1. Let $R$ be a unital commutative ring (i.e., a commutative ring with a non-zero multiplicative identity element, denoted by 1, which satisfies $1x = x = x1$ for all $x \in R$). We say that an element $x$ of $R$ is a unit if and only if there exists some element $x^{-1}$ of $R$ satisfying $xx^{-1} = 1 = x^{-1}x$.

(a) Show that the set of units of $R$ is a group with respect to the operation of multiplication.

(b) Let $x \in R$. Suppose that there exist $s, t \in R$ such that $sx = 1 = xt$. Prove that $x$ is a unit of $R$.

(c) Show that any proper ideal $I$ of $R$ cannot contain any units of $R$.

(d) Let $x$ be an element of $R$ that is not a unit of $R$. Show that the set $Rx$ of multiples of $x$ is a proper ideal of $R$, and that $x \in Rx$.

(e) Prove that a unital ring $R$ has a exactly one maximal ideal if and only if the set
\[ \{ x \in R : X \text{ is not a unit of } R \} \]
is an ideal of $R$.

2. (a) Let $R$ be a ring. Let $\hat{R}$ be the set of all infinite sequences
\[ (r_0, r_1, r_2, \ldots) \]
with $r_i \in R$ for all $i$, and let operations of addition and multiplication be defined on $\hat{R}$ by the formulae
\[
\begin{align*}
(r_0, r_1, r_2, \ldots) + (s_0, s_1, s_2, \ldots) &= (r_0 + s_0, r_1 + s_1, r_2 + s_2, \ldots), \\
(r_0, r_1, r_2, \ldots)(s_0, s_1, s_2, \ldots) &= (t_0, t_1, t_2, \ldots),
\end{align*}
\]
where $t_0 = r_0s_0$, $t_1 = r_0s_1 + s_0r_1$, and
\[ t_i = r_0s_i + r_1s_{i-1} + \cdots + r_{i-1}s_1 + r_is_0. \]
Show that $\hat{R}$, with these algebraic operations, is a ring.
(b) Explain why the polynomial ring $R[t]$ is isomorphic to the subring of $\hat{R}$ consisting of all sequences $(r_0, r_1, r_2, \ldots)$ in $\hat{R}$ with the property that $r_i \neq 0$ for at most finitely many values of $i$.

(c) Suppose that the ring $R$ has a non-zero multiplicative identity element 1. Show that $(1, 0, 0, \ldots)$ is a multiplicative identity element for the ring $\hat{R}$. By examining the formula for the product of two elements of $\hat{R}$, or otherwise, show that an element of $(r_0, r_1, r_2, \ldots)$ of $\hat{R}$ is a unit of $\hat{R}$ if and only if $r_0$ is a unit of $R$.

(d) Suppose that that $R$ is an integral domain. Prove that $\hat{R}$ is also an integral domain. [Hint: given non-zero elements $(r_0, r_1, r_2, \ldots)$ and $(s_0, s_1, s_2, \ldots)$ of $\hat{R}$ with product $(t_0, t_1, t_2, \ldots)$, consider $t_{m+n}$, where $m$ and $n$ are the smallest non-negative integers with the property that $r_m \neq 0$ and $s_n \neq 0$.]

(e) Suppose that $R$ is a field. Prove that $\hat{R}$ has exactly one maximal ideal, and that this maximal ideal consists of all elements $(r_0, r_1, r_2, \ldots)$ of $\hat{R}$ satisfying $r_0 = 0$.

(We can think of an element $(r_0, r_1, r_2, \ldots)$ of the ring $\hat{R}$ as representing a formal power series

$$r_0 + r_1 t + r_2 t^2 + \cdots$$

with coefficients in the ring $R$. Such formal power series are added and multiplied in the obvious fashion. The ring $\hat{R}$ is therefore referred to as the ring of formal power series in the indeterminate $t$ with coefficients in the ring $R$, and is customarily denoted by $R[[t]]$.)

3. Let $R$ be a unital commutative ring.

(a) Let $I$, $J$ and $K$ be ideals of $R$. Verify that

$$I + J = J + I, \quad IJ = JI, \quad (I + J) + K = I + (J + K),$$

$$(IJ)K = I(JK), \quad (I + J)K = IK + JK, \quad I(J + K) = IJ + IK.$$ 

(Here $I+J$ denotes the ideal of $R$ consisting of all elements of $R$ that are of the form $i + j$ for some $i \in I$ and $j \in J$, and $IJ$ denotes the ideal of $R$ consisting of all elements of $R$ that are of the form $i_1 j_1 + i_2 j_2 + \cdots + i_k j_k$ for some elements $i_1, i_2, \ldots, i_k$ of $I$ and $j_1, j_2, \ldots, j_k$ of $J$.) Explain why the set of ideals of a ring $R$ is not itself a unital commutative ring with respect to these operations of addition and multiplication.
(b) Let $I$ and $J$ be ideals of $R$ satisfying $I + J = R$. Show that $(I + J)^n \subset I + J^n$ for all natural numbers $n$ and hence prove that $I + J^n = R$ for all $n$. Thus show that $I^m + J^n = R$ for all natural numbers $m$ and $n$. (The ideal $J^n$ is by definition the set of all elements of $R$ that can be expressed as a finite sum of elements of $R$ of the form $a_1a_2 \cdots a_n$ with $a_i \in J$ for $i = 1, 2, \ldots, n$.)

(c) Let $I$ and $J$ be ideals of $R$ satisfying $I + J = R$. By considering the ideal $(I \cap J)(I + J)$, or otherwise, show that $IJ = I \cap J$.

4. Let $R$ be a unital commutative ring, and let $I$ be a finitely generated ideal of $R$. Show that there exists some natural number $m$ such that $I^m \subset \sqrt{I}$, where $\sqrt{I}$ is the radical of $I$. [Hint: let $\{x_1, x_2, \ldots, x_k\}$ be a finite set that generates the ideal $I$ and let $m = m_1 + m_2 + \cdots + m_k$, where $m_1, m_2, \ldots, m_k$ are chosen such that $x_i^{m_i} \in \sqrt{I}$ for $i = 1, 2, \ldots, k$.]

5. (a) Show that the cubic curve $\{(t, t^2, t^3) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is an algebraic set.

(b) Show that the cone $\{(s \cos t, s \sin t, s) \in \mathbb{A}^3(\mathbb{R}) : s, t \in \mathbb{R}\}$ is an algebraic set.

(c) Show that the unit sphere $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ in $\mathbb{A}^2(\mathbb{C})$ is not an algebraic set.

(d) Show that the curve $\{(t \cos t, t \sin t, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is not an algebraic set.

6. Let $K$ be a field, and let $\mathbb{A}^n$ denote $n$-dimensional affine space over the field $K$.

Let $V$ and $W$ be algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Show that the Cartesian product $V \times W$ of $V$ and $W$ is an algebraic set in $\mathbb{A}^{m+n}$, where

\[ V \times W = \{(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \in \mathbb{A}^{m+n} : (x_1, x_2, \ldots, x_m) \in V \text{ and } (y_1, y_2, \ldots, y_n) \in W\}. \]

7. Give an example of a proper ideal $I$ in $\mathbb{R}[X]$ with the property that $V[I] = \emptyset$. [Hint: consider quadratic polynomials in $X$.]

8. Show that the ideal $I$ of $K[X, Y, Z]$ generated by the polynomials $X^2 + Y^2 + Z^2$ and $XY + YZ + ZX$ is not a radical ideal.
9. Prove that a topological space $Z$ is irreducible if and only if every non-empty open set in $Z$ is connected.

10. Let $K$ be a field, and let $A^n$ denote $n$-dimensional affine space over the field $K$.

   (a) Consider the algebraic set
       $$\{(x, y, z) \in A^3 : xy = yz = zx = 0\}.$$

   Is this set irreducible? Is it connected (with respect to the Zariski topology)?

   (b) Consider the algebraic set
       $$\{(x, y) \in A^2(K) : (y - x)(y - x^2) = 0\},$$

   where $K$ is a field with at least 3 elements. Is this set irreducible? Is it connected (with respect to the Zariski topology)?