is a diagram depicting a collection of homomorphisms between various Abelian groups occurring on the diagram. The diagram is said to *commute* if, whenever there are two routes through the diagram from an Abelian group G to an Abelian group H, the homomorphism from G to H obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r, "g"] \arrow[d, "q"] & C \arrow[d, "r"] \\ D \arrow[r, "h"] & E \arrow[r, "k"] & F \end{tikzcd}$$

commutes if and only if $q \circ f = h \circ p$ and $r \circ g = k \circ q$.

Proposition 11.2 *Suppose that the following diagram of Abelian groups and homomorphisms*

$$\begin{tikzcd} G_1 \arrow[r, "\theta_1"] \arrow[d, "\psi_1"] & G_2 \arrow[r, "\theta_2"] \arrow[d, "\psi_2"] & G_3 \arrow[r, "\theta_3"] \arrow[d, "\psi_3"] & G_4 \arrow[r, "\theta_4"] \arrow[d, "\psi_4"] & G_5 \arrow[d, "\psi_5"] \\ H_1 \arrow[r, "\phi_1"] & H_2 \arrow[r, "\phi_2"] & H_3 \arrow[r, "\phi_3"] & H_4 \arrow[r, "\phi_4"] & H_5 \end{tikzcd}$$

commutes and that both rows are exact sequences. Then the following results follow:

(i) *if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism,*

(ii) *if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism.*

Proof First we prove (i). Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove (ii). Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required. ∎

The following result is an immediate corollary of Proposition 11.2.

Lemma 11.3 (Five-Lemma) *Suppose that the rows of the commutative diagram of Proposition 11.2 are exact sequences and that ψ_1,ψ_2,ψ_4 and ψ_5 are isomorphisms. Then ψ_3 is also an isomorphism.*

11.2 Chain Complexes

Definition A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i \colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i \colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1} \colon C_{i+1} \to C_i$.

Definition Let C_* and D_* be chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence $f_i\colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

Note that a collection of homomorphisms $f_i\colon C_i \to D_i$ defines a chain map $f_*\colon C_* \to D_*$ if and only if the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow \cdots \\
 & \Big\downarrow f_{i+1} & & \Big\downarrow f_i & & \Big\downarrow f_{i-1} & \\
\cdots \longrightarrow & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \longrightarrow \cdots
\end{array}$$

is commutative.

Let C_* and D_* be chain complexes, and let $f_*\colon C_* \to D_*$ be a chain map. Then $f_i(Z_i(C_*)) \subset Z_i(D_*)$ and $f_i(B_i(C_*)) \subset B_i(D_*)$ for all i. It follows from this that $f_i\colon C_i \to D_i$ induces a homomorphism $f_*\colon H_i(C_*) \to H_i(D_*)$ of homology groups sending $[z]$ to $[f_i(z)]$ for all $z \in Z_i(C_*)$, where $[z] = z + B_i(C_*)$, and $[f_i(z)] = f_i(z) + B_i(D_*)$.

Definition A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \Big\downarrow \partial_{i+2} & & \Big\downarrow \partial_{i+2} & & \Big\downarrow \partial_{i+2} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \Big\downarrow \partial_{i+1} & & \Big\downarrow \partial_{i+1} & & \Big\downarrow \partial_{i+1} & & \\
0 & \longrightarrow & A_i & \xrightarrow{p_i} & B_i & \xrightarrow{q_i} & C_i & \longrightarrow & 0\,. \\
 & & \Big\downarrow \partial_i & & \Big\downarrow \partial_i & & \Big\downarrow \partial_i & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \Big\downarrow \partial_{i-1} & & \Big\downarrow \partial_{i-1} & & \Big\downarrow \partial_{i-1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

Lemma 11.4 *Given any short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes, there is a well-defined homomorphism*

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$$

which sends the homology class $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i \colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} \colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_{i-1}$, by exactness. But then

$$p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),$$

and $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} \colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$, as required. ∎

Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ and $0 \longrightarrow A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \longrightarrow 0$ be short exact sequences of chain complexes, and let $\lambda_*: A_* \to A'_*$, $\mu_*: B_* \to B'_*$ and $\nu_*: C_* \to C'_*$ be chain maps. For each integer i, let $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ and $\alpha'_i: H_i(C'_*) \to H_{i-1}(A'_*)$ be the homomorphisms defined as in described in Lemma 11.4. Suppose that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{p_*} & B_* & \xrightarrow{q_*} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \nu_*} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{p'_*} & B'_* & \xrightarrow{q'_*} & C'_* & \longrightarrow & 0 \end{array}$$

commutes (i.e., $p'_i \circ \lambda_i = \mu_i \circ p_i$ and $q'_i \circ \mu_i = \nu_i \circ q_i$ for all i). Then the square

$$\begin{array}{ccc} H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\ \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \lambda_*} \\ H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*) \end{array}$$

commutes for all $i \in \mathbb{Z}$ (i.e., $\lambda_* \circ \alpha_i = \alpha'_i \circ \nu_*$).

Proposition 11.5 *Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ be a short exact sequence of chain complexes. Then the (infinite) sequence*

$$\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots$$

of homology groups is exact, where $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ is the well-defined homomorphism that sends the homology class $[z]$ of $z \in Z_i(C_*)$ to the homology class $[w]$ of any element w of $Z_{i-1}(A_*)$ with the property that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$.

Proof First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_*: H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_*: H_i(B_*) \to H_i(C_*)$. Let x be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x + \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus a represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x + \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now $\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w]$, where w is the unique element of $Z_i(A_*)$ satisfing $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let z be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b + p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$. Now let w be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required. ∎

11.3 The Mayer-Vietoris Sequence

Let K be a simplicial complex and let L and M be subcomplexes of K such that $K = L \cup M$. Let

$$i_q: C_q(L \cap M) \to C_q(L), \qquad j_q: C_q(L \cap M) \to C_q(M),$$

$$u_q: C_q(L) \to C_q(K), \qquad v_q: C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i: L \cap M \hookrightarrow L$, $j: L \cap M \hookrightarrow M$, $u: L \hookrightarrow K$ and $v: M \hookrightarrow K$. Then

$$0 \longrightarrow C_*(L \cap M) \xrightarrow{k_*} C_*(L) \oplus C_*(M) \xrightarrow{w_*} C_*(K) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L)$ and $c'' \in C_q(M)$. It follows from Lemma 11.4 that there is a well-defined homomorphism $\alpha_q: H_q(K) \to H_{q-1}(L \cap M)$ such that $\alpha_q([z]) = [\partial_q(c')] = -[\partial_q(c'')]$ for any $z \in Z_q(K)$, where c' and c''

are any q-chains of L and M respectively satisfying $z = c' + c''$. (Note that $\partial_q(c') \in Z_{q-1}(L \cap M)$ since $\partial_q(c') \in Z_{q-1}(L)$, $\partial_q(c'') \in Z_{q-1}(M)$ and $\partial_q(c') = -\partial_q(c'')$.) It now follows immediately from Proposition 11.5 that the infinite sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups is exact. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

12 The Topological Invariance of Simplicial Homology Groups

12.1 Contiguous Simplicial Maps

Definition Two simplicial maps $s\colon K \to L$ and $t\colon K \to L$ between simplicial complexes K and L are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

Lemma 12.1 *Let K and L be simplicial complexes, and let $s\colon K \to L$ and $t\colon K \to L$ be simplicial approximations to some continuous map $f\colon |K| \to |L|$. Then the simplicial maps s and t are contiguous.*

Proof Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required. ∎

Proposition 12.2 *Let $s\colon K \to L$ and $t\colon K \to L$ be simplicial maps between simplicial complexes K and L. Suppose that s and t are contiguous. Then the homomorphisms $s_*\colon H_q(K) \to H_q(L)$ and $t_*\colon H_q(K) \to H_q(L)$ coincide for all q.*

Proof Choose an ordering of the vertices of K. Then there are well-defined homomorphisms $D_q\colon C_q(K) \to C_{q+1}(L)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle.$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order (with respect to the chosen ordering of the vertices of K). Then

$$\partial_1(D_0(\langle \mathbf{v} \rangle)) = \partial_1(\langle s(\mathbf{v}), t(\mathbf{v}) \rangle) = \langle t(\mathbf{v}) \rangle - \langle s(\mathbf{v}) \rangle,$$

and thus $\partial_1 \circ D_0 = t_0 - s_0$. Also

$$\begin{aligned}
&D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) \\
&\quad = \sum_{i=0}^{q} (-1)^i D_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle) \\
&\quad = \sum_{i=0}^{q} \sum_{j=0}^{i-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{i=0}^{q} \sum_{j=i+1}^{q} (-1)^{i+j-1} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \ldots \mathbf{v}_q \rangle)) \\
&\quad = \sum_{j=0}^{q} (-1)^j \partial_{q+1}(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle) \\
&\quad = \sum_{j=0}^{q} \sum_{i=0}^{j-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q) \rangle + \sum_{j=1}^{q} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad - \sum_{j=0}^{q-1} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q) \rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{j=0}^{q} \sum_{i=j+1}^{q} (-1)^{i+j+1} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\quad = -D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q) \rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q) \rangle
\end{aligned}$$

and thus

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K)$ to $H_q(L)$, as required. ∎

12.2 The Homology of Barycentric Subdivisions

We shall show that the homology groups of a simplicial complex are isomorphic to those of its first barycentric subdivision.

We recall that the vertices of the first barycentric subdivision K' of a simplicial complex K are the barycentres $\hat{\sigma}$ of the simplices σ of K, and that K' consists of the simplices spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_q$, where $\sigma_0, \sigma_1, \ldots, \sigma_q \in K$ and σ_{j-1} is a proper face of σ_j for $j = 1, 2, \ldots, q$.

Lemma 12.3 *Let K' be the first barycentric subdivision of a simplicial complex K. Then a function $\zeta\colon \mathrm{Vert}\, K' \to \mathrm{Vert}\, K$ from the vertices of K' to those of K represents a simplicial approximation to the identity map of $|K|$ if and only if it sends the barycentre of any simplex of K to some vertex of that simplex.*

Proof If ζ represents a simplicial approximation to the identity map of $|K|$ then $\zeta(\hat{\sigma}) \in \sigma$ for any $\sigma \in K$, and hence $\zeta(\hat{\sigma})$ is a vertex of σ.

Conversely suppose that the function ζ sends the barycentre of any simplex of K to a vertex of that simplex. Let τ be a simplex of K'. Then it follows from the definition of K' that the interior of τ is contained in the interior of some simplex σ of K, and the vertices of τ are barycentres of faces of σ. Then ζ must map the vertices of τ to vertices of σ, and hence ζ represents a simplicial map from K' to K. Moreover this simplicial map is a simplicial approximation to the identity map, since the interior of τ is contained in σ and ζ maps the interior of τ into σ. ∎

It follows from Lemma 12.3 that there exist simplicial approximations $\zeta\colon K' \to K$ to the identity map of $|K|$: such a simplicial approximation can be obtained by choosing, for each $\sigma \in K$, a vertex $\mathbf{v}_\sigma$ of σ, and defining $\zeta(\hat{\sigma}) = \mathbf{v}_\sigma$.

Suppose that $\zeta\colon K' \to K$ and $\theta\colon K' \to K$ are both simplicial approximations to the identity map of $|K|$. Then ζ and θ are contiguous (Lemma 12.1), and therefore the homomorphisms ζ_* and θ_* of homology groups induced by ζ and θ must coincide. It follows that there is a well-defined natural homomorphism $\nu_K\colon H_q(K') \to H_q(K)$ of homology groups which coincides with ζ_* for any simplicial approximation $\zeta\colon K' \to K$ to the identity map of $|K|$.

Theorem 12.4 *The natural homomorphism $\nu_K\colon H_q(K') \to H_q(K)$ is an isomorphism for any simplicial complex K.*

Proof Let M be the simplicial complex consisting of some simplex σ together with all of its faces. Then $H_0(M) \cong \mathbb{Z}$, $H_0(M') \cong \mathbb{Z}$, and $H_q(M) =$

$0 = H_q(M')$ for all $q > 0$ (see Proposition 10.4 and the following example). Let $\mathbf{v}$ be a vertex of M. If $\theta: M' \to M$ is any simplicial approximation to the identity map of $|M|$ then $\theta(\mathbf{v}) = \mathbf{v}$. But the homology class of $\langle \mathbf{v} \rangle$ generates both $H_0(M)$ and $H_0(M')$. It follows that $\theta_*: H_0(M') \to H_0(M)$ is an isomorphism, and thus $\nu_M: H_q(M') \to H_q(M)$ is an isomorphism for all q.

We now use induction on the number of simplices in K to prove the theorem in the general case. It therefore suffices to prove that the required result holds for a simplicial complex K under the additional assumption that the result is valid for all proper subcomplexes of K.

Let σ be a simplex of K whose dimension equals the dimension of K. Then σ is not a face of any other simplex of K, and therefore $K \setminus \{\sigma\}$ is a subcomplex of K. Let M be the subcomplex of K consisting of the simplex σ, together with all of its faces. We have already proved the result in the special case when $K = M$. Thus we only need to verify the result in the case when M is a proper subcomplex of K. In that case $K = L \cup M$, where $L = K \setminus \{\sigma\}$.

Let $\zeta: K' \to K$ be a simplicial approximation to the identity map of $|K|$. Then the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively. Therefore the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_q(L'\cap M') \arrow[r] \arrow[d, "\zeta|L'\cap M'"] & C_q(L')\oplus C_q(M') \arrow[r] \arrow[d, "(\zeta|L')\oplus(\zeta|M')"] & C_q(K') \arrow[r] \arrow[d, "\zeta"] & 0 \\ 0 \arrow[r] & C_q(L\cap M) \arrow[r] & C_q(L)\oplus C_q(M) \arrow[r] & C_q(K) \arrow[r] & 0 \end{tikzcd}$$

commutes, and its rows are short exact sequences. But the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively, and therefore induce the natural homomorphisms ν_L, ν_M and $\nu_{L\cap M}$. We therefore obtain a commutative diagram

$$\begin{tikzcd} H_q(L'\cap M') \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_q(L')\oplus H_q(M') \arrow[r] \arrow[d, "\nu_L\oplus\nu_M"] & H_q(K') \arrow[r, "\alpha_q"] \arrow[d, "\nu_K"] & H_{q-1}(L'\cap M') \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_{q-1}(L')\oplus H_{q-1}(M') \arrow[d, "\nu_L\oplus\nu_M"] \\ H_q(L\cap M) \arrow[r] & H_q(L)\oplus H_q(M) \arrow[r] & H_q(K) \arrow[r, "\alpha_q"] & H_{q-1}(L\cap M) \arrow[r] & H_{q-1}(L)\oplus H_{q-1}(M) \end{tikzcd}$$

in which the rows are exact sequences, and are the Mayer-Vietoris sequences corresponding to the decompositions $K = L \cup M$ and $K' = L' \cup M'$ of K and K'. But the induction hypothesis ensures that the homomorphisms ν_L, ν_M and $\nu_{L\cap M}$ are isomorphisms, since L, M and $L \cap M$ are all proper subcomplexes of K. It now follows directly from the Five-Lemma (Lemma 11.3) that $\nu_K: H_q(K') \to H_q(K)$ is also an isomorphism, as required. ∎

We refer to the isomorphism $\nu_K: H_q(K') \to H_q(K)$ as the *canonical isomorphism* from the qth homology group of K' to that of K.

For each $j > 0$, we define the canonical isomorphism $\nu_{K,j}\colon H_q(K^{(j)}) \to H_q(K)$ from the homology groups of the jth barycentric subdivision $K^{(j)}$ of K to those of K itself to be the composition of the natural isomorphisms

$$H_q(K^{(j)}) \to H_q(K^{(j-1)}) \to \cdots \to H_q(K') \to H_q(K)$$

induced by appropriate simplicial approximations to the identity map of $|K|$. Note that if $i \leq j$ then $\nu_{K,i}^{-1} \circ \nu_{K,j}$ is induced by a composition of simplicial approximations to the identity map of $|K|$. But any composition of simplicial approximations to the identity map is itself a simplicial approximation to the identity map (Corollary 9.10). We deduce the following result.

Lemma 12.5 *Let K be a simplicial complex, let i and j be positive integers satisfying $i \leq j$. Then $\nu_{K,j} = \nu_{K,i} \circ \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$.*

12.3 Continuous Maps and Induced Homomorphisms

Proposition 12.6 *Any continuous map $f\colon |K| \to |L|$ between the polyhedra of simplicial complexes K and L induces a well-defined homomorphism $f_*\colon H_q(K) \to H_q(L)$ of homology groups such that $f_* = s_* \circ \nu_{K,i}^{-1}$ for any simplicial approximation $s\colon K^{(i)} \to L$ to the map f, where $s_*\colon H_q(K^{(i)}) \to H_q(L)$ is the homomorphism induced by the simplicial map s and $\nu_{K,i}\colon H_q(K^{(i)}) \to H_q(K)$ is the canonical isomorphism.*

Proof The Simplicial Approximation Theorem (Theorem 9.11) guarantees the existence of a simplicial approximation $s\colon K^{(i)} \to L$ to the map f defined on the ith barycentric subdivision $K^{(i)}$ of K for some sufficiently large i. Thus it only remains to verify that if $s\colon K^{(i)} \to L$ and $t\colon K^{(j)} \to L$ are both simplicial approximations to the map f then $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$.

Suppose that $i \leq j$. Then $\nu_{K,i}^{-1}\nu_{K,j} = \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$ (Lemma 12.5). Thus $s_* \circ \nu_{K,i}^{-1} = s_* \circ \zeta_* \circ \nu_{K,j}^{-1} = (s \circ \zeta)_* \circ \nu_{K,j}^{-1}$. Moreover $\zeta\colon K^{(j)} \to K^{(i)}$ and $s\colon K^{(i)} \to L$ are simplicial approximations to the identity map of $|K|$ and to $f\colon |K| \to |L|$ respectively, and therefore $s \circ \zeta\colon K^{(j)} \to L$ is a simplicial approximation to $f\colon |K| \to |L|$ (Corollary 9.10). But then $s \circ \zeta$ and t are simplicial approximations to the same continuous map, and thus are contiguous simplicial maps from $K^{(j)}$ to L (Lemma 12.1). It follows that $(s \circ \zeta)_*$ and t_* coincide as homomorphisms from $H_q(K^{(j)})$ to $H_q(L)$ (Lemma 12.2), and therefore $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$, as required. ∎

Proposition 12.7 *Let K, L and M be simplicial complexes and let $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ be continuous maps. Then the homomorphisms f_*, g_* and $(g \circ f)_*$ of homology groups induced by the maps f, g and $g \circ f$ satisfy $(g \circ f)_* = g_* \circ f_*$.*

Proof Let $t\colon L^{(m)} \to M$ be a simplicial approximation to g and let $s\colon K^{(j)} \to L^{(m)}$ be a simplicial approximation to f. Now the canonical isomorphism $\nu_{L,m}$ from $H_q(L^{(m)})$ to $H_q(L)$ is induced by some simplicial approximation to the identity map of $|L|$. It follows that $\nu_{L,m} \circ s_*$ is induced by some simplicial approximation to f (Corollary 9.10), and therefore $f_* = \nu_{L,m} \circ s_* \circ \nu_{K,j}^{-1}$. Also $g_* = t_* \circ \nu_{L,m}^{-1}$. It follows that $g_* \circ f_* = t_* \circ s_* \circ \nu_{K,j}^{-1} = (t \circ s)_* \circ \nu_{K,j}^{-1}$. But $t \circ s\colon K^{(j)} \to M$ is a simplicial approximation to $g \circ f$ (Corollary 9.10). Thus $(g \circ f)_* = g_* \circ f_*$, as required. ∎

Corollary 12.8 *If the polyhedra $|K|$ and $|L|$ of simplicial complexes K and L are homeomorphic then the homology groups of K and L are isomorphic.*

Proof Let $h\colon |K| \to |L|$ be a homeomorphism. Then $h_*\colon H_q(K) \to H_q(L)$ is an isomorphism whose inverse is $(h^{-1})_*\colon H_q(L) \to H_q(K)$. ∎

One can make use of induced homomorphisms in homology theory in order to prove the Brouwer Fixed Point Theorem (Theorem 9.14) in all dimensions. The Brouwer Fixed Point Theorem is a consequence of the fact that there is no continuous map $r\colon \Delta \to \partial\Delta$ from an n-simplex Δ to its boundary $\partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$ (Proposition 9.13). Such a continuous map would induce homomorphisms $r_*\colon H_q(\Delta) \to H_q(\partial\Delta)$ of homology groups for all non-negative integers q, and $r_* \circ i_*$ would be the identity automorphism of $H_q(\partial\Delta)$ for all q, where $i_*\colon H_q(\partial\Delta) \to H_q(\Delta)$ is induced by the inclusion map $i\colon \partial\Delta \hookrightarrow \Delta$. But this would imply that $r_*\colon H_q(\Delta) \to H_q(\partial\Delta)$ was surjective for all non-negative integers q,which is impossible, since $H_{n-1}(\Delta) = 0$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z}$ when $n \geq 2$ (and $H_{n-1}(\Delta) \cong \mathbb{Z}$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z} \oplus \mathbb{Z}$ when $n = 1$). We conclude therefore that there is no continuous map $r\colon \Delta \to \partial\Delta$ that fixes all points of $\partial\Delta$, and therefore the Brouwer Fixed Point Theorem is satisfied in all dimensions.

We next show that homotopic maps between the polyhedra of simplicial complexes induce the same homomorphisms of homology groups. For this we require the following result.

Lemma 12.9 *For any simplicial complex L there is some $\varepsilon > 0$ with the following property: given continuous maps $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ defined on the polyhedron of some simplicial complex K, where $f(\mathbf{x})$ is within* *a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$, there exists a simplicial map defined on $K^{(i)}$ for some sufficiently large i which is a simplicial approximation to both f and g.*

Proof An application of the Lebesgue Lemma shows that there exists $\varepsilon > 0$ such that the open ball of radius 2ε about any point of $|L|$ is contained in $\mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps. Suppose that $f(\mathbf{x})$ is within a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Another application of the Lebesgue Lemma (to the open cover of $|K|$ by preimages of open balls of radius ε) shows that there exists $\delta > 0$ such that any subset S of $|K|$ whose diameter is less than δ is mapped by f into an open ball of radius ε about some point of $|L|$, and is therefore mapped by g into an open ball of of radius 2ε about that point. But then $f(S) \subset \mathrm{st}_L(\mathbf{b})$ and $g(S) \subset \mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Now choose i such that $\mu(K^{(i)}) < \frac{1}{2}\delta$. As in the proof of the Simplicial Approximation Theorem (Theorem 9.11) we see that, for every vertex $\mathbf{a}$ of $K^{(i)}$, the diameter of $\mathrm{st}_{K^{(i)}}(\mathbf{a})$ is less than δ, and hence $f(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ and $g(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ for some vertex $s(\mathbf{a})$ of L. It then follows from Proposition 9.9 that the function $s\colon \mathrm{Vert}\, K^{(i)} \to \mathrm{Vert}\, L$ constructed in this manner is the required simplicial approximation to f and g. ∎

Theorem 12.10 *Let K and L be simplicial complexes and let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps from $|K|$ to $|L|$. Suppose that f and g are homotopic. Then the induced homomorphisms f_* and g_* from $H_q(K)$ to $H_q(L)$ are equal for all q.*

Proof Let $F\colon |K| \times [0,1] \to |L|$ be a homotopy with $F(\mathbf{x},0) = f(\mathbf{x})$ and $F(\mathbf{x},1) = g(\mathbf{x})$, and let $\varepsilon > 0$ be given. Using the well-known fact that continuous functions defined on compact metric spaces are uniformly continuous (which is easily proved using the Lebesgue Lemma), we see that there exists some $\delta > 0$ such that if $|s - t| < \delta$ then the distance from $F(\mathbf{x}, s)$ to $F(\mathbf{x}, t)$ is less than ε. Let $f_i(\mathbf{x}) = F(\mathbf{x}, t_i)$ for $i = 0, 1, \ldots, r$, where $t_0, t_1, \ldots, t_r$ have been chosen such that $0 = t_0 < t_1 < \cdots < t_r = 1$ and $t_i - t_{i-1} < \delta$ for all i. Then $f_{i-1}(\mathbf{x})$ is within a distance ε of $f_i(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Using Lemma 12.9, we see that the maps f_{i-1} and f_i from $|K|$ to $|L|$ have a common simplicial approximation, and thus f_{i-1} and f_i induce the same homomorphisms of homology groups, provided that $\varepsilon > 0$ has been chosen sufficiently small. It follows that the maps f and g induce the same homomorphisms of homology groups, as required. ∎

12.4 Homotopy Equivalence

Definition Let X and Y be topological spaces. A continuous map $f: X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g: Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y. The spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence from X to Y.

Lemma 12.11 *A composition of homotopy equivalences is itself a homotopy equivalence.*

Proof Let X, Y and Z be topological spaces, and let $f: X \to Y$ and $h: Y \to Z$ be homotopy equivalences. Then there exist continuous maps $g: Y \to X$ and $k: Z \to Y$ such that $g \circ f \simeq i_X$, $f \circ g \simeq i_Y$, $k \circ h \simeq i_Y$ and $h \circ k \simeq i_Z$, where i_X, i_Y and i_Z denote the identity maps of the spaces X, Y, Z. Then $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ i_Y \circ f = g \circ f \simeq i_X$ and $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ i_Y \circ k = h \circ k \simeq i_Z$. Thus $h \circ f: X \to Z$ is a homotopy equivalence from X to Z. ∎

Lemma 12.12 *Let $f: |K| \to |L|$ be a homotopy equivalence between the polyhedra of simplicial complexes K and L. Then, for each non-negative integer q, the induced homomorphism $f_*: H_q(K) \to H_q(L)$ of homology groups is an isomorphism.*

Proof There exists a continuous map $g: |L| \to |K|$ such that $g \circ f$ is homotopic to the identity map of $|K|$ and $f \circ g$ is homotopic to the identity map of $|L|$. It follows that the induced homomorphisms $(g \circ f)_*: H_q(K) \to H_q(K)$ and $(f \circ g)_*: H_q(L) \to H_q(L)$ are the identity automorphisms of $H_q(K)$ and $H_q(L)$ for each q. But $(g \circ f)_* = g_* \circ h_*$ and $(f \circ g)_* = g_* \circ h_*$. It follows that $f_*: H_q(K) \to H_q(L)$ is an isomorphism with inverse $g_*: H_q(L) \to H_q(K)$. ∎

Definition A subset A of a topological space X is said to be a *deformation retract* of X if there exists a continuous map $H: X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) \in A$ for all $x \in X$ and $H(a,1) = a$ for all $a \in A$.

Thus a subset A of a topological space X is a deformation retract of X if and only if there exists a function $r: X \to A$ such that $r(a) = a$ for all $a \in A$ and r is homotopic in X to the identity map of X.

Example The unit sphere S^{n-1} in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For if $H(\mathbf{x}, t) = (1 - t + t/|\mathbf{x}|)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$ then $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) \in S^{n-1}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $H(\mathbf{x}, 1) = \mathbf{x}$ when $\mathbf{x} \in S^{n-1}$.

If A is a deformation retract of a topological space X then the inclusion map $i: A \hookrightarrow X$ is a homotopy equivalence.

Theorem 12.13 *The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.*

Proof Let S^{m-1} and S^{n-1} denote the unit spheres in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then S^{m-1} and S^{n-1} are homeomorphic to the polyhedra of simplicial complexes K and L respectively. Let $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ be the inclusion map and let $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ be the map that sends $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ to $(1/|\mathbf{x}|)\mathbf{x}$. Then both $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ are homotopy equivalences.

Suppose that there were to exist a homeomorphism $h: \mathbb{R}^m \to \mathbb{R}^n$. Let $f(\mathbf{x}) = h(\mathbf{x}) - h(\mathbf{0})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Then $f: \mathbb{R}^m \setminus \{\mathbf{0}\} \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ would also be a homeomorphism, and therefore $r_n \circ f \circ i_m: S^{m-1} \to S^{n-1}$ would be a homotopy equivalence. Thus if $\mathbb{R}^m$ and $\mathbb{R}^n$ were homeomorphic then S^{m-1} and S^{n-1} would be homotopy equivalent, and therefore the homology groups of the simplicial complexes K and L would be isomorphic. But $H_q(K) \cong \mathbb{Z}$ when $q = 0$ and $q = m - 1$ and $H_q(K) = 0$ for all other values of q, whereas $H_q(L) \cong \mathbb{Z}$ when $q = 0$ and $q = n - 1$ and $H_q(L) = 0$ for all other values of q. Thus if $m \neq n$ then the homology groups of the simplicial complexes K and L are not isomorphic, and therefore $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic. ∎