

**Course 2BA1: Michaelmas Term 2006.**  
**Sample Problems concerning Sets and**  
**Functions**

1. (a) Prove that  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$  for all sets  $A$ ,  $B$  and  $C$ .  
(b) Prove that  $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$  for all sets  $A$ ,  $B$  and  $C$ .  
(c) Prove that  $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$  for all sets  $A$ ,  $B$  and  $C$ .  
(d) Prove that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  for all sets  $A$ ,  $B$  and  $C$ .
2. For each of the following relations, determine whether or not that relation is reflexive, symmetric, transitive, anti-symmetric, an equivalence relation, and/or a partial order, giving appropriate reasons for your answers:—
  - (a) the relation  $P$  on the set  $\mathbb{N}$  of natural numbers, where natural numbers  $m$  and  $n$  satisfy  $mPn$  if and only if  $m + n$  is divisible by 2;
  - (b) the relation  $Q$  on the set  $\mathbb{N}$  of natural numbers, where natural numbers  $m$  and  $n$  satisfy  $mQn$  if and only if  $m + n$  is divisible by 3.
  - (c) the relation  $R$  on the set  $\mathbb{N}$  of natural numbers, where natural numbers  $m$  and  $n$  satisfy  $mRn$  if and only if  $n = 2^k m$  for some integer  $k$  (which may be positive, zero or negative);
  - (d) the relation  $S$  on the set  $\mathbb{Z}$  of integers, where integers  $x$  and  $y$  satisfy  $xSy$  if and only if  $x^2 \leq y^2$ ;
  - (e) the relation  $Q$  on the set  $\mathbb{Z}$  of integers, where integers  $x$  and  $y$  satisfy  $xQy$  if and only if  $x - y = k^3$  for some integer  $k$ ;
  - (f) the relation  $S$  on the set  $\mathbb{Z}$  of integers, where integers  $x$  and  $y$  satisfy  $xSy$  if and only if  $xy$  is even.
  - (g) the relation  $P$  on the set  $\mathbb{Z}$  of integers, where integers  $x$  and  $y$  satisfy  $xPy$  if and only if  $xy$  is odd.

(h) the relation  $Q$  on the set  $\mathbb{R}$  of real numbers, where real numbers  $x$  and  $y$  satisfy  $xQy$  if and only if  $y^3 = x^3 - x + y$ .

(i) the relation  $P$  on the set  $\mathbb{R}$  of real numbers, where real numbers  $x$  and  $y$  satisfy  $xPy$  if and only if  $x^3 - y^3 + x - y \geq 0$ .

3. For each of the following functions, determine whether or not that function is injective and/or surjective, and whether or not it has a well-defined inverse, giving appropriate reasons for your answers:—

(a) the function  $f: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$  with  $f(1) = 2$ ,  $f(2) = 3$ ,  $f(3) = 2$  and  $f(4) = 4$ ;

(b) the function  $g: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$  with  $f(1) = 2$ ,  $f(2) = 3$ ,  $f(3) = 1$  and  $f(4) = 4$ ;

(c) the function  $h: [1, 2] \rightarrow [0, \frac{1}{2}]$  with

$$h(x) = \frac{x-1}{x},$$

where  $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$  and  $[0, \frac{1}{2}] = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{2}\}$ .

(d) the function  $f: [-1, 1] \rightarrow [-2, 2]$  with  $f(x) = x^3 - x$  for all  $x \in [-1, 1]$ .

(e) the function  $g: [1, 2] \rightarrow [0, 6]$  with  $g(x) = x^3 - x$  for all  $x \in [1, 3]$ .

(f) the function  $h: [0, 1] \rightarrow [-2, 2]$  with  $h(x) = x^3 - x$  for all  $x \in [0, 1]$ .

(g) the function  $f: [-1, 1] \rightarrow [-2, 2]$  with  $f(x) = x^3 + x$  for all  $x \in [-1, 1]$ .

(h) the function  $g: (-1, 1) \rightarrow \mathbb{R}$  with  $g(x) = \frac{1}{1-x^2}$  for all  $x \in (-1, 1)$ .